Ceteris paribus majority for social ranking

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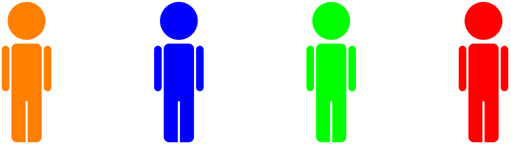
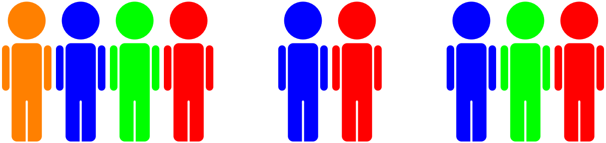
Problem definition

Individuals





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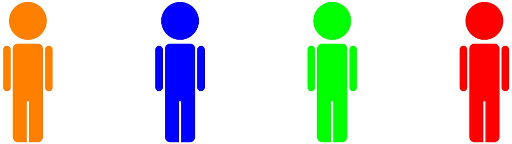

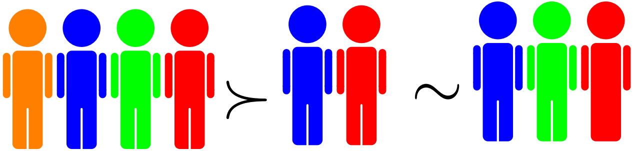

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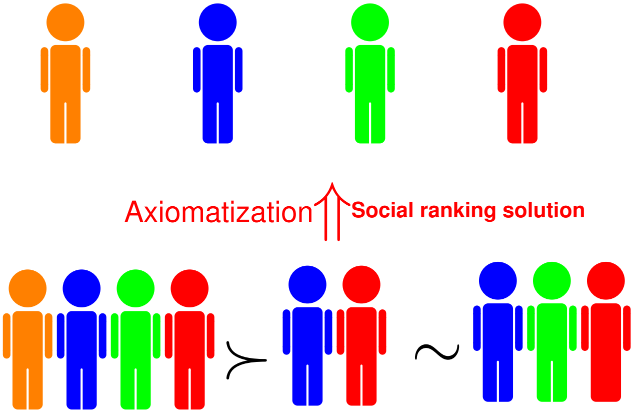

Outline

1. Social ranking problem

Objective

Input :

- A set of individuals : $N = \{1, \dots, n\}$
- A *power relation* $\succeq$ on 2^N :
  $S \succeq T$: The "team" S performs at least as good as T.

  We suppose $\succeq \in \mathcal{B}(2^N)$, set of all binary relation.

Output :

- A *solution* $R^{\succeq}$($I^{\succeq}$ the symmetric part, $P^{\succeq}$ the strict part), associates to every power relation ($\succeq$) a ranking (total preorder) over the set of individuals.

Critical information

Ceteris Paribus comparisons :

$$234 \succ 24 \succ 134 \succ 13 \sim 23 \succ 12 \succ 3 \succ 14$$

$$1 \;?\; 2$$

Ceteris Paribus Comparisons :

$$2\,\boxed{4} \succ 1\,\boxed{4}$$
$$1\,\boxed{3} \succ 2\,\boxed{3}$$
$$2\,\boxed{34} \succ 1\,\boxed{34}$$

Ranking two alternatives ?

Ceteris Paribus Majority solution

$$2345 \succ 245 \succ 1234 \succ 13 \sim 23 \succ 12 \succ 145 \succ 35 \succ 24 \succ 14$$

Coalition	Comparison
45	$2\,45 \succ 1\,45$
3	$1\,3 \sim 2\,3$
4	$2\,4 \succ 1\,4$

$$D_{12} = \{\}\ , |D_{12}| = d_{12} = 0$$

$$D_{21} = \{45, 4\},\ |D_{21}| = d_{21} = 2$$

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$$2 R^{\succeq} 1$$

Ceteris Paribus Majority rule because it utilizes comparison of Ceteris Paribus coalitions

Majority because it counts number of times each researcher is winner

Ceteris Paribus Majority

Definition (Ceteris Paribus Majority)

Let $\succeq \in \mathcal{B}(2^N)$. The *ceteris paribus majority relation* (CP-majority) is the binary relation $R^{\succeq} \subseteq N \times N$ such that for all $x, y \in N$:

$$xR^{\succeq}y \Leftrightarrow d_{xy}(\succeq) \geq d_{yx}(\succeq).$$

Property driven approach

Inspiring from classical social choice theory three axioms are defined :

- Equality of Coalitions
- Neutrality
- Positive Responsiveness

Equality of Coalitions

Equality of coalitions says that each coalition should influence the social ranking of two alternatives x and y equally.

$\succeq$	$\sqsupseteq$
$245 \succ 145$	$2345 \sqsupset 1345$
$13 \sim 23$	$14 \simeq 24$
$24 \succ 14$	$234 \sqsupset 134$

Therefore :

$$2R^{\succeq}1 \Leftrightarrow 2R^{\sqsupseteq}1$$

Equality of Coalitions

Definition (Equality of Coalitions)

Let $A \subseteq N$. A solution $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ satisfies the property of *Equality of Coalitions* (EC) if and only if for all power relations $\succeq, \sqsupseteq \in \mathcal{B}(2^N)$, $x, y \in A$ and bijection $\pi : 2^{N \setminus \{x,y\}} \rightarrow 2^{N \setminus \{x,y\}}$ such that $S \cup \{x\} \succeq S \cup \{y\} \Leftrightarrow \pi(S) \cup \{x\} \sqsupseteq \pi(S) \cup \{y\}$ for all $S \in 2^{N \setminus \{x,y\}}$, it holds that $xR_A^{\succeq} y \Leftrightarrow xR_A^{\sqsupseteq} y$.

Neutrality

Neutrality states that a solution should not favor any candidate in $A \subseteq N$: if the name of two elements in A is reversed, the social ranking remains the same.

The solution is not biased in favor of one researcher

Coalitions	$\succeq$	$\sqsupseteq$
45	$245 \succ 145$	$145 \sqsupset 245$
3	$13 \sim 23$	$23 \simeq 13$
4	$24 \succ 14$	$14 \sqsupset 24$

If $2 R_A^{\succeq} 1$ then $1 R_A^{\sqsupseteq} 2$

Neutrality

Definition (Neutrality)

Let $A \subseteq N$. A solution $R_A \colon \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ satisfies the property of *Neutrality* (N) if and only if for all power relations $\succeq, \sqsupseteq \in \mathcal{B}(2^N)$ and $x, y \in A$ such that $S \cup \{x\} \succeq S \cup \{y\} \Leftrightarrow S \cup \{y\} \sqsupseteq S \cup \{x\}$ for all $S \in 2^{N \setminus \{x,y\}}$, it holds that $x R_A^{\succeq} y \Leftrightarrow y R_A^{\sqsupseteq} x$.

Positive Responsiveness

Positive Responsiveness states that a solution should be coherent with changes of the power relation of coalitions.

Coalition	$\succcurlyeq$	$\sqsupseteq$
45	$245 \succ 145$	$245 \sqsupset 145$
3	$13 \sim 23$	$13 \sqsupset 23$
4	$24 \succ 14$	$24 \sqsupset 14$

$$1 R^{\succeq} 2 \Leftrightarrow 1 P^{\sqsupseteq} 2$$

Positive Responsiveness

Definition (Positive Responsiveness)

Let $A \subseteq N$. A solution $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ satisfies the property of *Positive Responsiveness* (PR) if and only if for all power relations $\succeq, \sqsupseteq \in \mathcal{B}(2^N)$, $x, y \in A$ with $x R_A^{\succeq} y$ and such that for some $T \in 2^{N \setminus \{x,y\}}$, $[T \cup \{x\} \sim T \cup \{y\}$ and $T \cup \{x\} \sqsupset T \cup \{y\}]$, or, $[T \cup \{y\} \succ T \cup \{x\}$ and $T \cup \{x\} \simeq T \cup \{y\}]$ and $S \cup \{x\} \succeq S \cup \{y\} \Leftrightarrow S \cup \{y\} \sqsupseteq S \cup \{x\}$ for all $S \in 2^{N \setminus \{x,y\}}$ with $S \neq T$, it holds that $x P_A^{\sqsupseteq} y$.

Characterization

Theorem

Let $A = \{x, y\} \subseteq N$ be a set with only two alternatives. A solution $R_A \colon \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ associates to each $\succeq \in \mathcal{B}(2^N)$ the corresponding CP-majority relation $R^{\succeq} \cap A \times A$ if and only if it satisfies axioms EC, N and PR.

Condorcet-like paradox

Suppose :

$$2 \succ 1 \succ 3 \succ 23 \succ 13 \succ 12 \succ 14 \succ 34 \succ 24 \succ 134 \sim 124 \sim 234$$

$$A = \{1, 2, 3\}$$

1 vs. 2	2 vs. 3	1 vs. 3
$2 \succ 1$ $23 \succ 13$ $14 \succ 24$ $134 \sim 234$	$2 \succ 3$ $13 \succ 12$ $34 \succ 24$ $124 \sim 134$	$1 \succ 3$ $23 \succ 12$ $14 \succ 34$ $124 \sim 234$

$$2 P_A^{\succ} 1,\ 3 P_A^{\succ} 2,\ 1 P_A^{\succ} 3$$

Restriction on the power relation

Question

Consider three alternatives $i, j, k \in N$. Under which restrictions on the power relation $\succeq \in \mathcal{B}(2^N)$ the social ranking solution results in a transitive ranking over individuals ?

Social Single Peakedness

Definition (Social single peakedness)

The (linear) power relation $\succ$ is socially single-peaked if there exists a linear order $\lhd$ on the set of items N such that for any $i, j, k \in N$ for which $i \lhd j \lhd k$ and any $S \in 2^{N \setminus \{i,j,k\}}$, none of the following conditions holds :

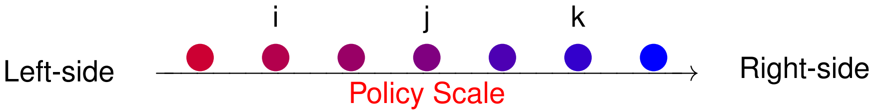
(sp$_1$) $S \cup \{i\} \succ S \cup \{j\}$ and $S \cup \{k\} \succ S \cup \{j\}$,

(sp$_2$) $S \cup \{i,k\} \succ S \cup \{i,j\}$ and $S \cup \{i,k\} \succ S \cup \{j,k\}$.

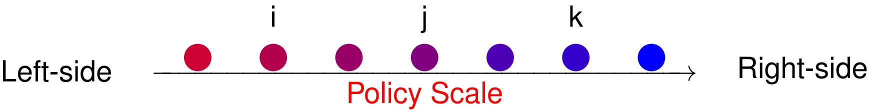
Policy Scale

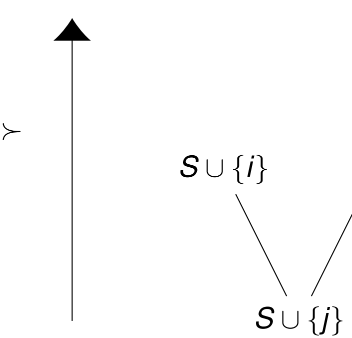
- Political interest can provide a scale to linearly order individuals.
- Researchers in a lab can be ordered linearly based on their level of experience.

Restriction one



Restriction one

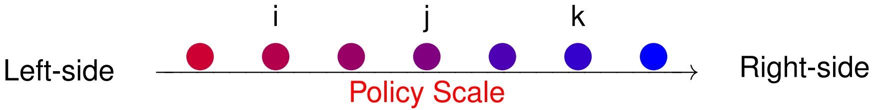


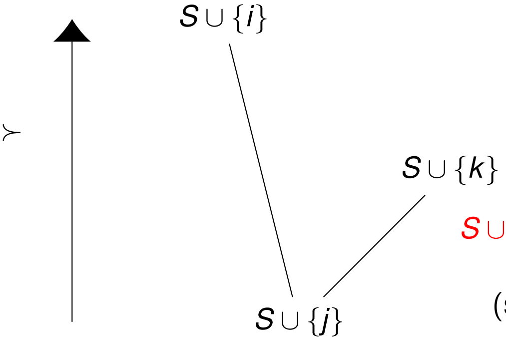


$S \cup \{k\} \succ S \cup \{i\} \succ S \cup \{j\}$

(sp_1)

Restriction one

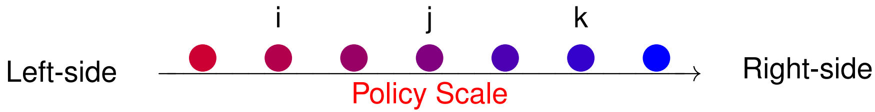


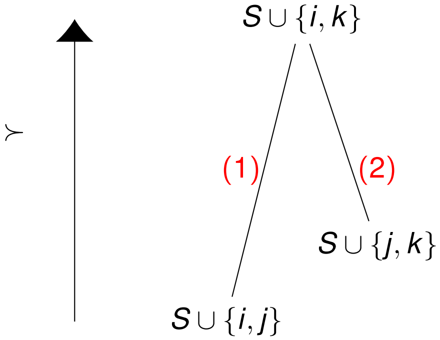


$S \cup \{i\} \succ S \cup \{k\} \succ S \cup \{j\}$

(sp_1)

Restriction two





$(1): S \cup \{i, k\} \succ S \cup \{i, j\}$
$(2): S \cup \{i, k\} \succ S \cup \{j, k\}$

(sp_2)

Theorem

Theorem

If the power relation $\succeq$ is socially single-peaked, then for any items $i, j, k \in N$, it does not hold that $iR^{\succ}jR^{\succ}kR^{\succ}i$ (i.e., the ceteris paribus majority solution does produce any non-transitive cycles).

Conclusion

- Big literature is available about the inverse problem : Ranking over individuals $\rightarrow$ ranking over teams
- **Ordinal ranking over teams** $\rightarrow$ **Ordinal ranking over individuals**
  - ✓ Equality of Coalitions, Neutrality, Positive Responsiveness.
  - ✓ Social single peakedness.

Future works

- Our way is to utilize more information in the power relaiton
  - Extending Shapley value and Banzhaf intex to ordinal case.
  - Considering possibility of forming coalitions in order to rank individuals.
- Complexity issues of applying the solution on real application.