Linear Programming

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- Book: Linear Programming, Vasek Chvatal, McGill University, W.H. Freeman and Company, New York, USA

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1 Preliminaries

1.1 Definitions

- Linear programming is a technique for the optimization of
  a linear objective function:
  \[ f(x) = c^T x = \sum_{j=1}^{n} c_j x_j \]

- \( f: \mathbb{R}^n \to \mathbb{R} \) is a linear function, it is convex since:
  \[ \forall x', x'' \in \text{dom}(f), \ 0 \leq \lambda \leq 1, \ \text{we have} \ f(\lambda x' + (1-\lambda)x'') = \lambda f(x') + (1-\lambda)f(x'') \]

- subject to linear inequality constraints:
  \[ g_i(x) \geq 0 \quad A_i x \geq b_i \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \]

- \( F \in \mathbb{R}^n \) is then a convex set defined by a set of linear inequalities:
  \[ g_i(x) \geq 0, \ i = 1, \ldots, m \]

- note that a linear inequality \( g_i(x) \geq 0 \) defines a half-space in \( \mathbb{R}^n \), whose support is the hyper-plane \( g_i(x) = 0 \).

\[ -0.6x_1 - x_2 + 3 \geq 0 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
- the feasible region is called a (convex) polyhedron (polyhedra), which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. In case the convex polyhedron is bounded then it is called (convex) polytope.

Example of 2D → Polytope

\[
\begin{align*}
+x_1 - x_2 & \geq -2 \\
-8x_1 - 2x_2 & \geq -19 \\
 x_1, x_2 & \geq 0
\end{align*}
\]

Example of 2D → Polyhedron

\[
\begin{align*}
x_1 + x_2 & \geq 2 \\
-x_1 + x_2 & \geq 0 \\
x_1, x_2 & \geq 0
\end{align*}
\]

- The objective function is a real-valued affine\(^1\) (linear) function defined on a polyhedron.

Example of objective function \( f(x_1, x_2) = x_1 + 2x_2 \):

\[
\begin{align*}
(x_1 = 0, x_2 = 2) \\
(x_1 = 1.5, x_2 = 3.5) \\
(x_1 = 1.5, x_2 = 3.5) \\
(x_1 = 2.375, x_2 = 0)
\end{align*}
\]

\[
\begin{align*}
(x_1 = 0, x_2 = 2) \\
(x_1 = 1, x_2 = 1) \\
(x_1 = 1, x_2 = 1) \\
(x_1 = 0, x_2 = 0)
\end{align*}
\]

\(^1\)An affine function is a function composed of a linear function plus a constant
A linear programming algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such a point exists.

Example of 3D Polytope

1.2 Notation

\[ Z(LP) = \max c^T x \]

\[ \text{subject to } Ax \leq b \]

\[ x \geq 0 \]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( c \in \mathbb{R}^n \).

INPUT DATA: \( A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \]
VARIABLES: \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \)

Equivalent way an writing an LP:

\[
Z(LP) = \max \sum_{j=1}^{n} c_j x_j \tag{4}
\]

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \tag{5}
\]

\[
x_j \geq 0 \quad j = 1, \ldots, n \tag{6}
\]

Example:

\[
A = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 250 \\ 4 \end{bmatrix} \quad c = \begin{bmatrix} \frac{1}{64} \\ 100 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
Z(LP) = \max x_1 + \frac{64}{100} x_2 \\
50x_1 + 31x_2 \leq 250 \\
-3x_1 + 2x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

- the optimal solution value is \( Z(LP) = \frac{984}{193} \approx 5.09 \)
- the optimal solution \( x^* = \begin{bmatrix} \frac{376}{193} \\ \frac{950}{193} \end{bmatrix} \)
- a feasible (suboptimal) solution is \( \bar{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \) and its corresponding solution value is 5

**Canonical** (\( n \) variables and \( m \) inequalities constraints) and **standard** (\( n + m \) variables and \( m \) equalities constraints) forms

\[
Z(LP) = \max \sum_{j=1}^{n} c_j x_j
\]

\[
\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \quad i = 1, \ldots, m
\]

\[
x_j \geq 0 \quad j = 1, \ldots, n
\]

\[
s_i \geq 0 \quad i = 1, \ldots, m
\]
to transform a problem from the Canonical to the Standard form is it necessary to add $m$ (non-negative) slack variables (one for each inequality) in order to transform inequalities into equalities.

- **Negative variable**: if $x_j < 0$, we can define $x_j^+ \geq 0$ such that $x_j = -x_j^+$
- **Free variable**: if $x_j \geq 0$, we can define $x_j^+, x_j^- \geq 0$ such that $x_j = x_j^+ - x_j^-$
- **Equivalences**:
  \[
  \min f(x) = - \max -f(x) \\
  \min f(x) + k = k + \min f(x) \\
  \min k \cdot f(x) = k \cdot \min f(x) \quad \text{with} \quad k \geq 0 \\
  \min -k \cdot f(x) = -k \cdot \max f(x) \quad \text{with} \quad k \geq 0
  \]

- Equalities can be transformed into inequalities constraints:
  \[
  \sum_{j=1}^{n} a_{ij} x_j = b_i \Rightarrow \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i
  \]

**Linear Programming properties and limits:**

1. **Proportionality**. The effect of the $h$-th variable is proportional to its amount $x_h$:
2. **Additivity.** There is no interaction between the variables in the objective function and constraints. We only sum the contribution of each variable.

3. **Divisibility.** Decision variables can get *non integer* values.

4. **Determinism.** All the model parameters are given constants, while, in real-world, parameters are affected by incertitude.
2 Examples

2.1 Example 1

- A production company has \( n \) warehouses \((i = 1, 2, \ldots, n)\) and it has to serve \( m \) shops \((j = 1, 2, \ldots, m)\). Each warehouses \( i \) can deliver a determined amount of product per week; \( a_i \) is called the capacity of the warehouse. Each shop \( j \) has a known weekly demand of \( b_j \). The cost of shipping one unit of product from warehouse \( i \) to the shop \( j \) has been estimated in \( c_{ij} \) (dollars per unit).

- The LP model to minimize the total cost to satisfy the demands of the shops reads as follows:

Decision variables:

- \( x_{ij} = \) units delivered from warehouse \( i \) to shop \( j \)

\[
\begin{align*}
Z(LP) = \min & \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \\
& \sum_{j=1}^{m} x_{ij} \leq a_i \quad i = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij} \geq b_j \quad j = 1, \ldots, m \\
& x_{ij} \geq 0 \quad i = 1, \ldots, n, j = 1, \ldots, m
\end{align*}
\]

- Now consider the follow case in which two are the warehouses \((n = 2)\) with weekly capacities \( a_1 = 1200.5 \) and \( a_2 = 1100.5 \). Three are the shops \((m = 3)\), the corresponding weekly demands are \( b_1 = 500.5, b_2 = 600.5 \) and \( b_3 = 1000.5 \). The following are the estimated shipping costs:

\[
\begin{array}{c|ccc}
  & j = 1 & j = 2 & j = 3 \\
  i = 1 & 300 & 200 & 100 \\
  i = 2 & 250 & 400 & 80 \\
\end{array}
\]

The LP model for this specific case reads as follows:
\[ Z(LP) = \min 300x_{11} + 200x_{12} + 100x_{13} + 250x_{21} + 400x_{22} + 80x_{23} \]

\[ x_{11} + x_{12} + x_{13} \leq 1200.5 \]
\[ x_{21} + x_{22} + x_{23} \leq 1100.5 \]
\[ x_{11} + x_{21} \geq 500.5 \]
\[ x_{12} + x_{22} \geq 600.5 \]
\[ x_{13} + x_{23} \geq 1000.5 \]
\[ x_{11}, x_{12}, x_{13}, \]
\[ x_{21}, x_{22}, x_{23} \geq 0 \]

- model solution using LibreOffice

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| Solution | 0.00 | 600.50 | 400.50 | 500.50 | 0.00 | 600.00 |

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<td>[ l_3 ]</td>
<td>1000.50 &gt;=</td>
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Figure 1: Model solution using LibreOffice

### 2.2 Example 2

- Consider a company which produces a metal tube and wants to plan the production for the next \( m \) weeks in order to satisfy the expected demand of \( d_t \) \((t = 1, 2, \ldots, m)\) meters of product.

- The company has an initial available quantity of \( p_0 \) meters.

- Each week the company can:
  - produce a max amount of \( p \) meters per week of metal tube at a unit cost of \( c \) $,
- store at a unit cost of \( \tilde{c} \) $ (unlimited amount),
- buy from the market at a unit cost of \( \bar{c} \) $ (unlimited amount).

• The LP model to plan the production at the minimum cost reads as follow:

Decision variables:

- \( x_t \) = meters of metal tube produced at week \( t \) (\( t = 1, 2, \ldots, m \))
- \( y_t \) = meters of metal tube bought form the market at week \( t \) (\( t = 1, 2, \ldots, m \))
- \( s_t \) = quantity stored at period \( t \) (\( t = 0, 1, 2, \ldots, m \))

\[
Z(LP) = \min c \sum_{t=1}^{m} x_t + \bar{c} \sum_{t=1}^{m} y_t + \tilde{c} \sum_{t=1}^{m} s_t \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_t \leq p \quad i = 1, \ldots, m \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_t + y_t + s_{t-1} - s_t = d_t \quad i = 1, \ldots, m \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad s_0 = p_0 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_t, y_t, s_t \geq 0 \quad i = 1, \ldots, m
\]

• Consider now the specific case of 4 weeks (\( m = 4 \)) and the following parameters:
  - the weakly demands \( d_t \) are 45.50, 58.70, 75.90 and 24.60
  - the initial available quantity \( p_0 \) is 11.60 meters
  - the maximum weakly production \( p \) is 41.50 meters
  - the production cost \( c \) is 400$, the market cost \( \bar{c} \) is 450$ and the storage cost \( \tilde{c} \) is 208$

• the LP model for this specific case reads as follows:
\[
Z(LP) = \text{min} \quad 400x_1 + 400x_2 + 400x_3 + 400x_4 \\
450y_1 + 450y_2 + 450y_3 + 450y_4 \\
20s_1 + 20s_2 + 20s_3 + 20s_4
\]
\[
x_1 \leq 41.50 \\
x_2 \leq 41.50 \\
x_3 \leq 41.50 \\
x_4 \leq 41.50 \\
x_1 + y_1 + s_0 - s_1 = 45.50 \\
x_2 + y_2 + s_1 - s_2 = 58.70 \\
x_3 + y_3 + s_2 - s_3 = 75.90 \\
x_4 + y_4 + s_3 - s_4 = 24.60 \\
s_0 = 11.60
\]
\[
x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, s_1, s_2, s_3, s_4 \geq 0
\]

- model solution using LibreOffice

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</tr>
<tr>
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<td>41.50 &lt;= 41.50</td>
</tr>
<tr>
<td></td>
<td>41.50 &lt;= 41.50</td>
</tr>
<tr>
<td></td>
<td>24.60 &lt;= 41.50</td>
</tr>
</tbody>
</table>

Figure 2: Model solution using LibreOffice
3 Simplex Algorithm

3.1 Revised Simplex Algorithm

Input: a Linear Program LP in the canonical form with \( \bar{n} \) variables and \( m \) constraints

\[
Z(LP) = \max \bar{c}^\top x \\
\bar{A}x \leq \bar{b} \\
x \geq 0
\]

where \( \bar{A} \in \mathbb{R}^{m-\bar{n}} \), \( \bar{b} \in \mathbb{R}^m \), \( \bar{c} \in \mathbb{R}^{\bar{n}} \), \( \bar{n} > m \) and rank \( A = m \).

- The LP has to be transformed in the standard form with \( n = \bar{n} + m \) variables and \( m \) constraints:

\[
Z(LP) = \max c^\top x \\
Ax = b \\
x \geq 0
\]

where \( A \in \mathbb{R}^{m-n} \), \( b \in \mathbb{R}^m \), \( c \in \mathbb{R}^n \).

- The simplex algorithm consists of writing the problem \( P \) into a sequence of simplex dictionaries, until the optimality can be proven.

- Main Idea:
  - assign the value of zero to \( n - m \) variables
  - obtain a system a \( m \) equations and \( m \) variables
  - express the value of the \( m \) variables and of the objective function in function of the \( n - m \) variables (set to zero)
  - in this way we can check if it is necessary to increase the value of the \( n - m \) variables set to zero or the solution we have is optimal

- To write a simplex dictionary is necessary to divide the variables into two sets:
  - the basic variables \( x_B \in B \) (\( n_B = m \) variables which form a basis)
  - the nonbasic variables \( x_{\bar{B}} \in \bar{B} \) (\( n_{\bar{B}} = n - n_B \) variables).

- we can then rewrite the constraints as follows:

\[
A_B x_B + A_{\bar{B}} x_{\bar{B}} = b \\
x_B = A_B^{-1}b - A_B^{-1}A_{\bar{B}} x_{\bar{B}}
\]
• we can then rewrite the objective function as follow
\[ c_B^T x_B + c_B^T x_{\bar{B}} = c_B^T A_B^{-1} b + (c_B^T - c_B^T A_B^{-1} A_{\bar{B}}) x_{\bar{B}} \]

**simplex dictionary**: system of linear equations in which the basic variables and the objective function is written in function of the nonbasic variables.

\[
\begin{align*}
    x_B &= \tilde{b} + \tilde{A} x_{\bar{B}} \\
    c^T x &= \psi + \tilde{c}_B^T x_{\bar{B}} \\
    \tilde{b} &= A_B^{-1} b \\
    \tilde{A} &= -A_B^{-1} A_{\bar{B}} \\
    \psi &= c_B^T A_B^{-1} b \\
    \tilde{c}_B^T &= c_B^T - c_B^T A_B^{-1} A_{\bar{B}}
\end{align*}
\]

• Since the nonbasic variables can be set to 0, we have
\[ x_B^* = \tilde{b} \quad \text{and} \quad c^T x = \psi \]

• It always possible to write the optimal simplex dictionary in this format.

• The optimal simplex dictionary corresponds to a basis in which all the reduced costs \( \tilde{c}_B^T \leq 0 \), i.e., it is not possible to improve the current solution by changing the basic and nonbasic variables.

**feasible basic solution** \( x_B^* \): a solution in which the basic variables have non negative values
\[ x_B^* = \tilde{b} \geq 0 \]

in case \( b \geq 0 \) an initial feasible basis is given by the slack variables (otherwise it is necessary the Phase 0 of the simplex algorithm).

• The revised simplex algorithm starts with a feasible basic solution and then it identifies a variable with a positive reduced cost to enter the current basis, called **pivot column** \( p \). This can be done solving the following linear system:
\[ y A_B = c_B^T \]

Once this system is solved we have \( y = c_B^T A_B^{-1} \), i.e., we can determine the pivot column selecting a nonbasic variable \( p \) such that \( y A_p < c_p \) (variable with a positive reduce cost).
The variable that exits the basis is determined increasing as much as possible the value of the entering variable while maintaining a feasible basic solution, i.e., non negative values for the basic variables. The variable which exists the basis is called pivot row. This can be done solving the following linear system:

\[ A_B d = A_p \]

Once this system is solved, we have \[ d = A_B^{-1} A_p = \tilde{A}_p \], i.e. the impact of the pivot column on the current basic variables values (\( x_B^* = \tilde{b} + \tilde{A}_p x_p \)).

Finally it is necessary to determine the largest value \( t \) such that

\[ \tilde{b} - td \geq 0 \]

\( t \) corresponds to the maximum possible value of the entering column that preserves the current feasible basic solution.

If there is no such \( t \) the problem is unbounded; otherwise at least one component of \( \tilde{b} - td = 0 \), identifying the leaving variable (pivot row).

The next feasible basic solution is then built assigning to the entering pivot column the value \( t \) and to the remaining basic variables the value \( \tilde{b} - td \) or, equivalently, \( x_B^* - td \).

Example with \( \bar{n} = 2 \) and \( m = 2 \): 

\[ Z(LP) = \max \quad x_1 + \frac{64}{100} x_2 \]
\[ 50x_1 + 31x_2 \leq 250 \]
\[ -3x_1 + 2x_2 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

\[ \bar{A} = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 250 \\ 4 \end{bmatrix} \quad \bar{c} = \begin{bmatrix} 1 \\ \frac{64}{100} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

standard form \( n = 4 \) and \( m = 2 \): 

\[ Z(LP) = \max \quad x_1 + \frac{64}{100} x_2 \]
\[ 50x_1 + 31x_2 + x_3 = 250 \]
\[ -3x_1 + 2x_2 + x_4 = 4 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

\[ A = \begin{bmatrix} 50 & 31 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 250 \\ 4 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ \frac{64}{100} \\ 0 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \]

first iteration
• Feasible basic solution associated to the basis $B = \{x_3, x_4\}$ ($\bar{B} = \{x_1, x_2\}$):

$$x^*_B = \begin{bmatrix} 250 \\ 4 \end{bmatrix}$$

• $A_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $c_B = [0 \ 0]$, $y = [y_1 \ y_2]$ then

$$yA_B = c_B^T \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases} \quad c_B^T A_B^{-1} y = [0 \ 0]$$

• $c_B^T = \begin{bmatrix} 1 & \frac{64}{100} \end{bmatrix}$, $A_B = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix}$

$$\tilde{c}_B^T = c_B^T - c_B^T A_B^{-1} A_B = \begin{bmatrix} 1 & \frac{64}{100} \end{bmatrix} - [0 \ 0] \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{64}{100} \end{bmatrix} \rightarrow x_1$ enters the basis

• $\psi = c_B^T A_B^{-1} b = [0 \ 0] \begin{bmatrix} 250 \\ 4 \end{bmatrix} = 0$

• $A_p = \begin{bmatrix} 50 \\ -3 \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ then

$$A_B d = A_p \begin{cases} d_1 = 50 \\ d_2 = -3 \end{cases} \quad A_B^{-1} A_p = d = \begin{bmatrix} 50 \\ -3 \end{bmatrix}$$

• find the maximum $t$ such that:

$$\begin{cases} x_3^* - t \cdot d_1 \geq 0 \\ x_4^* - t \cdot d_2 \geq 0 \end{cases} \quad \begin{cases} t \leq 5 \\ t \geq -\frac{4}{3} \end{cases} \rightarrow t = 5$ and $x_3$ leaves the basis

• the other basic variables take values $x^*_B - td$ then $x_4^* = 4 - (5)(-3) = 19$

second iteration

• Feasible basic solution associated to the basis $B = \{x_1, x_4\}$ ($\bar{B} = \{x_2, x_3\}$):

$$x^*_B = \begin{bmatrix} 5 \\ 19 \end{bmatrix}$$

• $A_B = \begin{bmatrix} 50 & 0 \\ -3 & 1 \end{bmatrix}$, $c_B = [1 \ 0]$, $y = [y_1 \ y_2]$ then

$$yA_B = c_B^T \begin{cases} 50y_1 - 3y_2 = 1 \\ y_2 = 0 \end{cases} \quad c_B^T A_B^{-1} y = \begin{bmatrix} \frac{1}{50} \\ 0 \end{bmatrix}$$
• $c_B^\top = \begin{bmatrix} \frac{64}{100} & 0 \end{bmatrix}$, $A_B = \begin{bmatrix} 31 & 1 \\ 2 & 0 \end{bmatrix}$

$\tilde{c}_B^\top = c_B^\top - c_B A_B^{-1} A_B = \begin{bmatrix} \frac{64}{100} \ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{50} \ 0 \end{bmatrix} \begin{bmatrix} 31 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{100} \ -\frac{2}{100} \end{bmatrix} \rightarrow x_2$ enters the basis

• $\psi = c_B A_B^{-1} b = \begin{bmatrix} \frac{1}{50} \ 0 \end{bmatrix} \begin{bmatrix} 250 \\ 4 \end{bmatrix} = 5$

• $A_P = \begin{bmatrix} 31 \\ 2 \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ then

$A_B d = A_p \begin{cases} 50d_1 = 31 \\ -3d_1 + d_2 = 2 \end{cases}$, $A_B^{-1} A_p = d = \begin{bmatrix} \frac{31}{50} \\ \frac{193}{50} \end{bmatrix}$

• find the maximum $t$ s.t.:

  \[
  \begin{cases}
  x_1^* - t \cdot d_1 \geq 0 \\
  x_4^* - t \cdot d_2 \geq 0
  \end{cases}
  \begin{cases}
  t \leq \frac{250}{31} \approx 8.06 \\
  t \leq \frac{950}{193} \approx 4.92
  \end{cases}
  \rightarrow t = \frac{950}{193}$ and $x_4$ leaves the basis

• the other basic variable take values $x_B^* - t d$ then $x_1^* = 5 - \frac{950}{193} \cdot \frac{31}{50} = 5 - \frac{29450}{9650} = \frac{376}{193} \approx 1.94$

third iteration

• Feasible basic solution associated to the basis $B = \{x_1, x_2\}$ ($\tilde{B} = \{x_3, x_4\}$):

  $x_B^* = \begin{bmatrix} \frac{376}{193} \approx 1.94 \\ \frac{950}{193} \approx 4.92 \end{bmatrix}$

• $A_B = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix}$, $c_B^\top = \begin{bmatrix} 1 \ \frac{64}{100} \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then

  $y A_B = c_B^\top \begin{cases} 50y_1 - 3y_2 = 1 \\ 31y_1 + 2y_2 = \frac{64}{100} \end{cases}$

  $c_B^\top A_B^{-1} = y = \begin{bmatrix} \frac{98}{4825} \\ \frac{1}{193} \end{bmatrix}$

• $c_B^\top = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\tilde{c}_B^\top = c_B^\top - c_B A_B^{-1} A_B = \begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{98}{4825} & \frac{1}{193} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{98}{4825} & -\frac{1}{193} \end{bmatrix} \rightarrow$ optimal!

• $\psi = c_B A_B^{-1} b = \begin{bmatrix} \frac{98}{4825} & \frac{1}{193} \end{bmatrix} \begin{bmatrix} 250 \\ 4 \end{bmatrix} = \frac{984}{193} \approx 5.09$
4 Duality

• Every maximization LP problem in the canonical form gives rise to a minimization LP problem called the dual problem. The two problems are linked in an interesting way. Every feasible solution in one yields a bound on the optimal value of the other. In fact, if one of the two problems has an optimal solution, then so does the other and the two optimal values coincide.

• How can we found lower bounds on the optimal value of the following LP?

\[
Z(LP) = \max 4x_1 + x_2 + 5x_3 + 3x_4 \\
= x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
= 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
= -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
= x_1, x_2, x_3, x_4 \geq 0
\]

we can try to get a quick estimate of of the optimal value \( z^* \) of its objective function. To get a reasonably good lower bound on \( z^* \) we need to figure out a reasonably good feasible solution.

• For example, the bound \( z^* \geq 5 \) comes from the feasible solution \((0, 0, 1, 0)\) or the bound \( z^* \geq 22 \) comes from the feasible solution \((3, 0, 2, 0)\).

• How can we found upper bounds on the optimal value? For example looking at the second constraint we can say that \( z^* \leq \frac{275}{3} \), which comes from multiplying the constraint by \( \frac{5}{3} \):

\[
\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}
\]

hence any feasible solution satisfies the inequality (overestimation of the objective function):

\[
4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}
\]

In particular, this inequality holds for the optimal solution, so \( z^* \leq \frac{275}{3} \). We can improve the bound for example summing the second and third constraints:

\[
4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58
\]

thus we have now a better bound, i.e., \( z^* \leq \frac{275}{3} \).

• The bounds are then given by linear combinations of the constraints. That is, we multiply the first constraint by some number \( y_1 \), the second by \( y_2 \), the second by \( y_3 \) and then we sum up. In the first case
\[ y_1 = 0, y_2 = \frac{5}{3}, y_3 = 0 \]

in the second case we had:

\[ y_1 = 0, y_2 = 1, y_3 = 1 \]

the resulting inequality is thus:

\[
(y_1+5y_2-y_3)x_1+(-y_1+y_2+2y_3)x_2+(-y_1+3y_2+3y_3)x_3+(3y_1+8y_2-5y_3)x_4 \leq y_1+55y_2+3y_3
\]  

(13)

of course each of the multipliers must be nonnegative otherwise the corresponding inequality would reverse its direction.

- Finally we want to use the right hand side of (13) as an upper bound on

\[
Z(LP) = \max \ 4x_1 + x_2 + 5x_3 + 3x_4
\]

The bound is then valid if and only if in (13) the coefficient of each variable \( x \) is at least as big as the corresponding coefficient in the objective function:

\[
\begin{align*}
+1 & \quad y_1 & \quad +5 & \quad y_2 & \quad -1 & \quad y_3 & \quad \geq 4 \\
-1 & \quad y_1 & \quad +1 & \quad y_2 & \quad +2 & \quad y_3 & \quad \geq 1 \\
-1 & \quad y_1 & \quad +3 & \quad y_2 & \quad +3 & \quad y_3 & \quad \geq 5 \\
+3 & \quad y_1 & \quad +8 & \quad y_2 & \quad -5 & \quad y_3 & \quad \geq 3
\end{align*}
\]

- now since we want to obtain the best possible bound, we obtain the following LP dual:

\[
Z(D(LP)) = \min \ y_1 + 55y_2 + 3y_3
\]

\[
\begin{align*}
y_1 & \quad +5y_2 & \quad -y_3 & \quad \geq 4 \\
-y_1 & \quad +y_2 & \quad +2y_3 & \quad \geq 1 \\
-y_1 & \quad +3y_2 & \quad +3y_3 & \quad \geq 5 \\
3y_1 & \quad +8y_2 & \quad -5y_3 & \quad \geq 3
\end{align*}
\]

\[
y_1, y_2, y_3 \geq 0
\]

Given a Linear Problem LP with \( n \) non-negative variables and \( m \) inequality constraints:

\[
Z(LP) = \max \sum_{i=1}^{m} b_i y_i \quad (14)
\]

\[
\sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad j = 1, \ldots, n \quad (15)
\]

\[
y_i \geq 0 \quad i = 1, \ldots, m \quad (16)
\]
The dual problem D(LP) with \(m\) non-negative variables and \(n\) inequalities constraints, reads as follows:

\[
Z(D(LP)) \min \sum_{j=1}^{n} c_j x_j \tag{17}
\]

\[
\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \ldots, m \tag{18}
\]

\[
x_j \geq 0 \quad j = 1, \ldots, n \tag{19}
\]

Example:

\[
Z(LP) = \max \quad x_1 + 2x_2 \\
50x_1 + 31x_2 \leq 250 \\
-3x_1 + 2x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

\[
D(Z(LP)) = \min \quad 250y_1 + 4y_2 \\
50y_1 - 3y_2 \geq 1 \\
31y_1 + 2y_2 \geq 2 \\
y_1, y_2 \geq 0
\]

**Theorem 4.1 (Weak Duality Theorem).** Let \((x_1^*, \ldots, x_n^*)\) be a feasible solution of the Primal Problem LP and let \((y_1^*, \ldots, y_m^*)\) be a feasible solution of the Dual Problem D(LP). Then the following hold

\[
\sum_{j=1}^{n} c_j x_j^* \leq \sum_{i=1}^{m} b_i y_i^*
\]

**Theorem 4.2 (Strong Duality Theorem).** If the Primal Problem LP has an optimal solution \((x_1^*, \ldots, x_n^*)\) then the Dual Problem D(LP) has an optimal solution \((y_1^*, \ldots, y_m^*)\) such that

\[
\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*
\]

**Dual Problem (inequalities and equalities)** Given a Linear Problem LP with \(n\) variables \((n_1\) non negative variables and \(n-n_1\) free variables) and \(m\) constraints \((m_1\) inequalities and \(m-m_1\) equalities):
\[ Z(\text{LP}) = \max \sum_{j=1}^{n} c_j x_j \quad (20) \]

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m_1 \quad (21) \]

\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = m_1 + 1, \ldots, m \quad (22) \]

\[ x_j \geq 0 \quad j = 1, \ldots, n_1. \quad (23) \]

The dual problem \( D(\text{LP}) \) with \( m \) variables (\( m_1 \) non negative variables and \( m - m_1 \) free variables) and \( n \) constraints (\( n_1 \) inequalities and \( n - n_1 \) equalities), reads as follows:

\[ Z(\text{D(LP)}) = \min \sum_{i=1}^{m} b_i y_i \quad (24) \]

\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = 1, \ldots, n_1 \quad (25) \]

\[ \sum_{i=1}^{m} a_{ij} y_i = c_j \quad j = n_1 + 1, \ldots, n \quad (26) \]

\[ y_i \geq 0 \quad i = 1, \ldots, m_1. \quad (27) \]

Example:

\[ Z(\text{LP}) = \max \quad x_1 + 2x_2 \]
\[ 50x_1 + 31x_2 \leq 250 \]
\[ -3x_1 + 2x_2 = 4 \]
\[ x_1 \geq 0 \]

\[ D(Z(\text{LP})) = \min \quad 250y_1 + 4y_2 \]
\[ 50y_1 - 3y_2 \geq 1 \]
\[ 31y_1 + 2y_2 = 2 \]
\[ y_1 \geq 0 \]
5 Optimality Conditions

Theorem 5.1 (Complementary Slackness (A)). Let \((x_1^*, \ldots, x_n^*)\) be a feasible solution of the Primal Problem LP and let \((y_1^*, \ldots, y_m^*)\) be a feasible solution of the Dual Problem D(LP). Necessary and sufficient conditions for the simultaneous optimality of \((x_1^*, \ldots, x_n^*)\) and \((y_1^*, \ldots, y_m^*)\) are

\[
\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{or (or both)} \quad x_j^* = 0 \quad \text{for every} \quad j = 1, \ldots, n
\]

and

\[
\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{or (or both)} \quad y_i^* = 0 \quad \text{for every} \quad i = 1, \ldots, m.
\]

Theorem 5.2 (Complementary Slackness (B)). Let \((x_1^*, \ldots, x_n^*)\) be a feasible solution of the Primal Problem LP and let \((y_1^*, \ldots, y_m^*)\) be a feasible solution of the Dual Problem D(LP). Necessary and sufficient conditions for the simultaneous optimality of \((x_1^*, \ldots, x_n^*)\) and \((y_1^*, \ldots, y_m^*)\) are

\[
x_j^*(c_j - \sum_{i=1}^{m} a_{ij} y_i^*) = 0 \quad \text{for every} \quad j = 1, \ldots, n.
\]

and

\[
y_i^*(\sum_{j=1}^{n} a_{ij} x_j^* - b_i) = 0 \quad \text{for every} \quad i = 1, \ldots, m.
\]

Theorem 5.3 (Complementary Slackness (C)). Let \((x_1^*, \ldots, x_n^*)\) be a feasible solution of the Primal Problem LP and let \((y_1^*, \ldots, y_m^*)\) be a feasible solution of the Dual Problem D(LP). Necessary and sufficient conditions for the simultaneous optimality of \((x_1^*, \ldots, x_n^*)\) and \((y_1^*, \ldots, y_m^*)\) are

\[
x_j^* + (c_j - \sum_{i=1}^{m} a_{ij} y_i^*) > 0 \quad \text{for every} \quad j = 1, \ldots, n.
\]

Theorem 5.4 (Complementary Slackness (D)). A feasible solution of the Primal Problem LP \((x_1^*, \ldots, x_n^*)\) is optimal if and only if there are numbers \((y_1^*, \ldots, y_m^*)\) such that (dual optimality):

\[
\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{whenever} \quad x_j^* > 0 \quad \text{for every} \quad j = 1, \ldots, n
\]

\[
y_i^* = 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_{ij} x_j^* < b_i \quad \text{for every} \quad i = 1, \ldots, m
\]

and such that (dual feasibility)
\[ \sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \text{ for every } j = 1, \ldots, n \]
\[ y_i^* \geq 0 \text{ for every } i = 1, \ldots, m \]

### 5.1 Complementary Slackness

**Theorem 5.5.** A feasible solution \((x_1^*, \ldots, x_n^*)\) of the Primal Problem \(P\) is optimal if and only if there are numbers \((y_1^*, \ldots, y_m^*)\) such that

\[ \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ whenever } x_j^* > 0 \]
\[ y_i^* = 0 \text{ whenever } \sum_{j=1}^{n} a_{ij} x_j^* < b_i \]

and such that

\[ \sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \text{ for all } j = 1, \ldots, n \]
\[ y_i^* \geq 0 \text{ for all } i = 1, \ldots, m. \]

Example with \(n = 2\) and \(m = 2\):

\[
Z(LP) = \max \quad x_1 + \frac{64}{100} x_2 \\
50x_1 + 31x_2 \leq 250 \\
-3x_1 + 2x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

\[
A = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 250 \\ 4 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ \frac{64}{100} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

• Feasible solution: \(x^* = \begin{bmatrix} \frac{376}{193} \\ \frac{950}{193} \end{bmatrix}\) optimal?

• \(y_i^* = 0\) if \(\sum_{j=1}^{n} a_{ij} x_j^* < b_i\) \(\begin{cases} 50 \cdot \frac{376}{193} + 31 \cdot \frac{950}{193} = 250 \\ -3 \cdot \frac{376}{193} + 2 \cdot \frac{950}{193} = 4 \end{cases} \rightarrow \text{no } y \text{ variables set to 0}\)

• \(\sum_{i=1}^{m} a_{ij} y_i^* = c_j\) if \(x_j^* > 0\) \(\begin{cases} 50y_1 - 3y_2 = 1 \\ 31y_1 + 2y_2 = \frac{64}{100} \end{cases}\)

• \(y_i^* \geq 0\) for all \(i = 1, \ldots, m \rightarrow \text{OK}\)
• \[ \sum_{i=1}^{m} a_{ij}y_i^* \geq c_j \quad \text{forall} \quad j = 1, \ldots, n \]

\[
\begin{align*}
50 \cdot \frac{98}{4825} - 3 \cdot \frac{1}{193} &= \frac{1}{64} \quad \rightarrow \text{Optimal}
\end{align*}
\]

• Feasible solution: \( x^* = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \)

\[ y_i^* = 0 \quad \text{if} \quad \sum_{j=1}^{n} a_{ij}x_j^* < b_i \quad \left\{ \begin{array}{l}
50 \cdot 0 + 31 \cdot 2 = 62 < 250 \\
-3 \cdot 0 + 2 \cdot 2 = 4
\end{array} \right. \rightarrow y_1 = 0
\]

\[ \sum_{i=1}^{m} a_{ij}y_i^* = c_j \quad \text{if} \quad x_j^* > 0 \quad \left\{ \begin{array}{l}
31y_1 + 2y_2 = \frac{64}{100} \\
y_1 = 0
\end{array} \right. \quad y^* = \begin{bmatrix} 0 \\ \frac{32}{100} \end{bmatrix}
\]

• \( y_i^* \geq 0 \quad \text{forall} \quad i = 1, \ldots, m \quad \rightarrow \text{OK} \)

\[ \sum_{i=1}^{m} a_{ij}y_i^* \geq c_j \quad \text{forall} \quad j = 1, \ldots, n \quad \left\{ \begin{array}{l}
50 \cdot 0 - 3 \cdot \frac{32}{100} = \frac{96}{100} < 1
\end{array} \right. \quad \rightarrow \text{Not Optimal}
\]
6 Linear-fractional programming

- **Linear-fractional programming (LFP)**, consider the following objective function subject to linear constraints:

\[
Z(\text{LFP}) = \max \frac{c^\top x + \alpha}{d^\top x + \beta}
\]

(28)

\[
Ax \leq b
\]

(29)

\[
x \geq 0
\]

(30)

- it is assumed that the constraints (29) generate a solution set \( X \) nonempty and bounded:

\[
X \equiv \{ x : Ax \leq b, x \geq 0 \}
\]

- in addition in order to avoid a division by 0 we must consider the additional logical constraints:

\[
d^\top x + \beta > 0 \quad \text{or} \quad d^\top x + \beta < 0.
\]

- the following transformation of variables is now introduced:

\[
y = tx \quad t = \frac{1}{d^\top x + \beta}
\]

- where \( t \geq 0 \) has to be chosen so that:

\[
d^\top y + \beta t = \gamma
\]

(31)

- where \( \gamma \neq 0 \) is a specified number in order to avoid the denominator to be 0.

- On multiplying numerator and denominator by \( t \) we obtain:

\[
\frac{t(c^\top x + \alpha)}{t(d^\top x + \beta)} = \frac{c^\top tx + \alpha t}{d^\top x + \beta} = c^\top y + \alpha t
\]

- Finally on multiplying also the system of inequalities by \( t \) and taking the new equality (31) into account, we obtain the linear programming problem (LP):

\[
Z(\text{LP}) = \max c^\top y + \alpha t
\]

(32)

\[
Ay - bt \leq 0
\]

(33)

\[
d^\top y + \beta t = \gamma
\]

(34)

\[
t, y \geq 0
\]

(35)
Lemma 6.1. Every \((y, t)\) satisfying the constraints ((33)-(35)) has \(t > 0\).

Proof. Suppose \((\hat{y}, 0)\) satisfies the constraints. Let \(\hat{x}\) be any element of \(X\). Then \(x_\mu \equiv \hat{x} + \mu \hat{y}\) is \(\in X\) for \(\mu > 0\) since

\[ A\hat{y} \leq 0, \quad \hat{y} \geq 0. \]

But then \(X\) is unbounded contrary to the regularity hypothesis imposed on \(X\). \(\Box\)

Theorem 6.2. If:

- (i) \(0 < \text{sgn}(\gamma)^2 = \text{sgn}(d^T x^* + \beta)\) where \(x^*\) an optimal solution of the original LFP
- (ii) \((y^*, t^*)\) is an optimal solution of \(\bar{L}\bar{P}\)

then \(\frac{y^*}{t^*}\) is an optimal solution of the original LFP.

Proof. Suppose the theorem were false, i.e., assume that there exists an optimal \(x^* \in X\) such that:

\[
\frac{c^T x^* + \alpha}{d^T x^* + \beta} > \frac{c^T (\frac{y^*}{t^*}) + \alpha}{d^T (\frac{y^*}{t^*}) + \beta}
\]

By condition (i), we have

\[ d^T x^* + \beta = \theta \gamma \]

for some \(\theta > 0\). Consider now

\(\hat{y} = \theta^{-1} x^*, \quad \hat{t} = \theta^{-1} \).

Then

\[ \theta^{-1}(d^T x^* + \beta) = d^T \hat{y} + \hat{t}\beta = \gamma. \]

and \((\hat{y}, \hat{t})\) also satisfies \(A\hat{y} - \hat{b} \leq 0, \quad \hat{y}, \hat{t} \geq 0\). But

\[
\frac{c^T x^* + \alpha}{d^T x^* + \beta} = \frac{\theta^{-1}(c^T x^* + \alpha)}{\theta^{-1}(d^T x^* + \beta)} = \frac{c^T \hat{y} + \alpha \hat{t}}{d^T \hat{y} + \beta \hat{t}} = \frac{c^T \hat{y} + \alpha \hat{t}}{\gamma}
\]

Also

\[
\frac{c^T (\frac{y^*}{t^*}) + \alpha}{d^T (\frac{y^*}{t^*}) + \beta} = \frac{c^T y^* + \alpha t^*}{d^T y^* + \beta t^*} = \frac{c^T y^* + \alpha t^*}{\gamma}
\]

2In mathematics, the sign function or signum function (from signum, Latin for “sign”) is a mathematical function that extracts the sign of a real number. In mathematical expressions the sign function is often represented as sgn
Since, by hypothesis (i), \( \gamma > 0 \) we have:

\[
\mathbf{c}^\top \hat{y} + \alpha \hat{t} > \mathbf{c}^\top \mathbf{y}^* + \alpha t^*
\]
a contradiction to \((\mathbf{y}^*, t^*)\) optimal for (LP).

\[\square\]

**Theorem 6.3.** For any \( X \) nonempty and bounded, to solve the problem LFP it suffices to solve the two ordinary linear programming problems:

\[
Z(\bar{\text{LP}}_1) = \max \mathbf{c}^\top \mathbf{y} + \alpha t
\]

(36)

\[
\mathbf{A} \mathbf{y} - b t \leq 0
\]

(37)

\[
d^\top \mathbf{y} + \beta t = 1
\]

(38)

\[
t, \mathbf{y} \geq 0
\]

(39)

and

\[
Z(\bar{\text{LP}}_2) = \max -\mathbf{c}^\top \mathbf{y} - \alpha t
\]

(40)

\[
\mathbf{A} \mathbf{y} - b t \leq 0
\]

(41)

\[
-d^\top \mathbf{y} - \beta t = 1
\]

(42)

\[
t, \mathbf{y} \geq 0
\]

(43)

If one knows the sign of the denominator for the functional at an optimum, one need only to solve a single ordinary linear programming problem (e.g., in the cases in which the function \( d^\top \mathbf{x} + \beta \) is always positive or always negative).

**Example:**

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} & \mathbf{b} &= \begin{bmatrix} 250 \\ 4 \end{bmatrix} & \mathbf{c} &= \begin{bmatrix} 1 \\ \frac{64}{100} \end{bmatrix} & d &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \alpha &= 0 & \beta &= 5 & \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*}
\]

\[
Z(\text{FLP}) = \max \frac{x_1 + \frac{64}{100}x_2}{2x_1 + 3x_2 + 5}
\]

\[
\begin{align*}
50x_1 + 31x_2 & \leq 250 \\
-3x_1 + 2x_2 & \leq 4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

- the function \( 2x_1 + 3x_2 + 5 \) is always > 0
- then we only have to solve \( \bar{\text{LP}}_1 \)
\[
Z(\bar{\text{LP}_1}) = \max \quad y_1 + \frac{64}{100} y_2 \\
50y_1 + 31y_2 - 250t \leq 0 \\
-3y_1 + 2y_2 - 4t \leq 0 \\
2y_1 + 3y_2 + 5t = 1 \\
y_1, y_2, t \geq 0
\]

- the optimal solution value is \(Z(\bar{\text{LP}_1}) \approx 0.33\)

- the optimal solution \(y^* = \begin{bmatrix} 0.33 \\ 0 \end{bmatrix} \quad t^* = 0.067\)

- using the relation: \(x^* = \frac{y^*}{t^*}\), the optimal solution \(x^* = \begin{bmatrix} 5 \\ 0 \end{bmatrix}\)

- the optimal solution value recomputed is \(Z(\text{FLP}) \approx 0.33\)
7 Additional Elements

7.1 Scenario based Linear Programming

• consider a set of $q$ objective functions in terms of maximization (each one linked to a different possible scenario $S$)

$$\max \sum_{j=1}^{n} c_j^k x_j \quad (k = 1, \ldots, q)$$

• consider now the problem $P^S$ in which one wants to minimize the maximum between the $q$ objective functions

$$Z(P^S) = \min \max_{k=1,\ldots,q} \sum_{j=1}^{n} c_j^k x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \ldots, m$$

$$x_j \geq 0 \quad j = 1, \ldots, n$$

• in order to get an LP problem one can introduce a continuous variable $z$ and $q$ additional constraints. The resulting LP reads as follows:

$$Z(LP^S) = \min \quad z$$

$$\sum_{j=1}^{n} c_j^k x_j \leq z \quad k = 1, \ldots, q$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \ldots, m$$

$$x_j \geq 0 \quad j = 1, \ldots, n$$

Example with 3 scenarios $q = 2$:

$$A = \begin{bmatrix} 50 & 31 \\ -3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 250 \\ 4 \end{bmatrix} \quad c^1 = \begin{bmatrix} 1 \\ \frac{64}{100} \end{bmatrix} \quad c^2 = \begin{bmatrix} \frac{64}{100} \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Z(LP^S) = \min \quad z$$

$$x_1 + \frac{64}{100} x_2 \leq z$$

$$\frac{64}{100} x_1 + x_2 \leq z$$

$$50 x_1 + 31 x_2 \geq 250$$

$$-3 x_1 + 2 x_2 \geq 4$$

$$x_1, x_2, z \geq 0$$
• the optimal solution value is $Z(LP^S) = \approx 6.17$

• the optimal solution is $x^* = \begin{bmatrix} \approx 1.95 \\ \approx 4.92 \end{bmatrix}$

$z^* \approx 6.17$