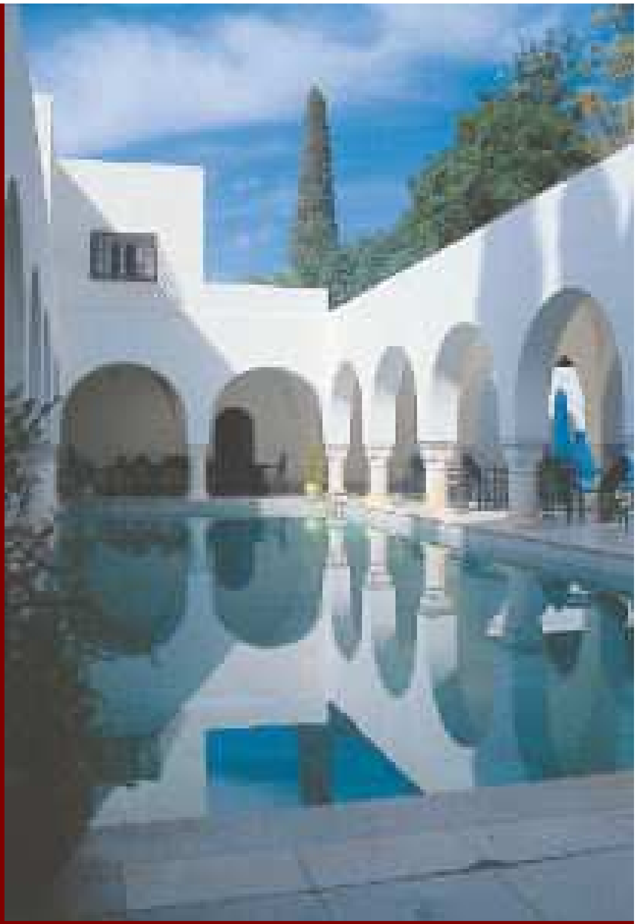
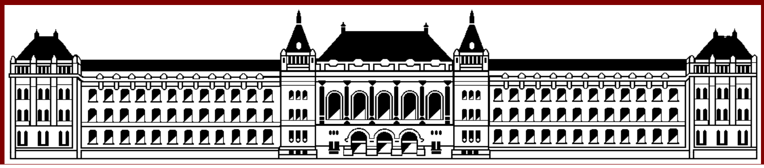
Applications of combinatorial optimization in statics (rigidity of frameworks)

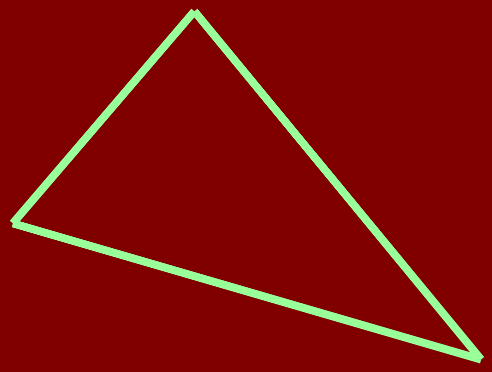


Hammamet, 2010

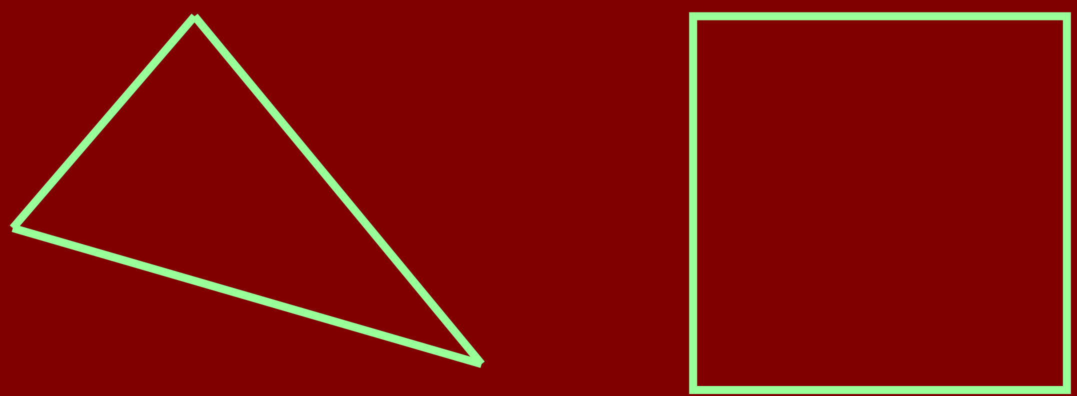
András Recski

Budapest University of Technology and Economics

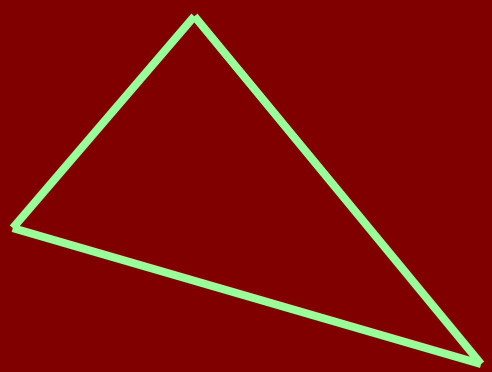




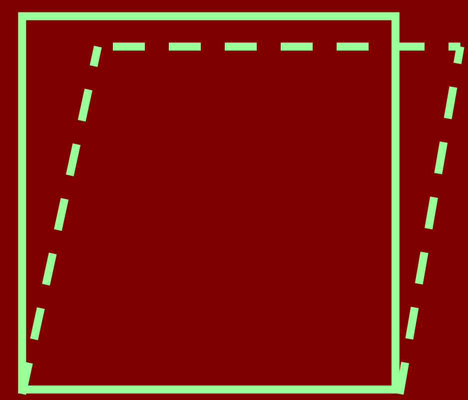
Rigid



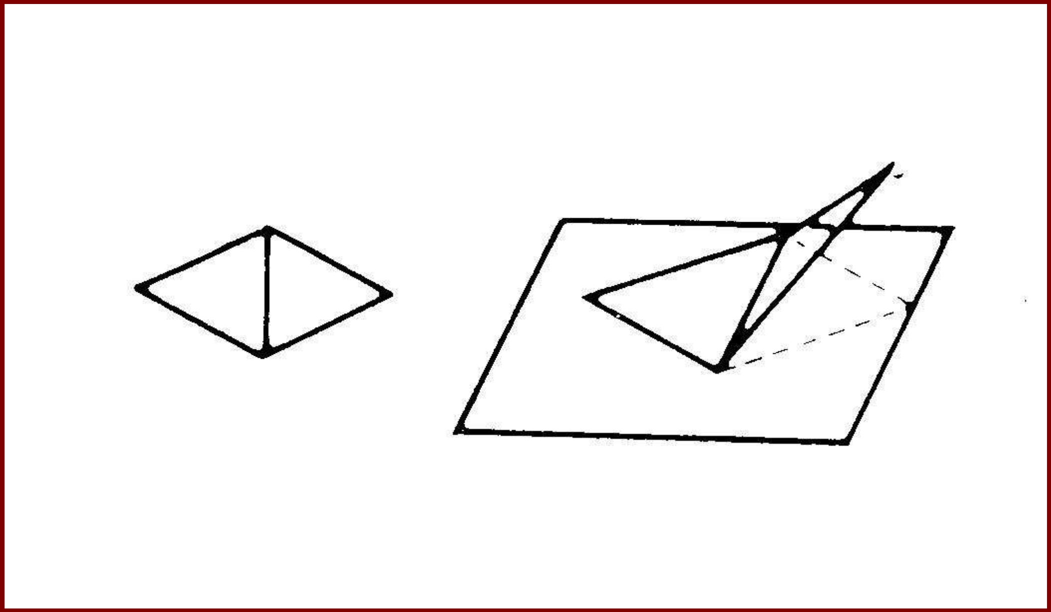
Rigid



Rigid

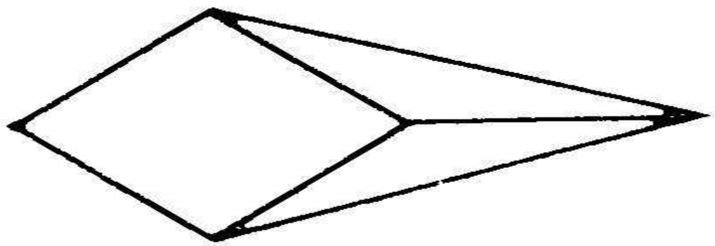


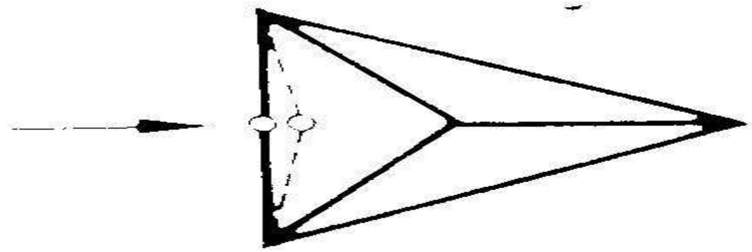
Non-rigid (mechanism)



Rigid in the plane

Non-rigid in the space

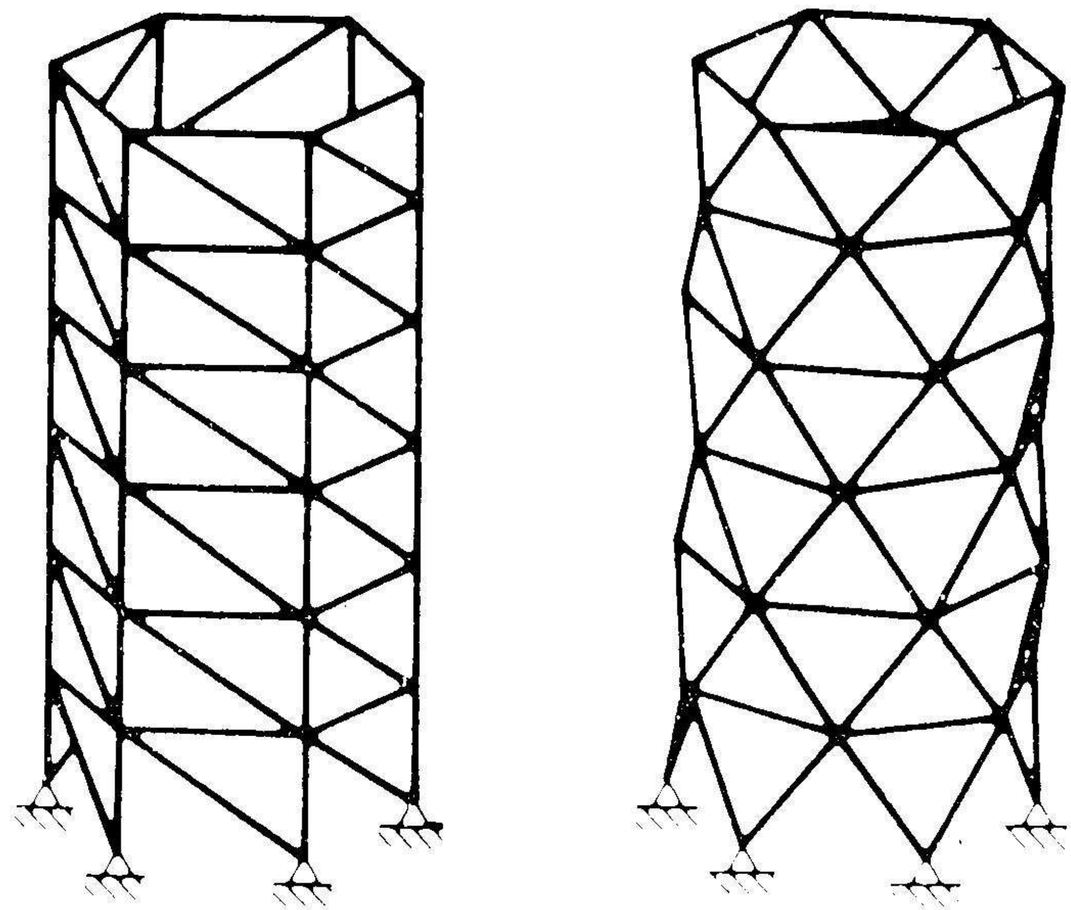




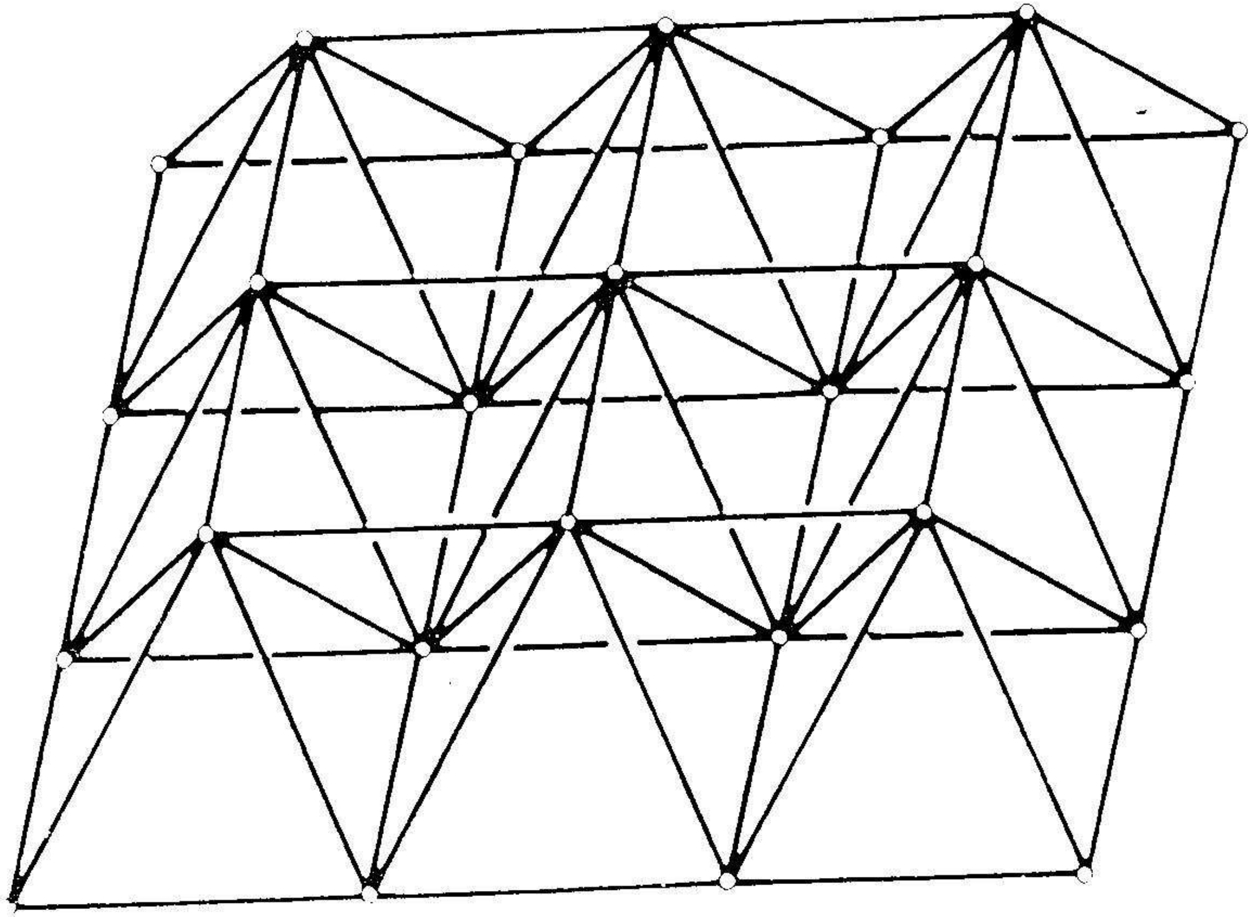
Rigid

Non-rigid (has an *infinitesimal* motion)

(although the graphs of the two frameworks are isomorphic)



Rigid Non-rigid



When is *this* framework rigid?

- For certain graphs (like C_4) every realization leads to non-rigid frameworks.
- For others, some of their realizations lead to rigid frameworks.

These latter type of graphs are called *generic rigid.*

What can combinatorialists do?

- They either study „very symmetric” structures, like square or cubic grids,

- or prefer „very asymmetric” ones, that is, the generic structures.

- Deciding the rigidity of a frame-work (that is, of an actual real-ization of a graph) is a problem in linear algebra.

- Deciding whether a graph is ge-neric rigid, is a combinatorial problem.
- Special case: minimal generic rigid graphs (when the deletion of any edge destroys rigidity).

The 1-dimensional case is easy:

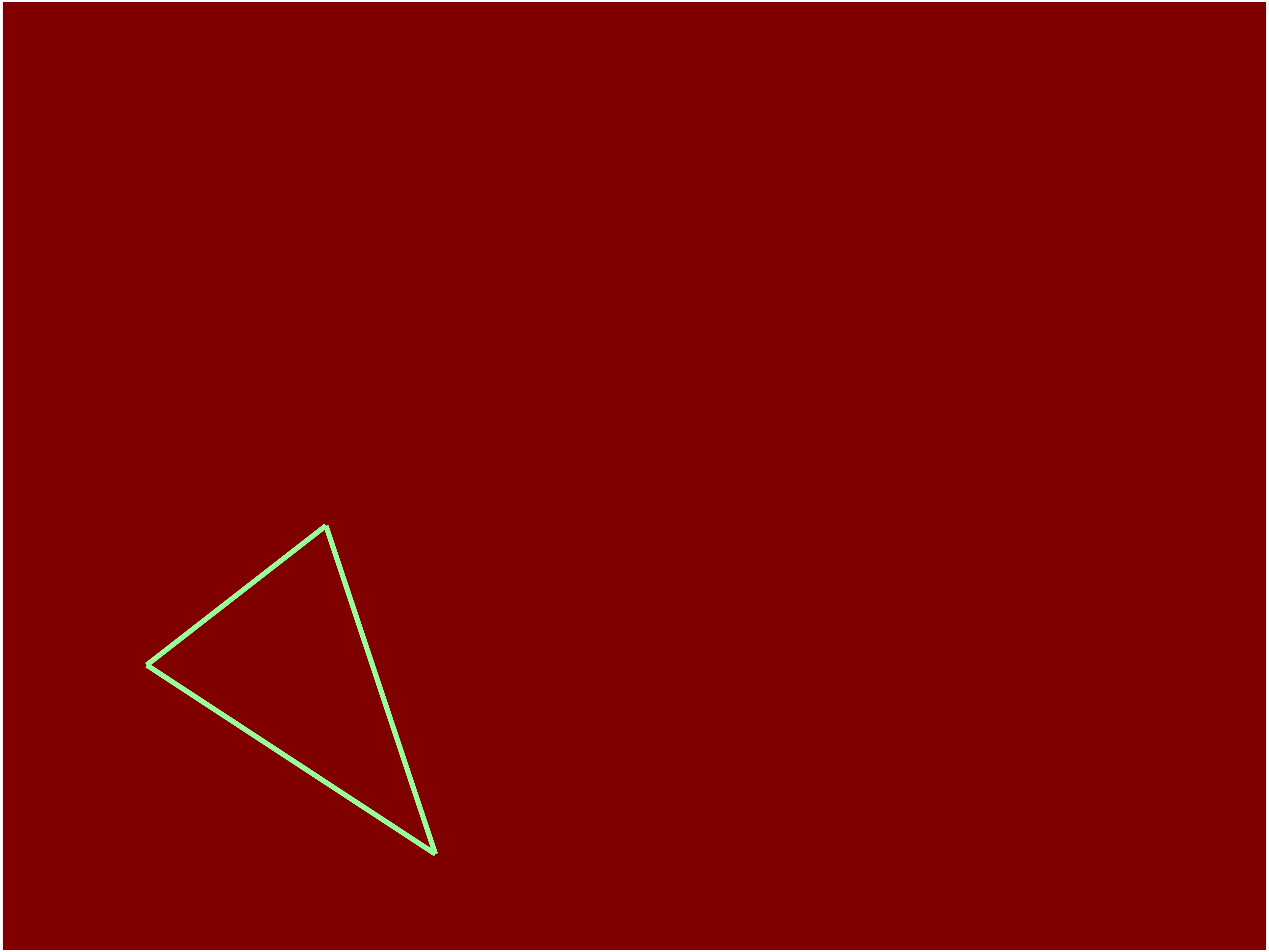
A 1-dimensional framework is rigid if and only if its graph is connected.

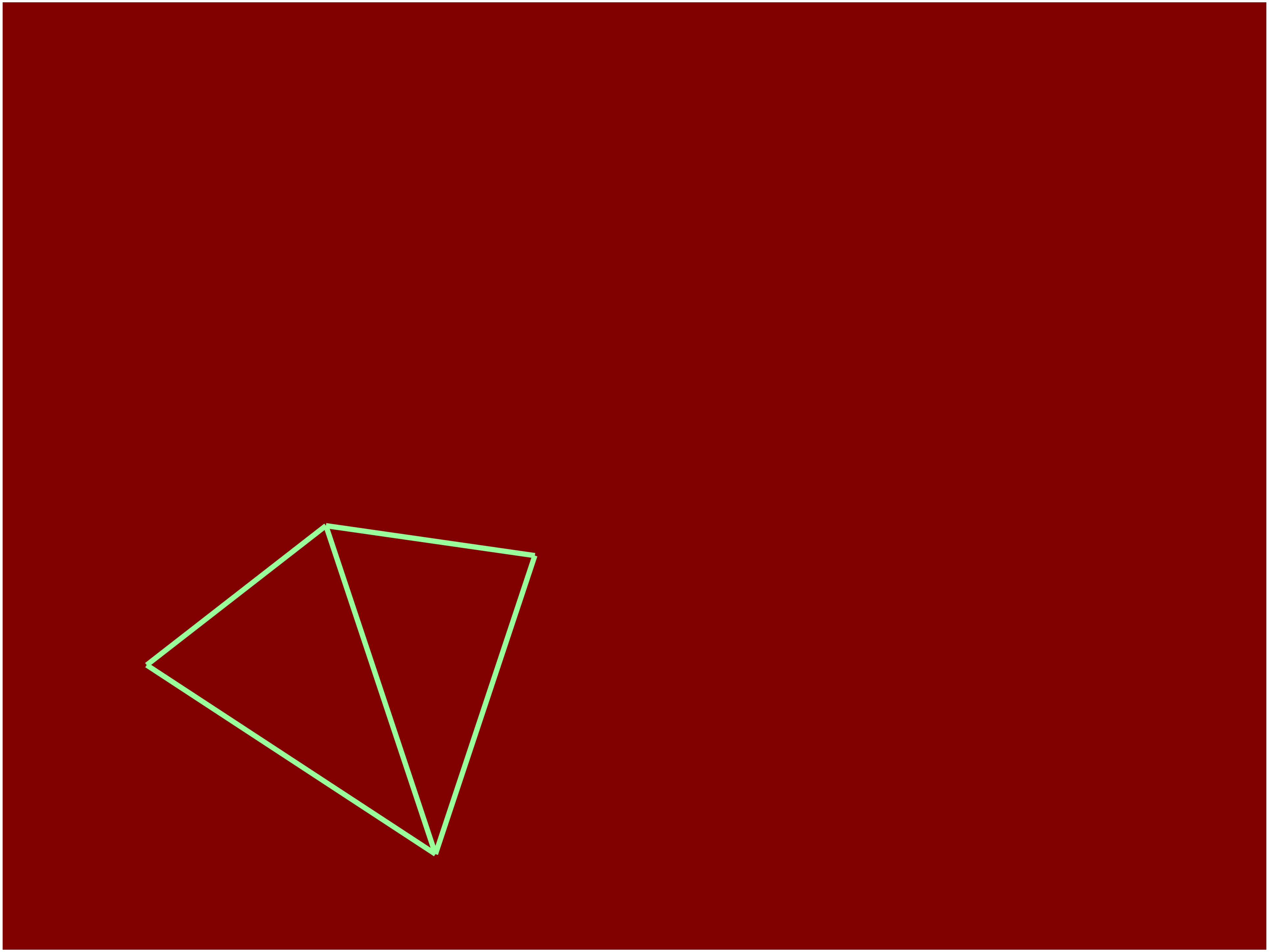
In particular, a graph corresponds to a 1-dimensional minimally rigid framework if and only if it is a tree.

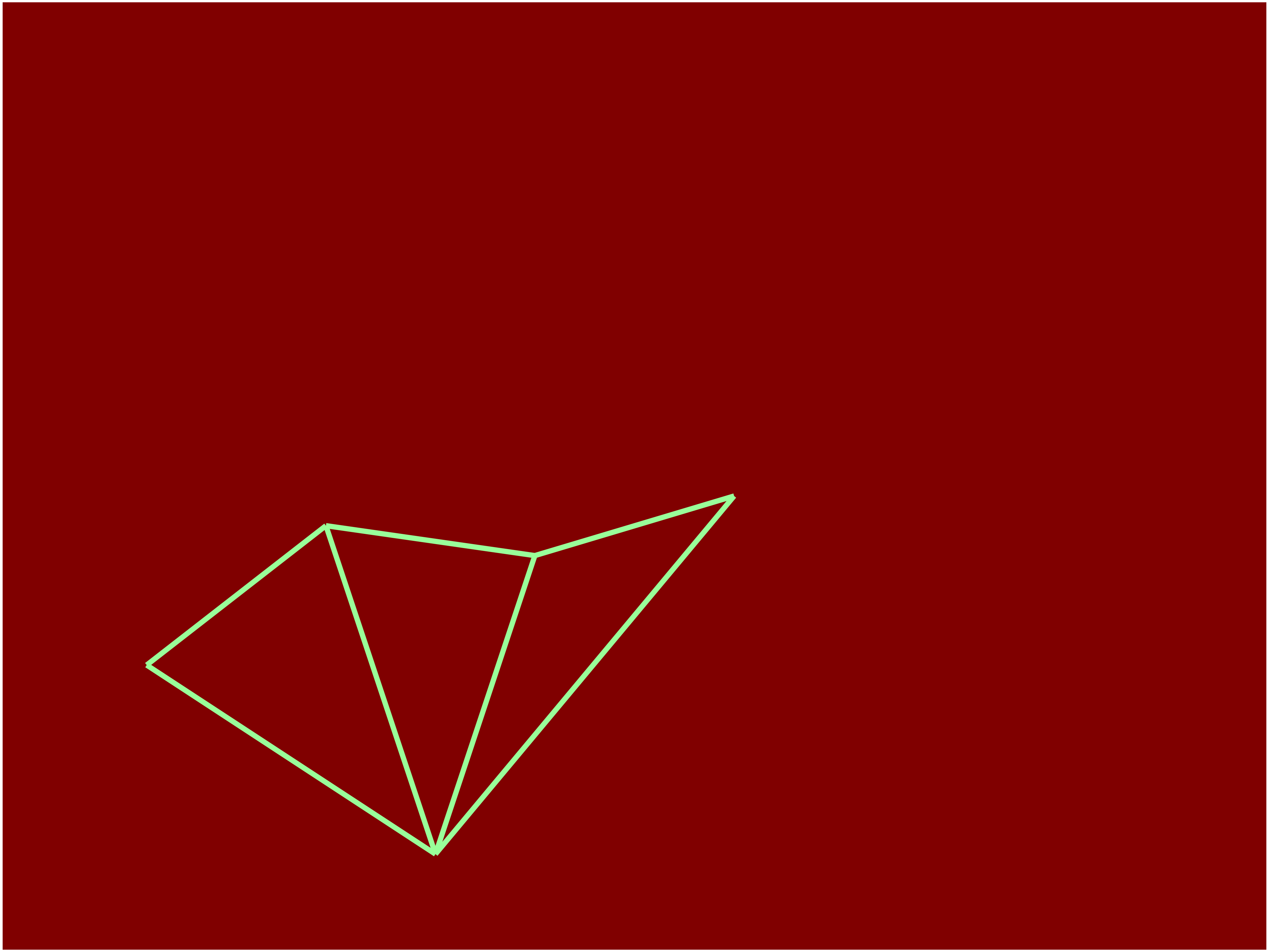
We wish to characterize those graphs which are (minimally) generic rigid in the plane.

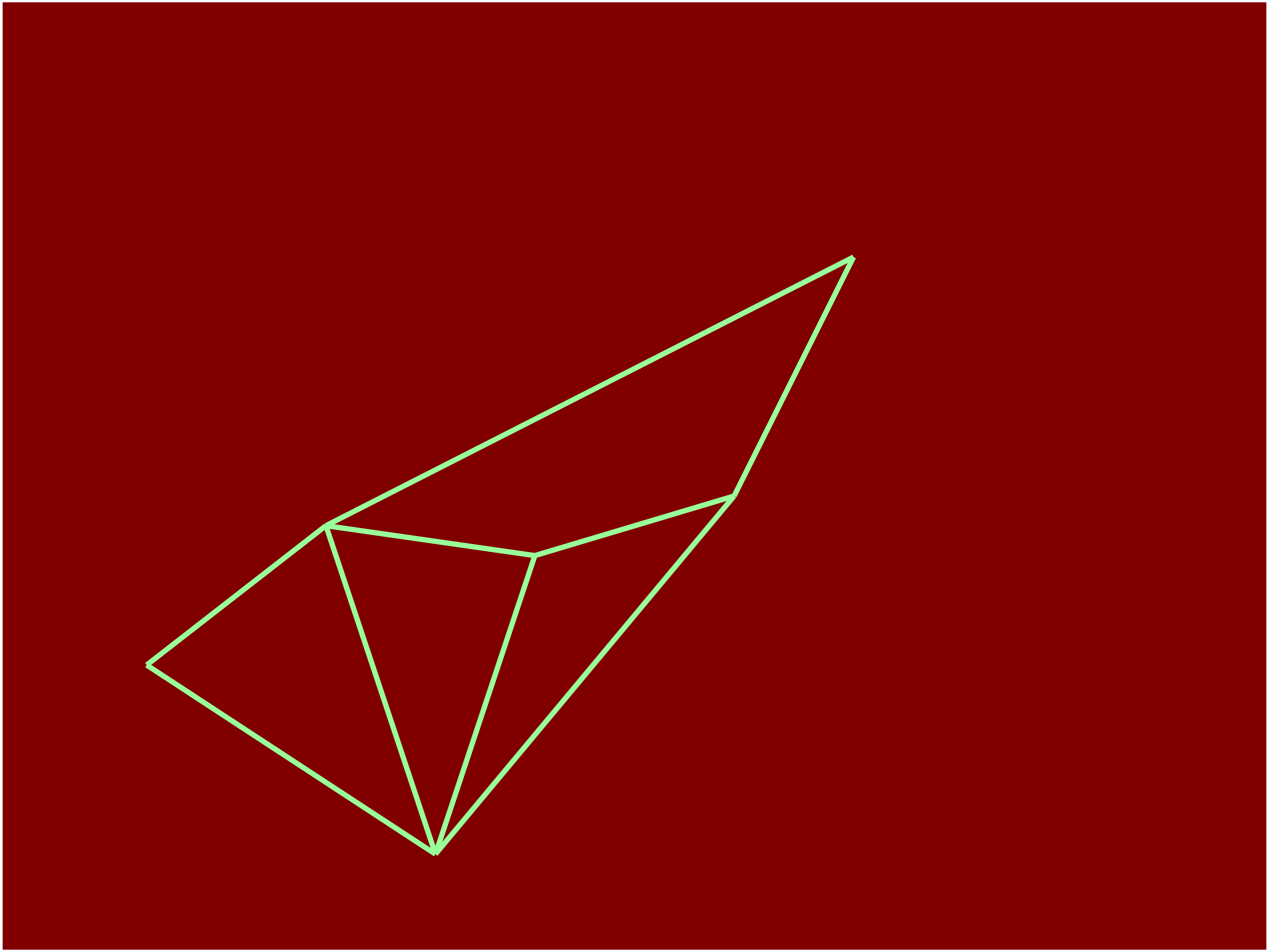
Let us find some examples at first.

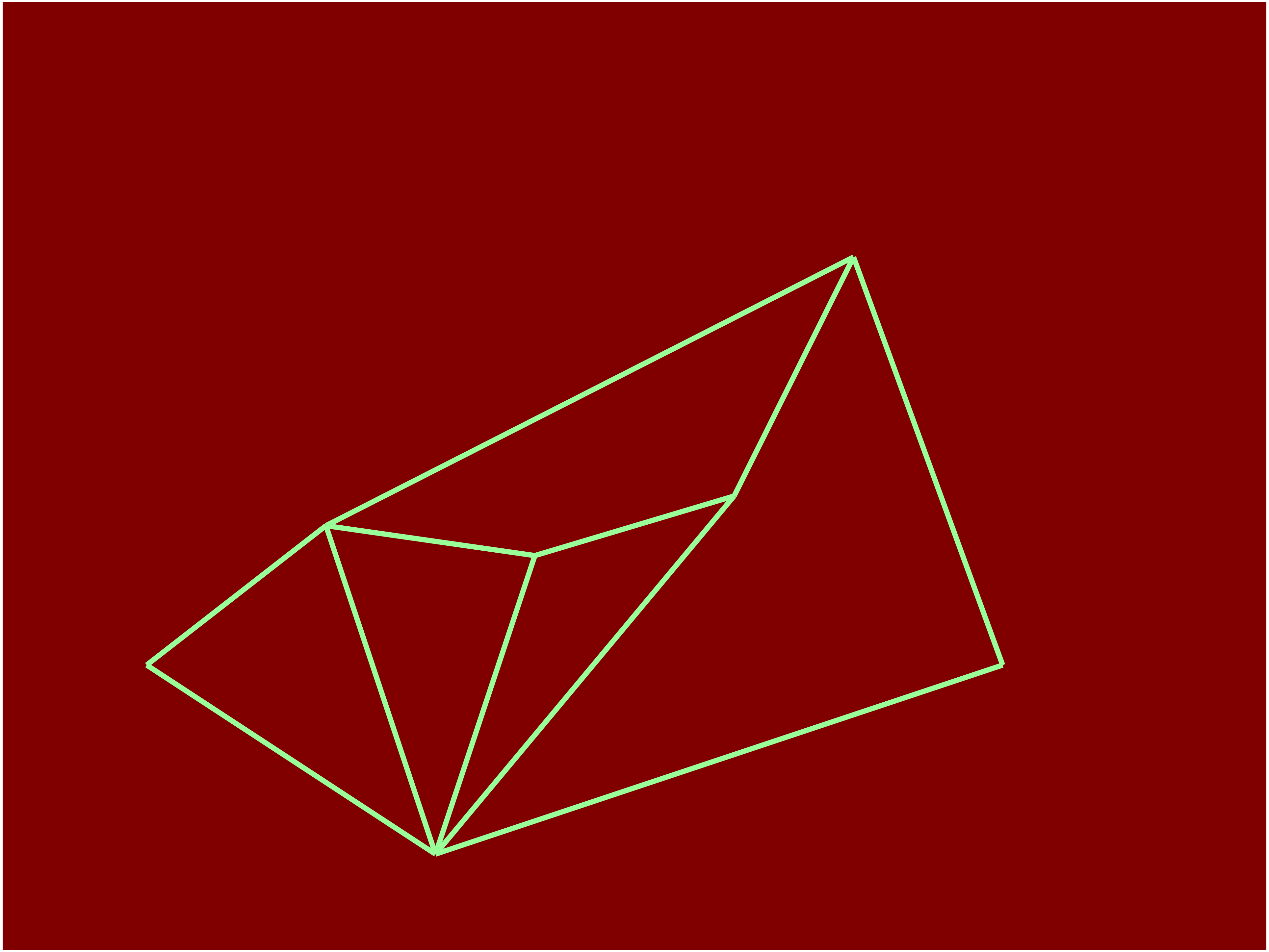
The simple trusses:



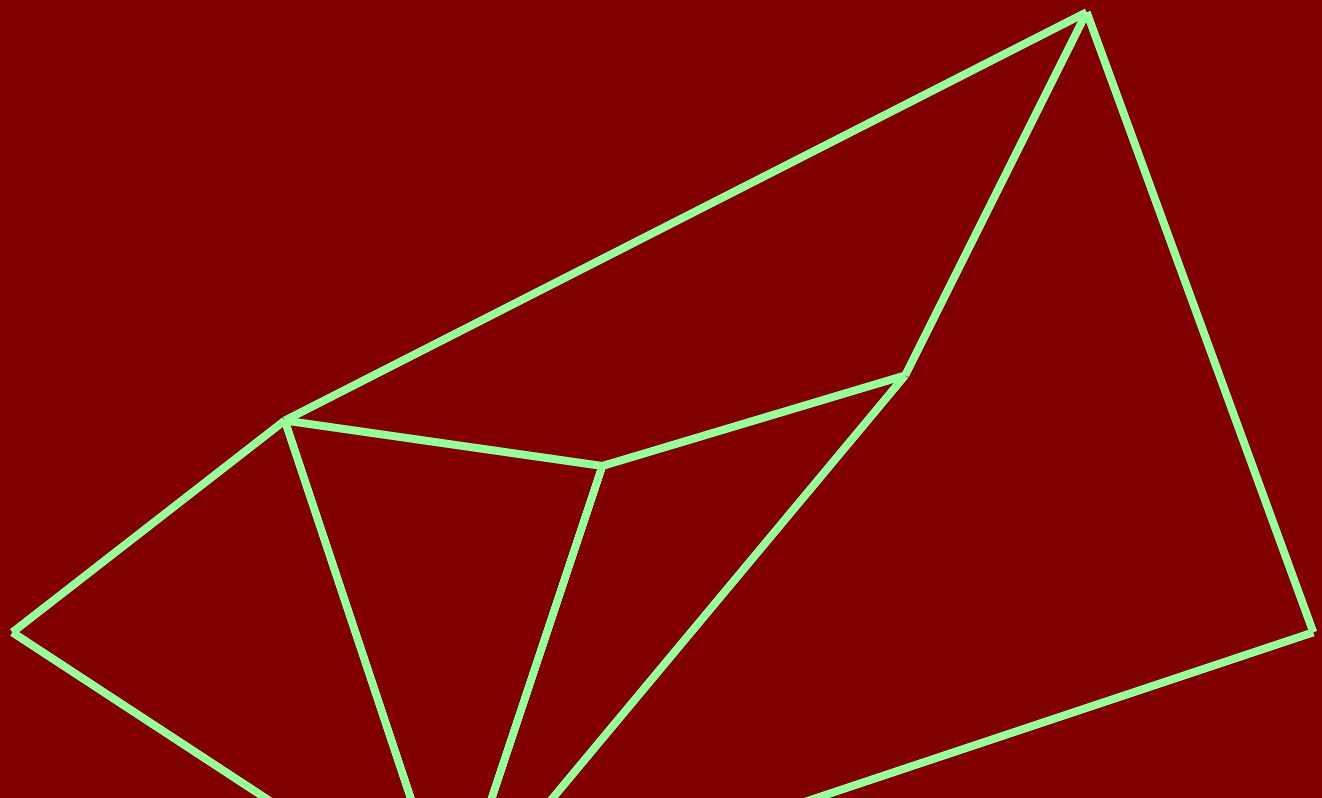








Simple trusses (in the plane)



All the simple trusses satisfy $e = 2n - 3$ ($n \geq 3$)

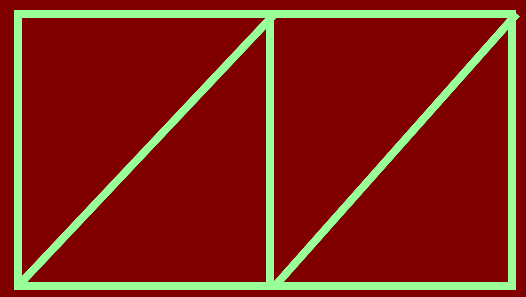
and they are *minimally rigid,*

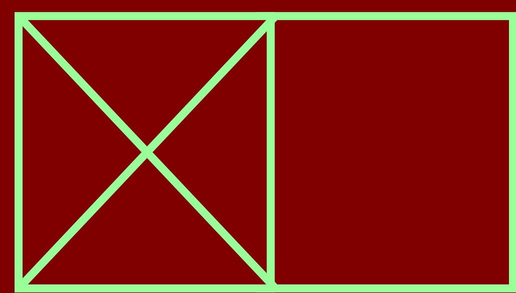
but not every minimally rigid framework is a simple truss (consider the Kuratowski graph $K_{3,3}$, for example).

A famous minimally rigid structure:

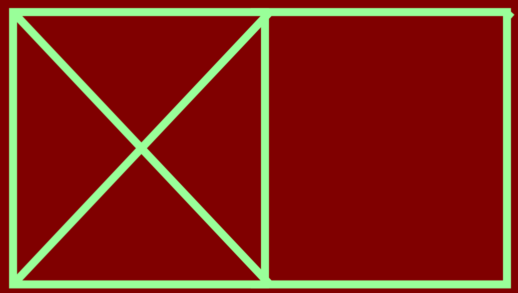
Szabadság Bridge, Budapest

Does $e = 2n\text{-}3$ imply that the framework is minimally rigid?





Certainly not:



If a part of the framework is „overbraced”, there will be a nonrigid part somewhere else…

Maxwell (1864):

If a graph G is minimal generic rigid in the plane then,

in addition to $e = 2n - 3$,

the relation $e' \leq 2n' - 3$ must hold for every (induced) sub-graph G' of G.

Laman (1970):

A graph G is minimal generic rigid in the plane if and only if $e = 2n - 3$ and the relation $e' \leq 2n' - 3$ holds for every (induced) subgraph G' of G.

This is a „good characterization" of minimal generic rigid graphs in the plane, but we do not wish to check some 2^n subgraphs…

Lovász and Yemini (1982):

A graph G is minimal generic rigid in the plane if and only if

$e = 2n - 3$ and doubling any edge the resulting graph, with $2(n\text{-}1)$ edges, is the union of two edge-disjoint trees.

A slight modification (R.,1984):

A graph G is minimal generic rigid in the plane if and only if $e = 2n - 3$ and joining any two vertices with a new edge the resulting graph, with $2(n-1)$ edges, is the union of two edge-disjoint trees.

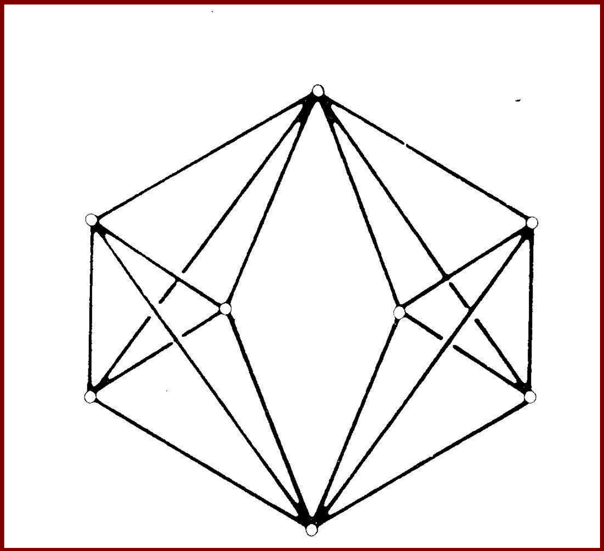
Maxwell (1864):

If a graph *G* is minimal generic rigid in the space then,

in addition to $e = 3n - 6$,

the relation $e' \leq 3n' - 6$ must hold for every (induced) sub-graph *G*' of *G*.

However, the 3-D analogue of Laman’s theorem is not true:



The double banana graph
(Asimow – Roth, 1978)

Rigid rods are resistant to compressions and tensions:

$$\|\mathbf{x}_i - \mathbf{x}_k\| = c_{ik}$$

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Cables are resistant to tensions only: $\|\mathbf{x}_i-\mathbf{x}_k\| \leq c_{ik}$

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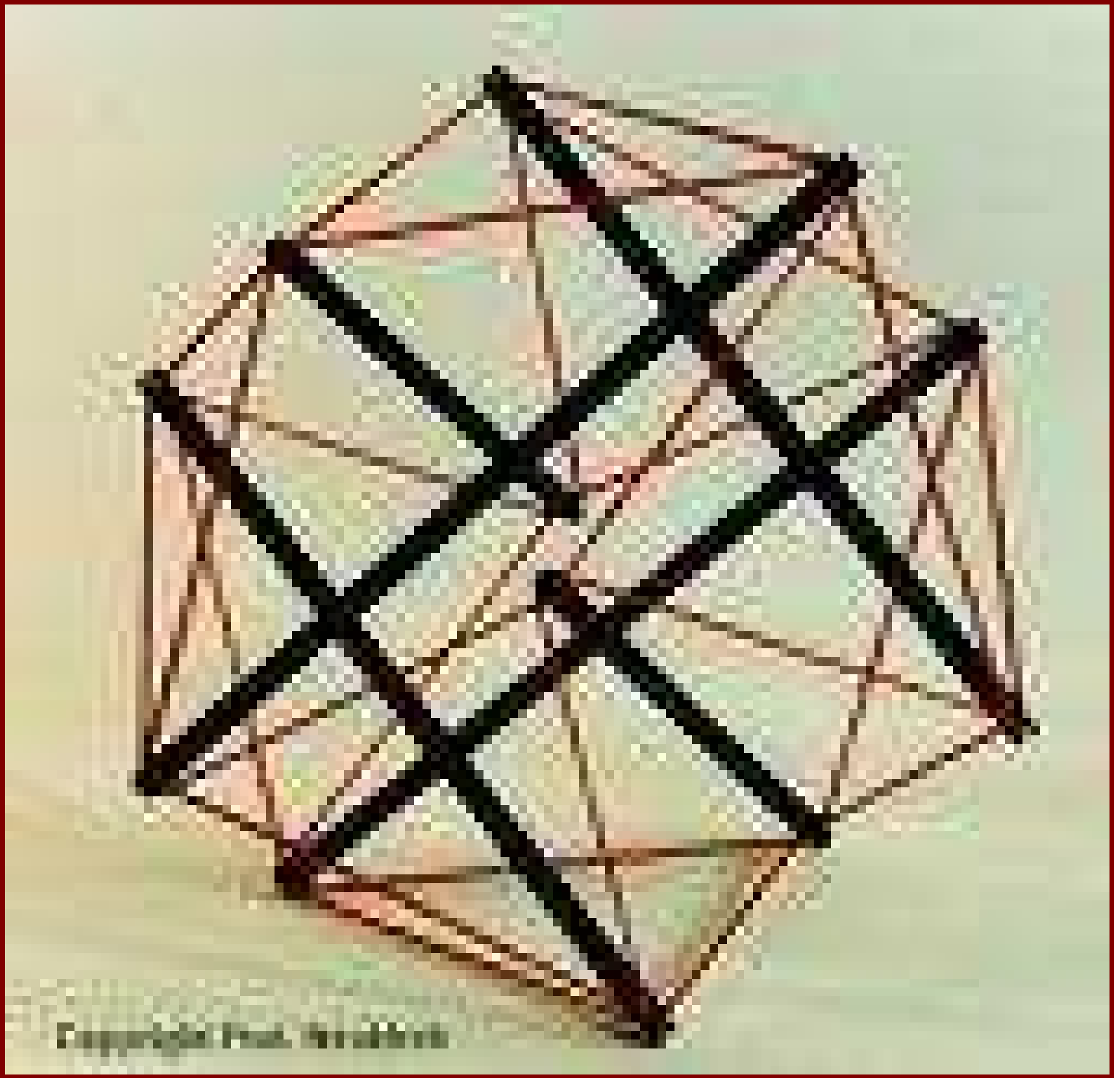
$$\|\mathbf{x}_i - \mathbf{x}_k\| = c_{ik}$$

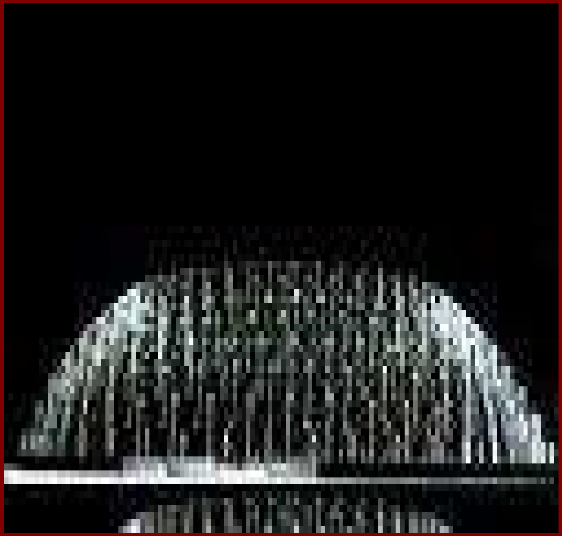
Cables are resistant to tensions only: $\|\mathbf{x}_i - \mathbf{x}_k\| \leq c_{ik}$

Struts are resistant to compressions only:

$$\|\mathbf{x}_i - \mathbf{x}_k\| \geq c_{ik}$$

Frameworks composed from rods (bars), cables and struts are called *tensegrity frame-works*.





Frameworks composed from rods (bars), cables and struts are called *tensegrity frameworks*.

A more restrictive concept is the *r-tensegrity framework*, where rods are not allowed, only cables and struts. (The letter r means rod-free or restricted.)

We wish to generalize the above results for tensegrity frameworks:

When is a graph minimal generic rigid in the plane as a tensegrity framework (or as an r-tensegrity framework)?

Which is the more difficult problem?

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If rods are permitted then why should one use anything else?

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If rods are permitted then why should one use anything else?

„Weak” problem: When is a graph minimal generic rigid in the plane as an r-tensegrity framework?

„Strong” problem: When is a graph *with a given tripartition* minimal generic rigid in the plane as a tensegrity framework?

The 1-dimensional case is still easy

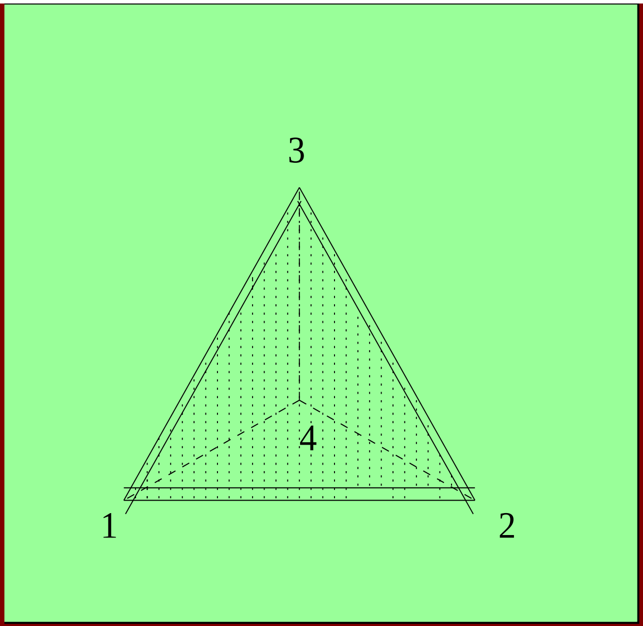
R. – Shai, 2005:

Let the cable-edges be red, the strut-edges be blue (and replace rods by a pair of parallel red and blue edges).

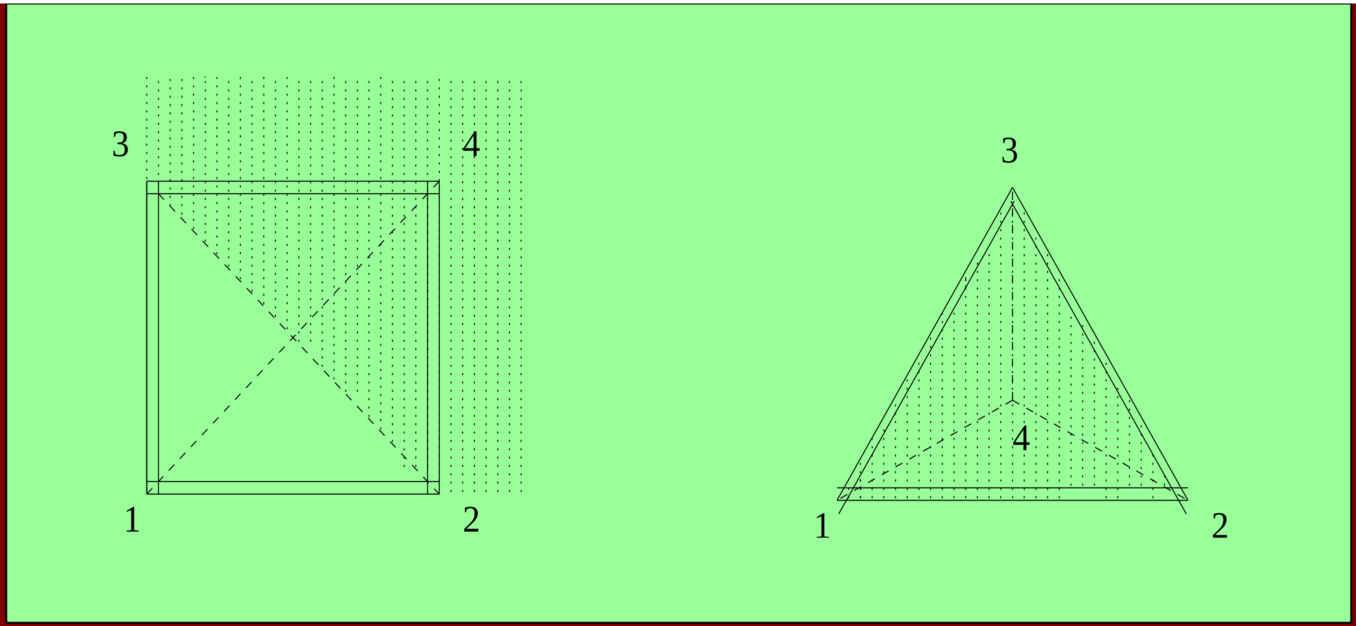
The graph with the given tripartition is realizable as a rigid tensegrity framework in the 1-dimensional space if and only if

- it is 2-edge-connected and
- every 2-vertex-connected component contains edges of both colours.

An example to the
2-dimensional case:

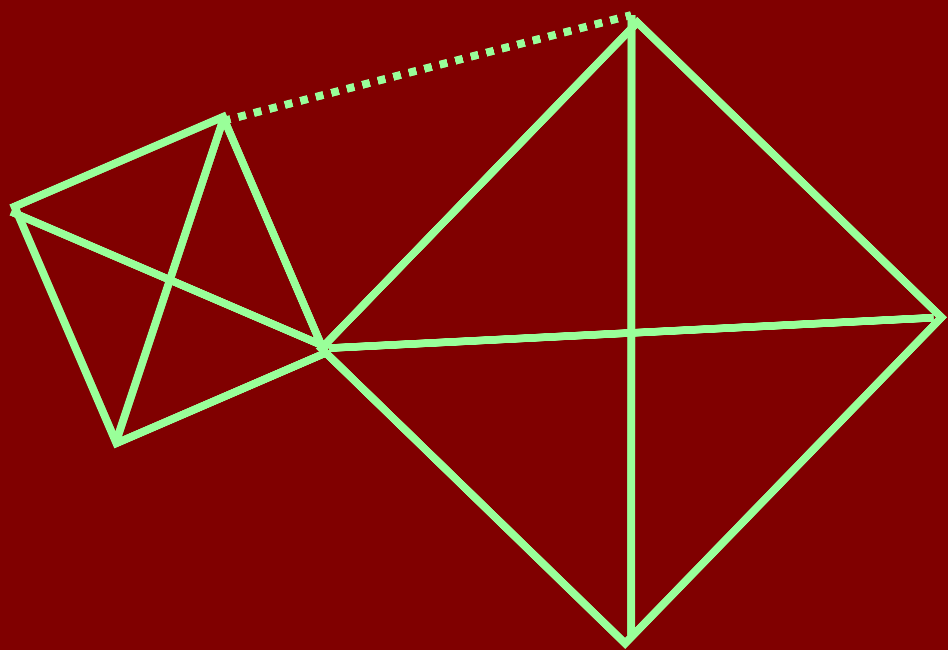


The graph K_4 can be realized as a rigid tensegrity framework with struts {1,2}, {2,3} and {3,1} and with cables for the rest (or *vice versa*) if ‘4’ is in the convex hull of {1,2,3} …



…or with cables for two independent edges and struts for the rest (or *vice versa*) if none of the joints is in the convex hull of the other three.

Critical rods cannot be replaced by cables or struts if we wish to preserve rigidity



Jordán – R. – Szabadka, 2007

A graph can be realized as a rigid *d*-dimensional r-tensegrity framework

if and only if

it can be realized as a rigid *d*-dimensional rod framework and none of its edges are critical.

Corollary (Laman – type):

A graph G is minimal generic rigid in the plane as an r-tensegrity framework if and only if

$e = 2n - 2$ and the relation $e' \leq 2n' - 3$ holds for every proper subgraph G' of G.

Corollary (Laman – type):

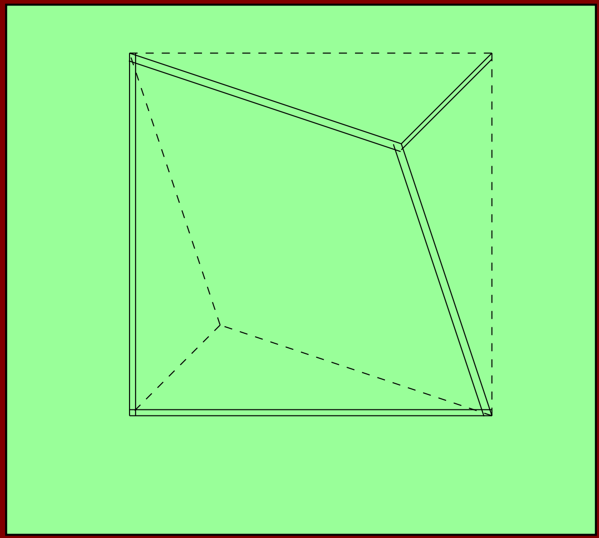
A graph G is minimal generic rigid in the plane as an r-tensegrity framework if and only if $e = 2n - 2$ and the relation $e' \leq 2n' - 3$ holds for every proper subgraph G' of G.

Corollary (Lovász-Yemini – type):

A graph is minimal generic rigid in the plane as an r-tensegrity framework if and only if it is the union of two edge-disjoint trees and remains so if any one of its edges is moved to any other position.

- A graph is generic rigid in the 1-dimensional space as an r-tensegrity framework if and only if it is 2-edge-connected.

- For the generic rigidity in the plane as an r-tensegrity framework, a graph must be 2-vertex-connected and 3-edge-connected. Neither 3-vertex-connectivity nor 4-edge-connectivity is necessary.

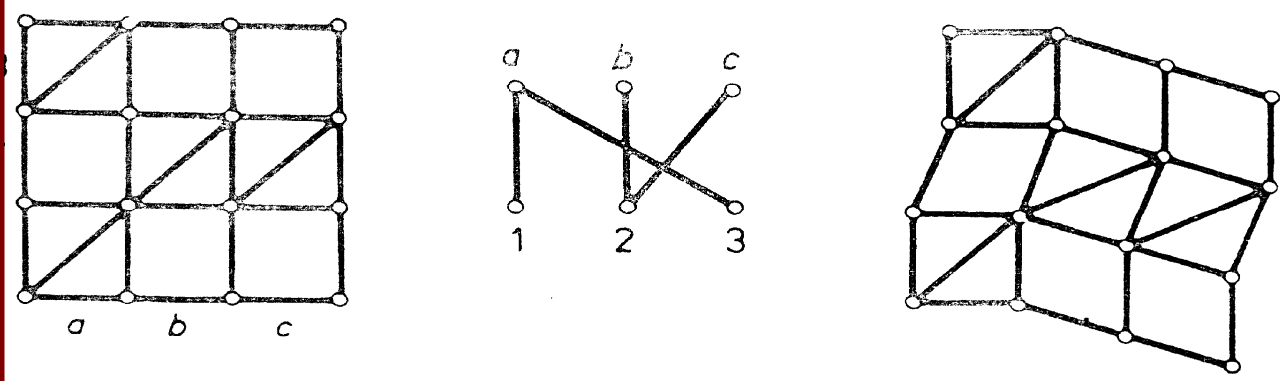


Let us return to the bar and joint frameworks

What can combinatorialists do?

- They either study „very symmetric” structures, like square or cubic grids,
- or prefer „very asymmetric” ones, that is, the generic structures.

Square grids with diagonals



Rigidity of square grids

- Bolker and Crapo, 1977: A set of diagonal bars makes a $k \times \ell$ square grid rigid if and only if the corresponding edges form a ***connected*** subgraph in the bipartite graph model.

- Baglivo and Graver, 1983: In case of diagonal cables, ***strong connectedness*** is needed in the (directed) bipartite graph model.

Minimum # diagonals needed:

$B = k + \ell - 1$ diagonal bars

$C = 2 \cdot \max(k, \ell)$ diagonal cables

(If $k \neq \ell$ then $C - B > 1$)

Rigidity of one-story buildings

Bolker and Crapo, 1977: If each external vertical wall contains a diagonal bar then instead of studying the roof of the building one may consider a $k \times \ell$ square grid ***with its four corners pinned down.***

Rigidity of one-story buildings

Bolker and Crapo, 1977: A set of diagonal bars makes a k X ℓ square grid (with corners pinned down) rigid if and only if the corresponding edges in the bipartite graph model form either a ***connected*** subgraph or a ***2-component asymmetric forest***.

For example, if $k = 8$, $\ell = 18$, $k' = 4$, $\ell' = 9$, then the 2-component forest is symmetric ($L = K$, where $\ell' / \ell = L$, $k' / k = K$).

Minimum # diagonals needed:

$B = k + \ell - 2$ diagonal bars

$C = k + \ell - 1$ diagonal cables
(except if $k = \ell = 1$ or $k = \ell = 2$)

(Chakravarty, Holman, McGuinness and R., 1986)

Rigidity of one-story buildings

Which $(k + \ell - 1)$-element sets of cables make the $k \times \ell$ square grid (with corners pinned down) rigid?

Let X, Y be the two colour classes of the directed bipartite graph. An XY-path is a directed path starting in X and ending in Y.

If X_0 is a subset of X then let $N(X_0)$ denote the set of those points in Y which can be reached from X_0 along XY-paths.

R. and Schwärzler, 1992:

A $(k + \ell - 1)$-element set of cables makes the k X ℓ square grid (with corners pinned down) rigid if and only if

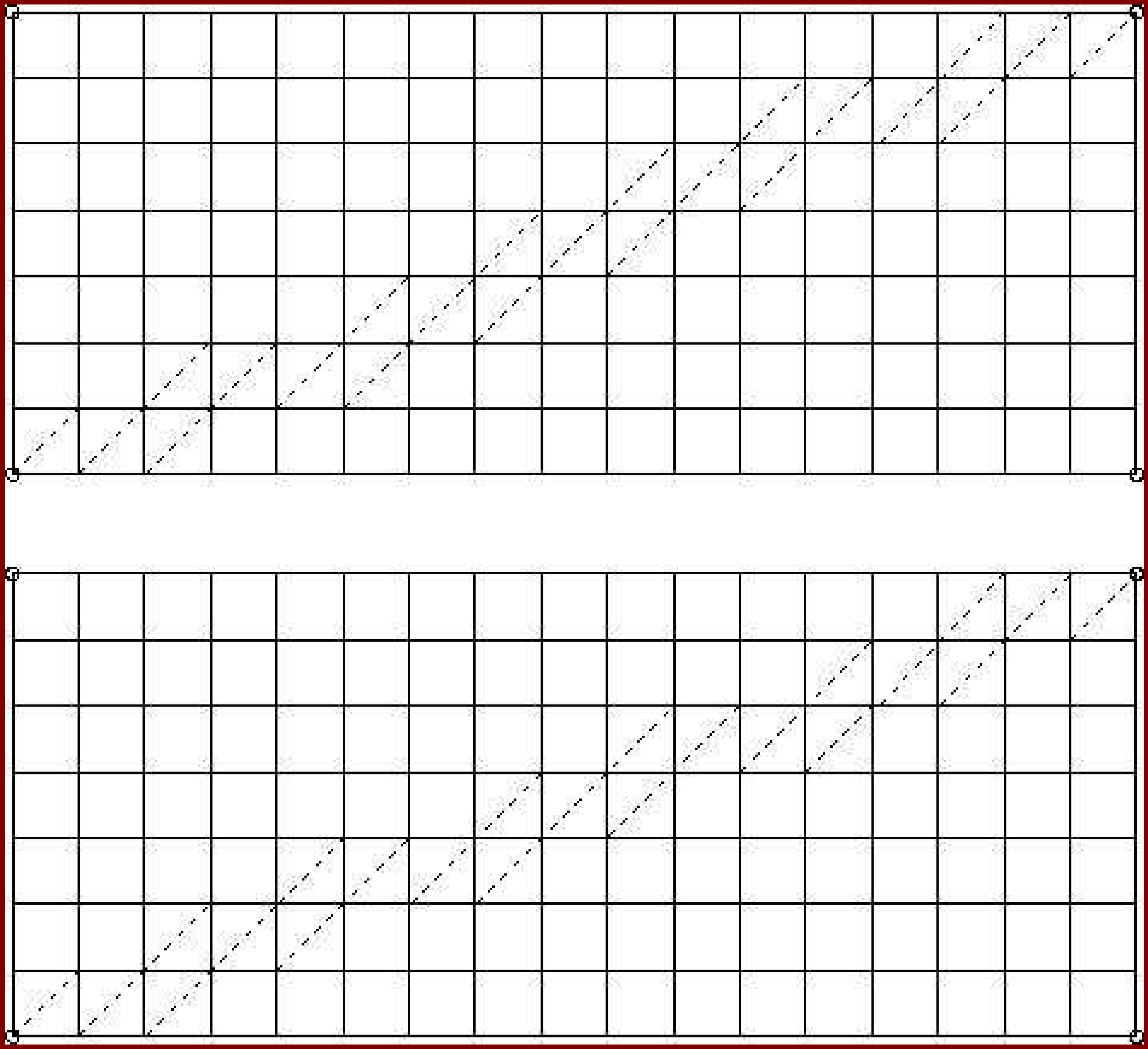
$$|N(X_0)| \cdot k > |X_0| \cdot \ell$$

holds for every proper subset X_0 of X and

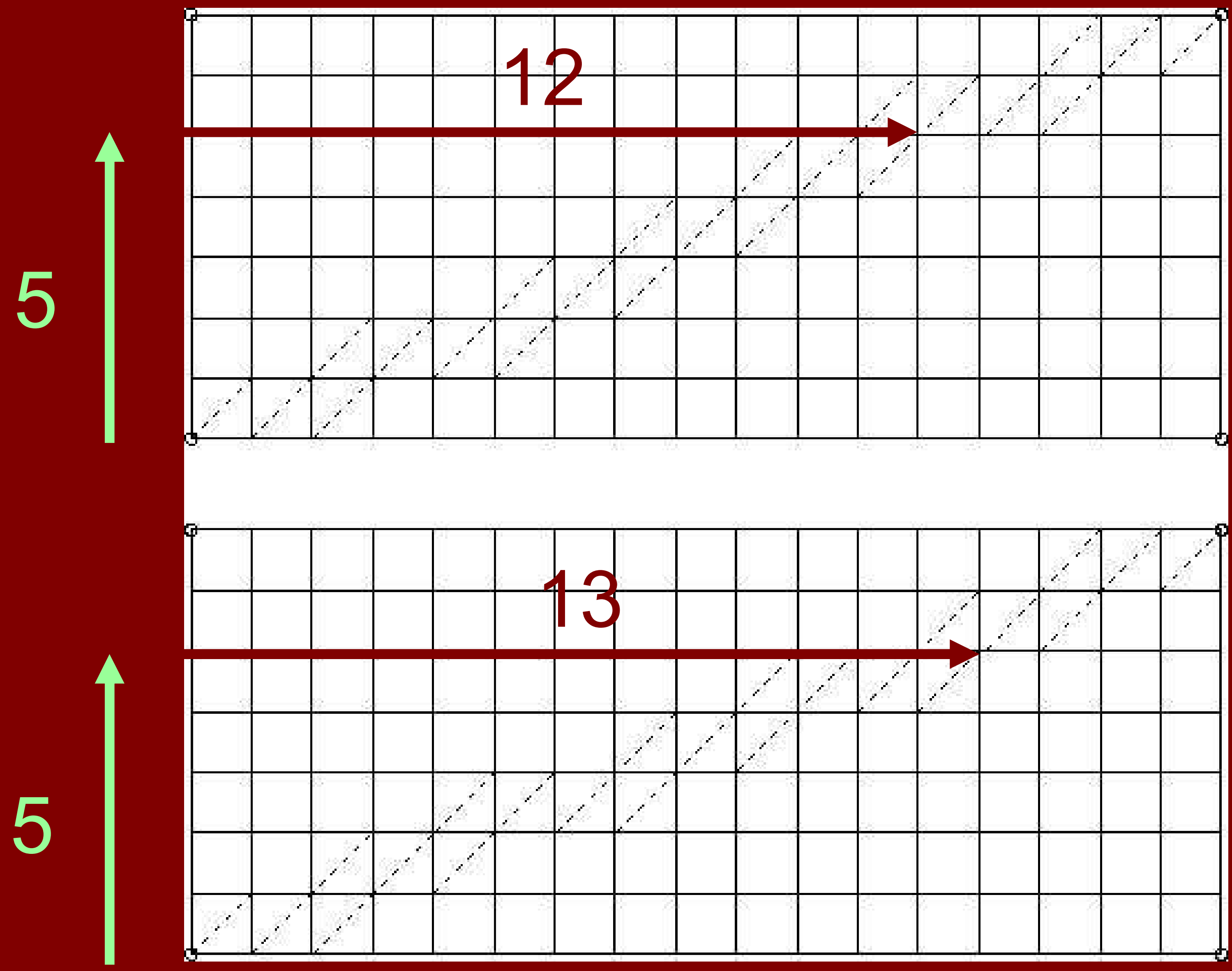
$$|N(Y_0)| \cdot \ell > |Y_0| \cdot k$$

holds for every proper subset Y_0 of Y.

Which one-story building is rigid?



Which one-story building is rigid?

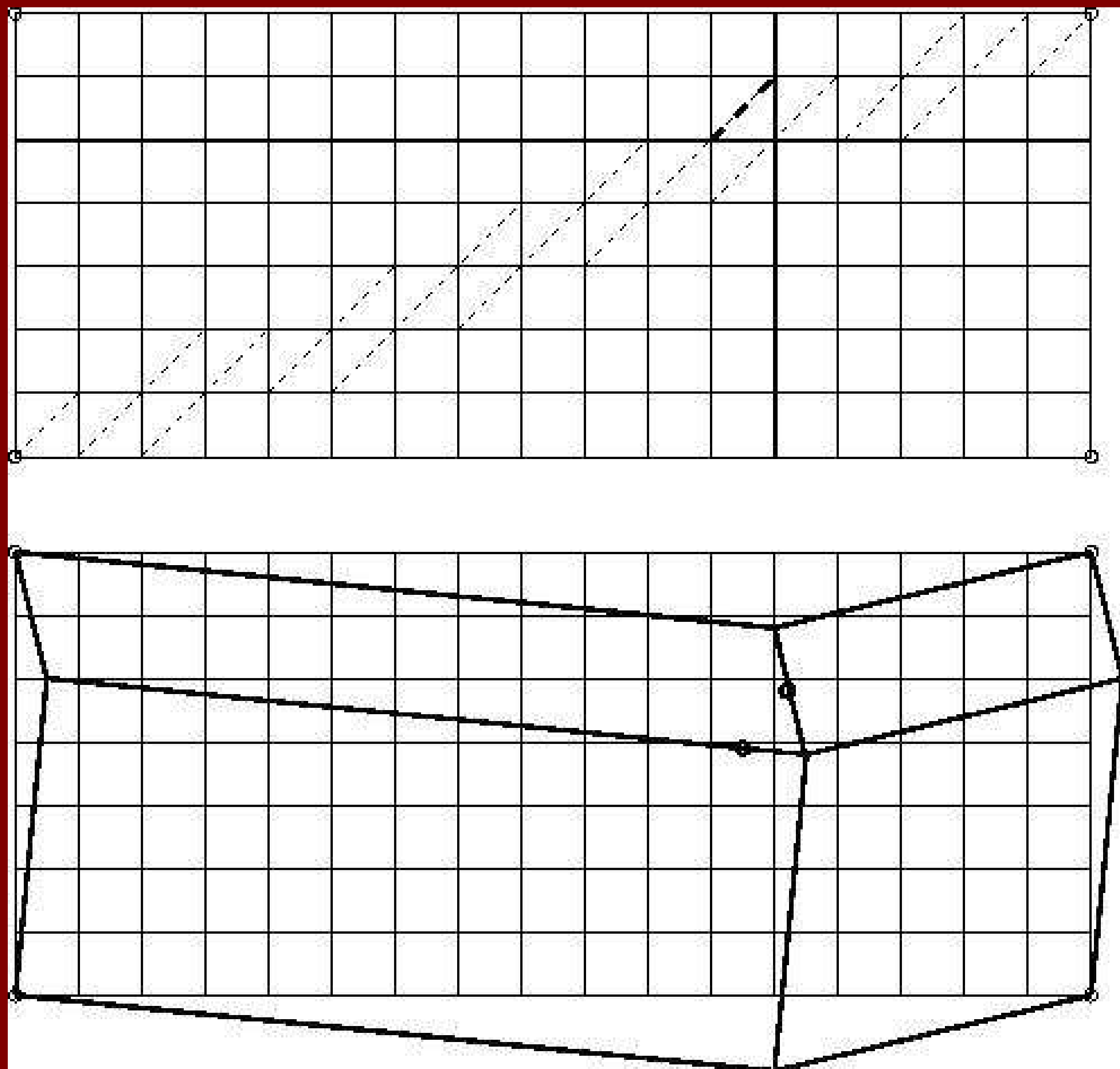


Solution:

Top: $k = 7, \ell = 17, k_0 = 5, \ell_0 = 12, L < K$
(0.7059 < 0.7143)

Bottom: $k = 7, \ell = 17, k_0 = 5, \ell_0 = 13, L > K$
(0.7647 > 0.7143)

where $\ell_0 / \ell = L, \quad k_0 / k = K$.



Hall, 1935 (König, 1931):

A bipartite graph with colour classes X, Y has a perfect matching if and only if

$$|N(X_0)| \geq |X_0|$$

holds for every ~~proper~~ subset X_0 of X and

$$|N(Y_0)| \geq |Y_0|$$

holds for every ~~proper~~ subset Y_0 of Y.

Hetyei, 1964:

A bipartite graph with colour classes X, Y has perfect matchings ***and every edge is contained in at least one*** if and only if

$$|N(X_0)| > |X_0|$$

holds for every proper subset X_0 of X and

$$|N(Y_0)| > |Y_0|$$

holds for every proper subset Y_0 of Y.

Thank you for your attention

recski@cs.bme.hu