Cutting plane methods for integer and combinatorial optimization

Pierre Bonami, Gérard Cornuéjols, Andrea Lodi

Hammamet, March 22-23, 2010

April 10, 2010

Part I

Cutting Planes for Mixed-Integer Linear Programs

1. Mixed-Integer Linear Programming (MILP): notation.
   The Linear Programming (LP) relaxation.
   Strengthening the LP relaxation by cutting planes.
   How much cuts are important in the MILP software?

2. Cutting Planes for MILPs.
   Families of cutting planes and their relationships.

3. Advanced topics.
   Closures and separation.

1. MILP motivation

Motivation

Mixed integer linear programming is today one of the most widely used techniques for dealing with optimization problems:

- Many optimization problems arising from practical applications (such as, e.g., scheduling, project planning, transportation, telecommunications, economics and finance, timetabling) can be easily formulated as MILPs.

- Academic and commercial MILP solvers now available on the market can solve hard MILPs in practice.

1. MILP notation

We consider an MILP of the form

$$\text{(MILP)} \quad \begin{aligned} & \max cx \\ & Ax \le b \\ & x \ge 0 \\ & x_j \in \mathbb{Z}, \forall j \in I \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $I \subseteq \{1, \ldots, n\}$, corresponding to the mixed integer set

$$S := \{x \in \mathbb{R}^n : Ax \le b, x \ge 0, x_j \in \mathbb{Z}, \ \forall j \in I\} \tag{2}$$

and to the underlying polyhedron

$$P := \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}. \tag{3}$$

We assume for sake of simplicity $S \neq \emptyset$

1. MILP complexity

- In the general case, MILP is $\mathcal{NP}$-hard, while Linear Programming (LP) is polynomially solvable and can be efficiently solved in practice.
- The classical approach for handling with MILP is the branch-and-bound algorithm.
- Branch-and-bound can be improved by strengthening the LP relaxation of each node of the search tree by means of valid inequalities (cuts). $\Rightarrow$ branch-and-cut.

1. Strengthening the LP relaxation

Generality

We are interested in general-purpose cutting planes: cutting planes which can be derived without assuming any special structure for the polyhedron P.

Validity

An inequality $\alpha x \leq \beta$ is said to be valid for S if it is satisfied by all $x \in S$.

Obtaining a valid inequality for a continuous set

Given P, any valid inequality for it is obtained as $uAx \leq \beta$, where $u \in \mathbb{R}^m_+$ and $\beta \geq ub$. (Farkas Lemma)

1. Strengthening the LP relaxation (cont.d)

Separation

Given a family of valid inequalities $\mathcal{F}$ and a solution $x^* \in P \setminus S$, the **Separation problem for $\mathcal{F}$:** is defined as:

> *Find an inequality $\alpha x \leq \beta$ of $\mathcal{F}$ valid for S such that $\alpha x^* > \beta$ or show that none exists.*

Iterative strengthening

1. solve the problem $\{\max cx \; : \; x \in P\}$ and get x^*
2. if $x^* \in S$ then **STOP**
3. solve the separation problem, add $\alpha x \leq \beta$ to P and go to 1.

1. Are cutting planes fundamental in practice?

Table: Computing times (as geometric means) for 12 CPLEX versions on a testbed of 1,734 MILP instances: normalization wrt CPLEX 11.0.

CPLEX version	year	better	worse	time
11.0	2007	–	–	1.00
10.0	2005	201	650	1.91
9.0	2003	142	793	2.73
8.0	2002	117	856	3.56
7.1	2001	63	930	4.59
6.5	1999	71	997	**7.47**
6.0	1998	55	1060	21.30
5.0	1997	45	1069	22.57
4.0	1995	37	1089	26.29
3.0	1994	34	1107	34.63
2.1	1993	13	1137	56.16
1.2	1991	17	1132	67.90

2. Basic rounding

A basic rounding argument

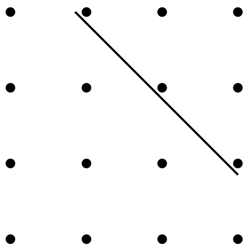
If $x \in \mathbb{Z}$ and $x \leq f$ $f \notin \mathbb{Z}$, then $x \leq \lfloor f \rfloor$.

Using rounding

Consider an inequality $\alpha x \leq \beta$ such that $\alpha_j \in \mathbb{Z},\ j = 1, \ldots, n$ in the pure integer case $I = \{1, \ldots, n\}$. If $\alpha x \leq \beta$, then $\alpha x \leq \lfloor \beta \rfloor$ is valid as well.

Example

$x \in \mathbb{Z}^2$ such that $x_1 + x_2 \leq 1.9$



2. Basic rounding

A basic rounding argument

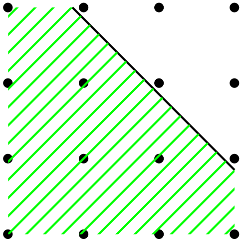
If $x \in \mathbb{Z}$ and $x \leq f$ $f \notin \mathbb{Z}$, then $x \leq \lfloor f \rfloor$.

Using rounding

Consider an inequality $\alpha x \leq \beta$ such that $\alpha_j \in \mathbb{Z},\ j = 1, \ldots, n$ in the pure integer case $I = \{1, \ldots, n\}$. If $\alpha x \leq \beta$, then $\alpha x \leq \lfloor \beta \rfloor$ is valid as well.

Example

$x \in \mathbb{Z}^2$ such that $x_1 + x_2 \leq 1.9$



2. Basic rounding

A basic rounding argument

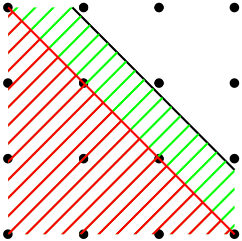
If $x \in \mathbb{Z}$ and $x \leq f$ $f \notin \mathbb{Z}$, then $x \leq \lfloor f \rfloor$.

Using rounding

Consider an inequality $\alpha x \leq \beta$ such that $\alpha_j \in \mathbb{Z}$, $j = 1, \ldots, n$ in the pure integer case $I = \{1, \ldots, n\}$. If $\alpha x \leq \beta$, then $\alpha x \leq \lfloor \beta \rfloor$ is valid as well.

Example

$x \in \mathbb{Z}^2$ such that $x_1 + x_2 \leq 1.9 \Rightarrow x_1 + x_2 \leq \lfloor 1.9 \rfloor = 1$



2. Chvátal-Gomory cuts, [Gomory 1958, Chvátal 1973]

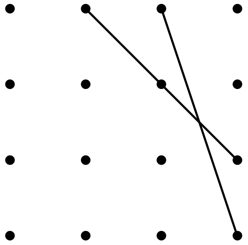
Theorem

If $x \in \mathbb{Z}^n$ satisfies $Ax \leq b$, then the inequality $uAx \leq \lfloor ub \rfloor$ is valid for S for all $u \geq 0$ such that $uA \in \mathbb{Z}^m$.

Example

Consider the polyhedron given by the two inequalities

$$x_1 + x_2 \leq 2$$
$$3x_1 + x_2 \leq 5$$



2. Chvátal-Gomory cuts, [Gomory 1958, Chvátal 1973]

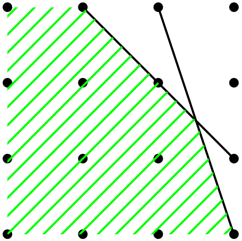
Theorem

If $x \in \mathbb{Z}^n$ satisfies $Ax \leq b$, then the inequality $uAx \leq \lfloor ub \rfloor$ is valid for S for all $u \geq 0$ such that $uA \in \mathbb{Z}^m$.

Example

Consider the polyhedron given by the two inequalities

$$x_1 + x_2 \leq 2$$
$$3x_1 + x_2 \leq 5$$



2. Chvátal-Gomory cuts, [Gomory 1958, Chvátal 1973]

Theorem

If $x \in \mathbb{Z}^n$ satisfies $Ax \leq b$, then the inequality $uAx \leq \lfloor ub \rfloor$ is valid for S for all $u \geq 0$ such that $uA \in \mathbb{Z}^m$.

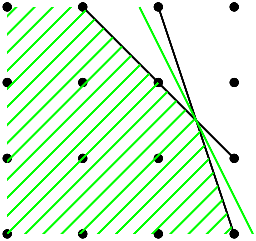
Example

Consider the polyhedron given by the two inequalities

$$x_1 + x_2 \leq 2$$
$$3x_1 + x_2 \leq 5$$

Let $u_1 = u_2 = 1/2$, then

$$2x_1 + x_2 \leq 3.5$$



2. Chvátal-Gomory cuts, [Gomory 1958, Chvátal 1973]

Theorem

If $x \in \mathbb{Z}^n$ satisfies $Ax \leq b$, then the inequality $uAx \leq \lfloor ub \rfloor$ is valid for S for all $u \geq 0$ such that $uA \in \mathbb{Z}^m$.

Example

Consider the polyhedron given by the two inequalities

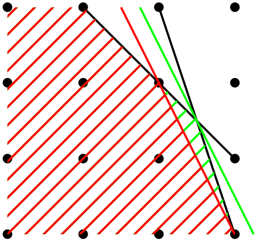
$$x_1 + x_2 \leq 2$$
$$3x_1 + x_2 \leq 5$$

Let $u_1 = u_2 = 1/2$, then

$$2x_1 + x_2 \leq 3.5$$

and rounding we obtain

$$2x_1 + x_2 \leq 3$$



2. A simple disjunctive argument

Simple

If $x \in \mathbb{R}^n$, $x \geq 0$ and x satisfies both $\sum_{i=1}^n a_i^1 x_i \geq 1$ **or** $\sum_{i=1}^n a_i^2 x_i \geq 1$, then x satisfies

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i \geq 1$$

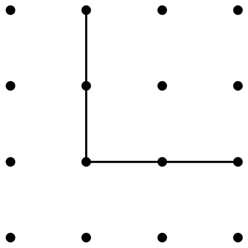
Example

If $x \geq 0$ satisfies both

$$\frac{x_1}{2} + \frac{x_2}{2} \geq 1$$

and

$$\frac{3}{5}x_1 + \frac{x_2}{5} \geq 1$$



2. A simple disjunctive argument

Simple

If $x \in \mathbb{R}^n$, $x \geq 0$ and x satisfies both $\sum_{i=1}^n a_i^1 x_i \geq 1$ **or** $\sum_{i=1}^n a_i^2 x_i \geq 1$, then x satisfies

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i \geq 1$$

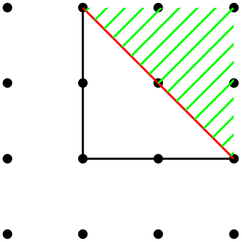
Example

If $x \geq 0$ satisfies both

$$\frac{x_1}{2} + \frac{x_2}{2} \geq 1$$

and

$$\frac{3}{5}x_1 + \frac{x_2}{5} \geq 1$$



2. A simple disjunctive argument

Simple

If $x \in \mathbb{R}^n$, $x \geq 0$ and x satisfies both $\sum_{i=1}^n a_i^1 x_i \geq 1$ **or** $\sum_{i=1}^n a_i^2 x_i \geq 1$, then x satisfies

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i \geq 1$$

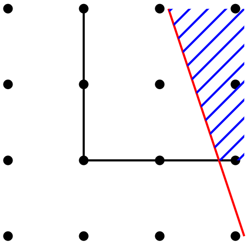
Example

If $x \geq 0$ satisfies both

$$\frac{x_1}{2} + \frac{x_2}{2} \geq 1$$

and

$$\frac{3}{5} x_1 + \frac{x_2}{5} \geq 1$$



2. A simple disjunctive argument

Simple

If $x \in \mathbb{R}^n$, $x \geq 0$ and x satisfies both $\sum_{i=1}^n a_i^1 x_i \geq 1$ **or** $\sum_{i=1}^n a_i^2 x_i \geq 1$, then x satisfies

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i \geq 1$$

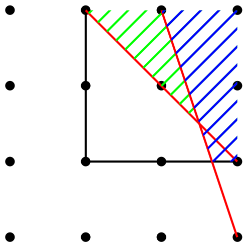
Example

If $x \geq 0$ satisfies both

$$\frac{x_1}{2} + \frac{x_2}{2} \geq 1$$

and

$$\frac{3}{5} x_1 + \frac{x_2}{5} \geq 1$$



2. A simple disjunctive argument

Simple

If $x \in \mathbb{R}^n$, $x \geq 0$ and x satisfies both $\sum_{i=1}^n a_i^1 x_i \geq 1$ **or** $\sum_{i=1}^n a_i^2 x_i \geq 1$, then x satisfies

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i \geq 1$$

Example

If $x \geq 0$ satisfies both

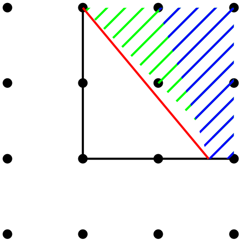
$$\frac{x_1}{2} + \frac{x_2}{2} \geq 1$$

and

$$\frac{3}{5} x_1 + \frac{x_2}{5} \geq 1$$

then it satisfies

$$\frac{3}{5} x_1 + \frac{x_2}{2} \geq 1$$



2. Mixed-Integer Gomory Cuts [Gomory 1963]

Mixed-Integer Set in equality form

$$\{x \in \mathbb{R}^n_+ \ : \ \sum_{j \in I} d_j x_j + \sum_{j \notin I} g_j x_j = b\} \tag{4}$$

Let $b = \lfloor b \rfloor + f_0, \ f_0 \in]0,1[$ and $d_j = \lfloor d_j \rfloor + f_j, \ f_j \in [0,1[$
Then, for some integer k,

$$\sum_{j:f_j \le f_0} f_j x_j + \sum_{j:f_j > f_0} (f_j - 1) x_j + \sum_{j \notin I} g_j x_j = k + f_0,$$

which in turn implies, because $k \le -1$ **OR** $k \ge 0$, that **EITHER**

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{j:f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j \notin I} \frac{g_j}{f_0} x_j \ge 1,$$

OR

$$-\sum_{j:f_j \le f_0} \frac{f_j}{1 - f_0} x_j + \sum_{j:f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j \notin I} \frac{g_j}{1 - f_0} x_j \ge 1,$$

2. Mixed-Integer Gomory Cuts (cont.d)

Mixed-Integer Set in equality form

Applying the disjunctive argument previously introduced, one can write a valid inequality which is indeed called Mixed-Integer Gomory cut (MIG):

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{j:g_j>0} \frac{g_j}{f_0} x_j - \sum_{j:g_j<0} \frac{g_j}{1-f_0} x_j \ge 1. \qquad (5)$$

Remark

When $I = \emptyset$ the MIG cut reduces to

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{1-f_j}{1-f_0} x_j \ge 1$$

which is of course stronger that the corresponding CG cut:

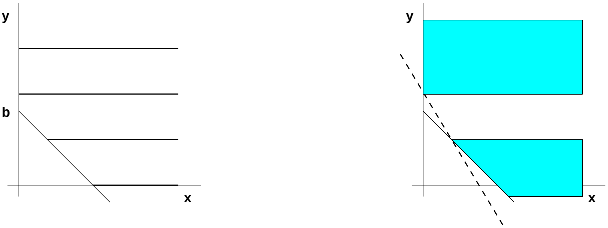
$$\sum_{j=1}^{n} \frac{f_j}{f_0} x_j \ge 1.$$

2. Mixed Integer Rounding Cuts (MIR) [Nemhauser & Wolsey 1988]

Basic MIR Principle

Consider the 2-variable mixed-integer set

$$\{x + y \geq b,\ x \in \mathbb{R}_+,\ y \in \mathbb{Z}\}$$



MIR Cut

The inequality

$$\frac{x}{b - \lfloor b \rfloor} + y \geq \lceil b \rceil$$

is clearly valid and together with the original inequality defines the convex hull of the mixed-integer set.

2. Mixed Integer Rounding Cuts (cont.d)

General MIR cuts

With a slight change of notation for a mixed-integer set in "$\leq$" form

$$\{x \in \mathbb{R}^n_+ \; : \; \sum_{j \in I} d_j x_j + \sum_{j \notin I} g_j x_j \leq b\}, \tag{6}$$

the general MIR cut is written as ($(\cdot)^+$ denotes the $\max\{0, \cdot\}$)

$$\sum_{j \in I} \lfloor d_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} x_j + \frac{1}{1 - f_0} \sum_{j : g_j < 0} g_j x_j \leq \lfloor b \rfloor, \tag{7}$$

Sketch of proof

Relax set (6) as (valid because $x \in \mathbb{R}^n_+$)

$$\sum_{j : f_j \leq f_0} \lfloor d_j \rfloor x_j + \sum_{j : f_j > f_0} d_j x_j + \sum_{j : g_j < 0} g_j x_j \leq b.$$

Then, one can group things so as to reduce to a set $-x + y \leq b$ (as before but with "$\leq$" form) and use the basic MIR derivation for such a case.

2. Relationship among cuts

Theorem (MIG versus MIR)

Given a mixed-integer set in "$\leq$" form (6),
the MIR inequality (7) is identical to the MIG inequality (5)
derived from the mixed-integer set in "$=$" form (4)
obtained by adding a slack variable.

Hint

All cuts derived so far make use of a disjunctive argument.