An Ordinal Banzhaf Index for Social Ranking

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Power index :

Society : Company, Parliament,...

Power index indicates the influence of individuals in the society, it can be cardinal or ordinal.

Power Relation (Binary relation)

$\succ$

$\sim$

Power Relation (Binary relation)

 $\succ$  $\sim$ 

Social ranking solution\ Axiomatization

Total preorder  ?  ? 

Objective

Input :

- A set of individuals : $N = \{1, \ldots, n\}$
- A *power relation* $\succeq$ on 2^N :
  $S \succeq T$: The "team" S performs at least as good as T.

  We suppose $\succeq \in \mathcal{B}(2^N)$, set of all binary relations.

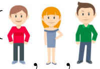
Output :

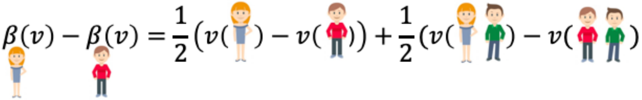
- A *solution* $R^{\succeq}$ ($I^{\succeq}$ the symmetric part, $P^{\succeq}$ the strict part), associates to every power relation ($\succeq$) a ranking (total preorder) over the set of individuals.

Ordinal Banzhaf solution : Motivation

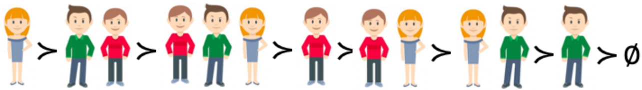
- The Banzhaf value $\beta(v)$ of TU-game (N, v) is the n-vector $\beta(v) = (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$, such that for each $i \in N$:

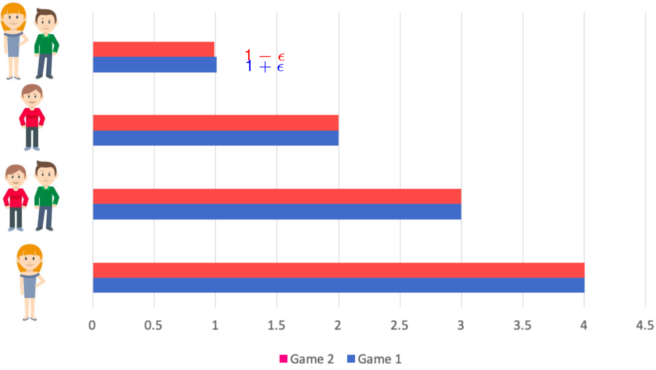
$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in 2^N, i \notin S} \big(v(S \cup \{i\}) - v(S)\big).$$

- For a cooperative game $(\{$$\}, v)$ in order to rank any two individuals  and  we have :



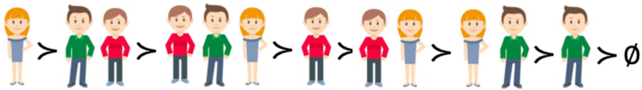
Motivation : Robustness

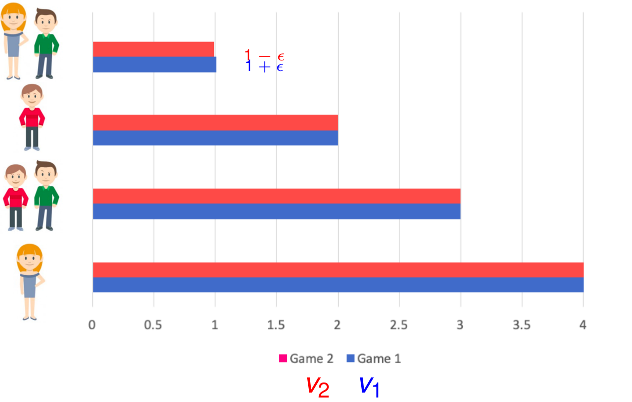




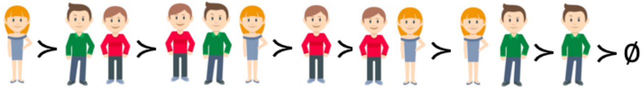
v_2 v_1

Motivation : Robustness

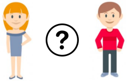




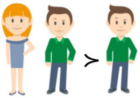
Solutions : Ordinal Banzhaf Solution

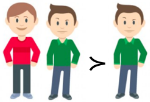


Ordinal Banzhaf rule :

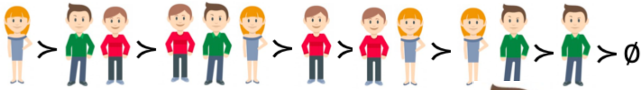


Informative part : Marginalistic comparisons

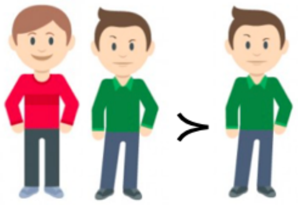




Ordinal Marginal Contribution

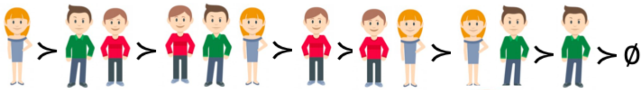


Ordinal Marginal contribution of Red w.r.t

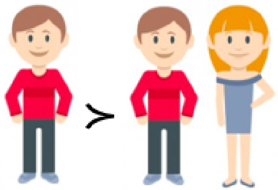


> 0

Ordinal Marginal Contribution



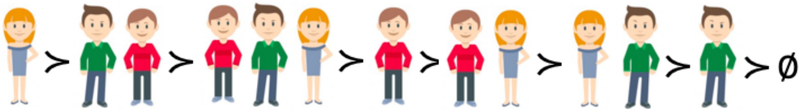
Ordinal Marginal contribution of Girl w.r.t 



< 0

Banzhaf Score

- To compute the Banzhaf Score of individual Red we compute the difference between the times its ordinal marginal contribution is positive and number of times it is negative.



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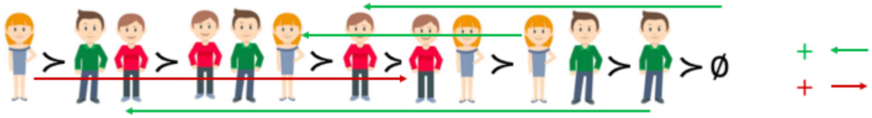
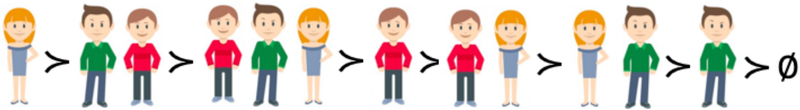
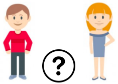


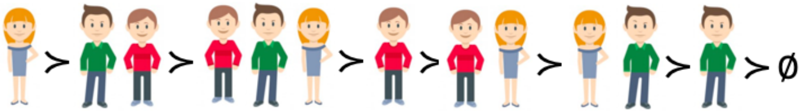
$$3 - 1 = 2$$

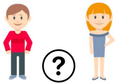
Ordinal Banzhad solution





Ordinal Banzhad solution

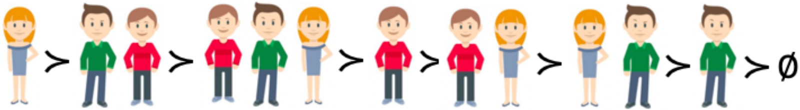




Banzhaf Scores :

2 0

Ordinal Banzhad solution



Banzhaf Scores :

2 0

Definition (Ordinal Banzhaf relation)

Let $\succeq \in \mathcal{B}(2^N)$ and $A \subseteq N$. The *ordinal Banzhaf relation* is the binary relation $\hat{R}^{\succeq}_A \subseteq A \times A$ such that for all $i, j \in A$:

$$i \hat{R}^{\succeq}_A j \Leftrightarrow s^{\succeq}_i \geq s^{\succeq}_j .$$

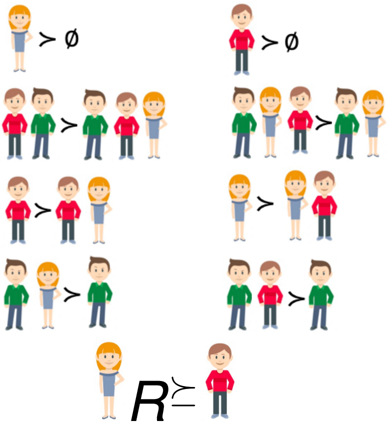
- From the definition it immediately follows that the ranking result is always transitive.

Axiomatization

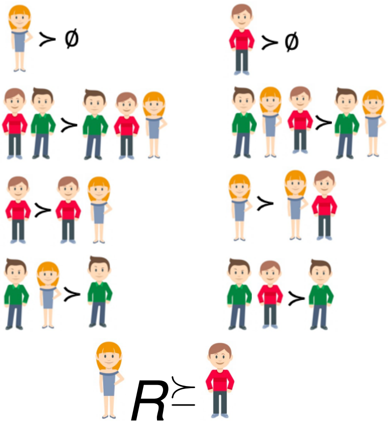
Ordinal Banzhaf solution :

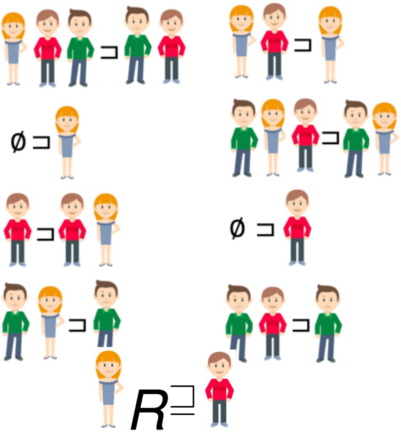
- Coalitional Anonymity
- Neutrality
- Monotonicity

Coalitional Anonymity



Coalitional Anonymity





Coalitional Anonymity

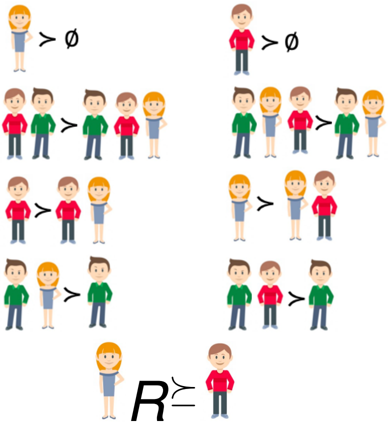
Definition (Coalitional Anonymity(CA))

Let $A \subseteq N$. A solution $R_A : C \subseteq \mathcal{B}(2^N) \to \mathcal{T}(A)$ satisfies the *coalitional anonymity* axiom on C if and only if for all power relations $\succeq, \sqsupseteq \in C$, for all players $i, j \in A$ and bijections $\pi^i : U_i \to U_i$ and $\pi^j : U_j \to U_j$ such that $S \cup \{i\} \succeq S \Leftrightarrow \pi^i(S) \cup \{i\} \sqsupseteq \pi^i(S)$ for all $S \in U_i$ and $S \cup \{j\} \succeq S \Leftrightarrow \pi^j(S) \cup \{j\} \sqsupseteq \pi^j(S)$ for all $S \in U_j$, then it holds that $iR_A^{\succeq} j \Leftrightarrow iR_A^{\sqsupseteq} j$.

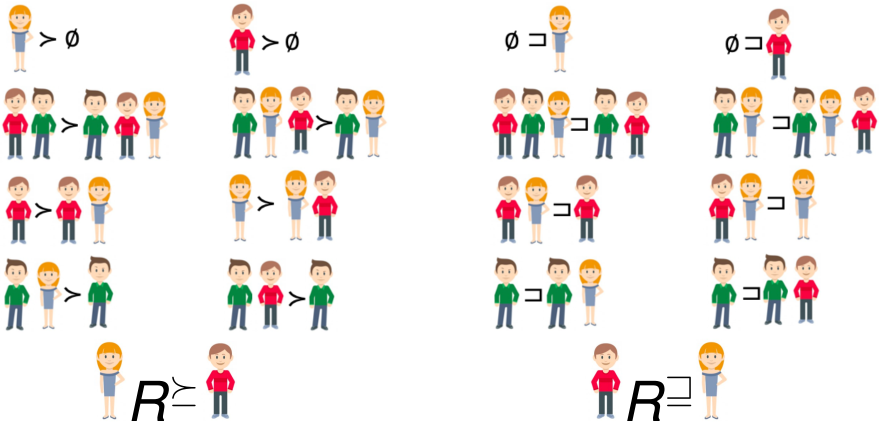
In this definition :

- $U_i = \{S \in 2^N \ s.t \ i \notin S\}$.

Neutrality



Neutrality



Neutrality(Ordinal Banzhaf)

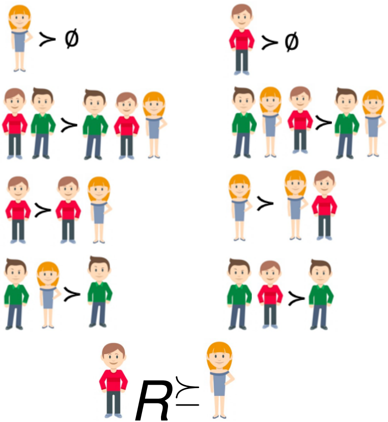
Definition (Neutrality (N))

Let $A \subseteq N$. A solution $R_A : C \subseteq \mathcal{B}(2^N) \to \mathcal{T}(A)$ satisfies the *neutrality* axiom on C if and only if for all power relations $\succeq, \sqsupseteq \in C$ and each bijection $\sigma : N \to N$ such that $\sigma(A) = A$ and $S \succeq T \Leftrightarrow \sigma(S) \sqsupseteq \sigma(T)$ for all $S, T \in 2^N$, then it holds that $iR_A^{\succeq}j \Leftrightarrow \sigma(i)R_A^{\sqsupseteq}\sigma(j)$ for every $i, j \in A$.

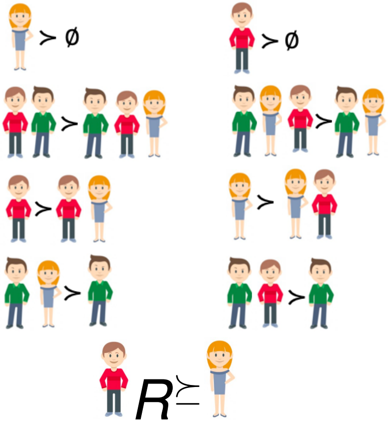
In this definition :

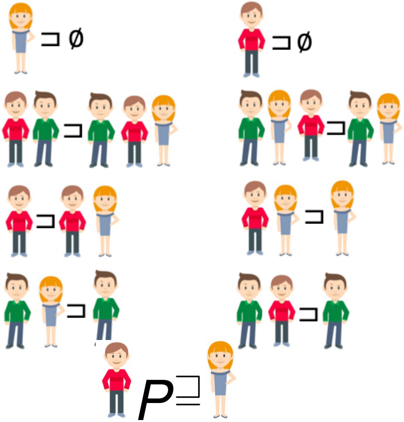
- $\sigma : N \to N$ s.t
  $S = \{i, j, k, ..., t\} \subseteq N \Rightarrow \sigma(S) = \{\sigma(i), \sigma(j), \sigma(k), ..., \sigma(t)\}$.

Monotonicity



Monotonicity





Monotonicity

Definition (Monotonicity (M))

Let $A \subseteq N$. A solution $R_A : C \subseteq \mathcal{B}(2^N) \to \mathcal{T}(A)$ satisfies the *monotonicity* axiom on C if and only if for all power relations $\succeq, \sqsupseteq \in C$ and $i, j \in A$ such that :

- there exists a coalition $S \in U_i$ such that $S \succ S \cup i$ and $S \cup i \sqsupset S$, and
- $T \cup i \succ T \Leftrightarrow T \cup i \sqsupset T$ and $T \cup j \succ T \Leftrightarrow T \cup j \sqsupset T$ for all the other coalitions $T \in 2^N$, $T \neq S$,

then it holds that $iR_A^{\succeq} j \Rightarrow iP_A^{\sqsupseteq} j$.

In this definition :

- $U_i = \{S \in 2^N \; s.t \; i \notin S\}$.

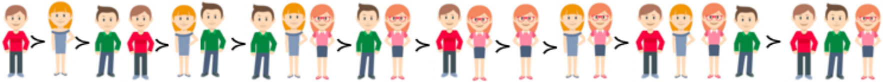
Characterization : Ordinal Banzhaf Solution

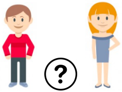
- For the set of all linear orders we have proved that :

Theorem

Let $A \subseteq N$. A solution $R_A : \mathcal{A}(2^N) \to \mathcal{T}(A)$ is the ordinal Banzhaf solution if and only if it satisfies the three axioms CA, N and M on $\mathcal{A}(2^N)$.

Ceteris Paribus Majority rule(IJCAI18)





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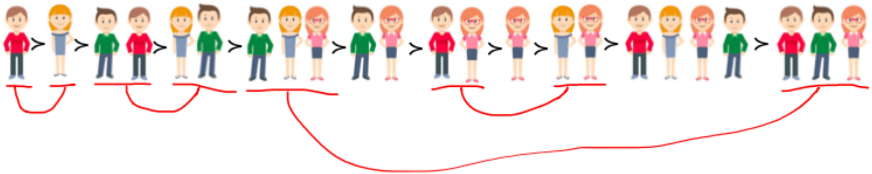

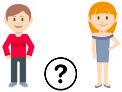

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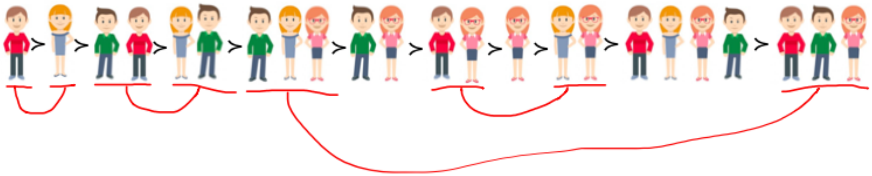
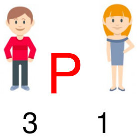
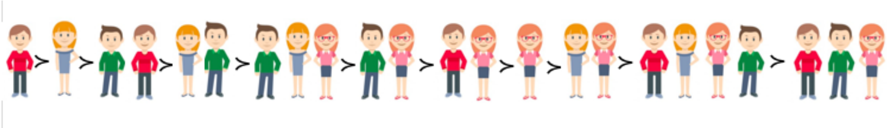




Family of weighted majority rules

- Ceteris Paribus majority rule and Ordinal Banzhaf solution belong to the same family of weighted majority rules.

HOW ? ?



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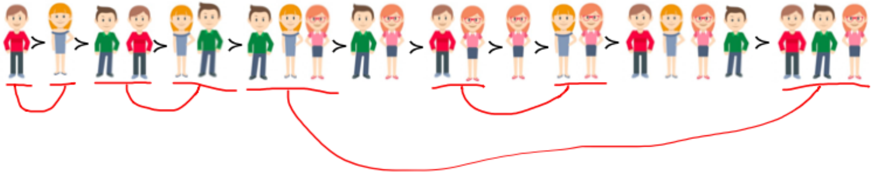

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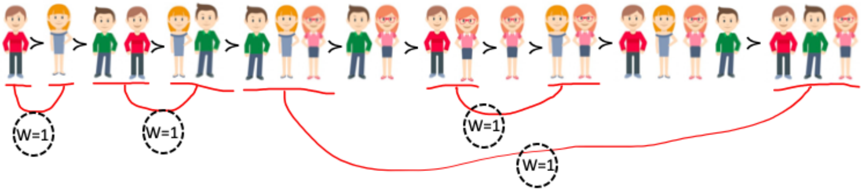
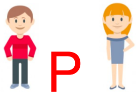


Ceteris Paribus Majority :



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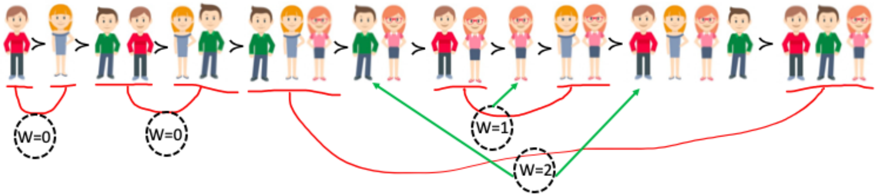
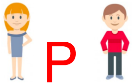


Ordinal Banzhaf Solution

Banzhaf Distance



Family of weighted majority relations

Definition (Weighted majority relation)

Let $\succeq \in \mathcal{B}(2^N)$, $A \subseteq N$ and let $\mathbf{w} = [w_{ij}^S]_{i,j \in A, S \in 2^N : i,j \notin S}$ be a *weight scheme* such that $w_{ij}^S \geq 0$ for all $i, j \in A$ and $S \in U_{ij}$. The *weighted majority relation* associated to $\mathbf{w}$ is the binary relation $R_A^{\succeq,\mathbf{w}} \subseteq A \times A$ such that for all $i, j \in A \subseteq N$:

$$iR_A^{\succeq,\mathbf{w}}j \;\; \Leftrightarrow \;\; \sum_{S \in U_{ij}} w_{ij}^S \bar{d}_{ij}^S(\succeq) \geq 0.$$

In this definition :

- $d_{ij}(\succeq)$ is the number of times that i performs better than j in power relation $\succeq$.
- $U_{ij} = \{S \in 2^N \text{ } s.t \text{ } i, j \notin S\}$.

Future work

- More general axioms composing two parts of Normative and Informative.
- Find out other members of the *Family of weighted majority*.
- Looking for practical applications to model it as the mentioned problem.
- To compare the solutions, applying the inverse of the problem to retrieve the power relation.