Boolean Expression using Binary Variables

• An investor wants to determine the most profitable selection of $k$ out of $n$ possible investment opportunities respecting a maximum budget of $B$. Each investment has the following features:
  
  - investment profit $p_j$ ($j = 1, \ldots, n$)
  - investment cost $w_j$ ($j = 1, \ldots, n$)

• Write a ILP model to maximize the total profit (it is not allowed to select a fraction of an investment). Identify the decisions that must be taken and the corresponding decision variables that have to be used. Identify and comment the objective function of the problem and the constraints.

• Now consider the following case in which there are 10 investment opportunities and the budget is $B = 1000$. The estimated profits and costs are:

<table>
<thead>
<tr>
<th></th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
<th>$j = 6$</th>
<th>$j = 7$</th>
<th>$j = 8$</th>
<th>$j = 9$</th>
<th>$j = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>300</td>
<td>200</td>
<td>100</td>
<td>110</td>
<td>250</td>
<td>200</td>
<td>180</td>
<td>130</td>
<td>140</td>
<td>230</td>
</tr>
<tr>
<td>$w$</td>
<td>250</td>
<td>400</td>
<td>80</td>
<td>40</td>
<td>150</td>
<td>200</td>
<td>210</td>
<td>230</td>
<td>145</td>
<td>160</td>
</tr>
</tbody>
</table>

Write the ILP model for this specific case.

• Model the following logical constraints using Boolean Expressions and Binary Variables:
  
  - If investment 2 is chosen, then investment 3 must be chosen
  - If investment 2 is chosen, then investment 4 cannot be chosen
  - If investment 1 and 6 are chosen, then investment 7 must be chosen
  - If investment 1 or 6 are chosen, then investment 8 must be chosen
  - If investments 2 and 3 are chosen, then investment 9 cannot be chosen
  - If investments 2 or 3 are chosen, then investment 10 cannot be chosen
the problem can be modeled using the following binary variables:

\[ x_j = \begin{cases} 
1 & \text{if investment } i \text{ is chosen} \\
0 & \text{otherwise} 
\end{cases} \quad (j = 1, \ldots, n) \]

\[ Z(\text{ILP}) = \max \sum_{j=1}^{n} p_j x_j \]  
(1)

\[ \sum_{j=1}^{n} w_j x_i \leq W \]  
(2)

\[ \sum_{j=1}^{n} x_j = k \]  
(3)

\[ x_j \in \{0, 1\} \quad j = 1, \ldots, n \]  
(4)

- If investment 2 is executed, then investment 3 must be executed \((x_2 \Rightarrow x_3)\) (attention: this also implies that if I don’t execute investment 3 then I don’t execute investment 2 either)

\[
(\text{NOT } x_2) \text{ OR } x_3 \\
1 - x_2 + x_3 \geq 1 \\
x_3 \geq x_2
\]

- If investment 2 is executed, then investment 4 cannot be executed \((x_2 \Rightarrow \text{NOT } x_4)\) (attention: this also implies that if I execute investment 4 then I cannot execute investment 2)

\[
(\text{NOT } x_2) \text{ OR } (\text{NOT } x_4) \\
1 - x_2 + 1 - x_4 \geq 1 \\
x_2 + x_4 \leq 1
\]

- If investment 1 and 6 are executed, then investment 7 must be executed \((x_1 \text{ AND } x_6 \Rightarrow x_7)\)

\[
(\text{NOT } (x_1 \text{ AND } x_6) \text{ OR } (x_7)) \\
(\text{NOT } x_1) \text{ OR } (\text{NOT } x_6) \text{ OR } (x_7) \\
1 - x_1 + 1 - x_6 + x_7 \geq 1 \\
x_1 + x_6 - x_7 \leq 1
\]

- If investment 1 or 6 are executed, then investment 8 must be executed \((x_1 \text{ OR } x_6 \Rightarrow x_8)\)

\[
(\text{NOT } (x_1 \text{ OR } x_6) \text{ OR } x_8) \\
((\text{NOT } x_1) \text{ AND } (\text{NOT } x_6)) \text{ OR } x_8 \\
((\text{NOT } x_1) \text{ OR } (x_8)) \text{ AND } ((\text{NOT } x_6) \text{ OR } (x_8)) \\
\begin{cases} 
1 - x_1 + x_8 \geq 1 \\
1 - x_6 + x_8 \geq 1 \\
x_8 \geq x_1 \\
x_8 \geq x_6
\end{cases}
\]
• If investments 2 and 3 are executed, then investment 9 cannot be executed \((x_2 \text{ AND } x_3 \Rightarrow \text{NOT } x_9)\)

\[
\text{NOT } (x_2 \text{ AND } x_3) \text{ OR } (\text{NOT } x_9) \\
(\text{NOT } x_2) \text{ OR } (\text{NOT } x_3) \text{ OR } (\text{NOT } x_9) \\
1 - x_2 + 1 - x_3 + 1 - x_9 \geq 1 \\
x_2 + x_3 + x_9 \leq 2
\]

• If investments 2 or 3 are executed, then investment 10 cannot be executed \((x_2 \text{ OR } x_3 \Rightarrow \text{NOT } x_{10})\)

\[
\text{NOT } (x_2 \text{ OR } x_3) \text{ OR } (\text{NOT } x_{10}) \\
((\text{NOT } x_2) \text{ AND } (\text{NOT } x_3)) \text{ OR } (\text{NOT } x_{10}) \\
((\text{NOT } x_2) \text{ OR } (\text{NOT } x_{10})) \text{ AND } ((\text{NOT } x_3) \text{ OR } (\text{NOT } x_{10})) \\
\begin{cases}
1 - x_{10} + 1 - x_2 \geq 1 \\
1 - x_{10} + 1 - x_3 \geq 1 \\
x_{10} + x_2 \leq 1 \\
x_{10} + x_3 \leq 1
\end{cases}
\]