Intersection cut and disjunctive cuts

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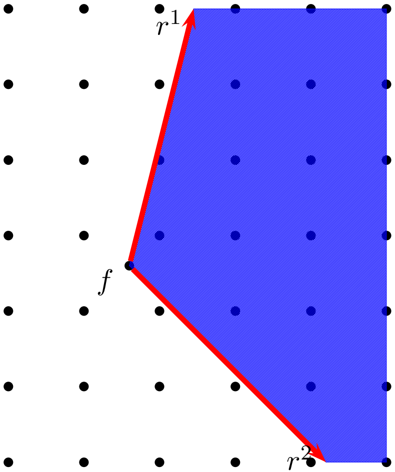
Intersection cuts [Balas 1970]

- Solve LP relaxation. Optimal tableau:

$$x_i = f_i + \sum_{j \in N} r^j x_j \qquad \forall i \in B.$$

$$x_j \geq 0 \qquad \forall j \in N$$

- If $f_i \in \mathbb{Z}$, $\forall i \in B \cap I$ problem solved.
- Suppose $f \notin \mathbb{Z}^2$.



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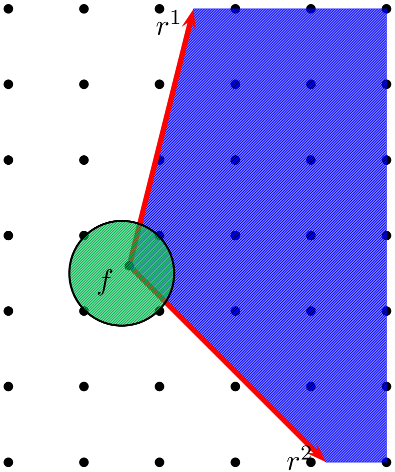
- Suppose $f \notin \mathbb{Z}^2$.
  - Consider a convex set S containing f in its interior but no integral point.



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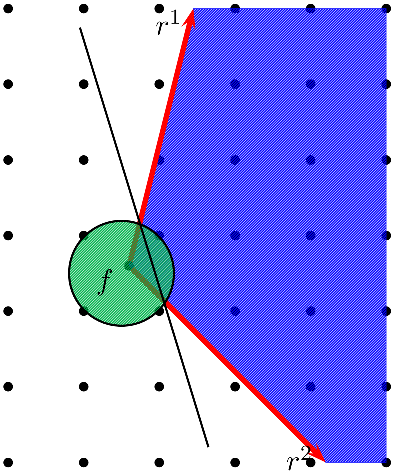
- If $f_i \in \mathbb{Z}, \forall i \in B \cap I$ problem solved.
- Suppose $f \notin \mathbb{Z}^2$.
  - Consider a convex set S containing f in its interior but no integral point.
  - Compute the intersection of the rays with the boundary of S.
  - The inequality passing through these point is valid and cuts off f.



Algebraic derivation of intersection cuts [Balas 1970]

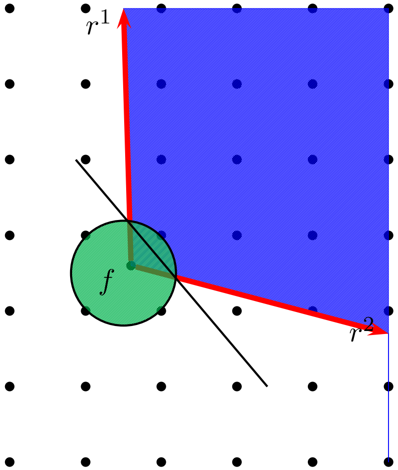
Given a convex set S with no integral point in its interior and $f \in \text{int}(S)$, and a simplex tableau:

$$x_i = f_i + \sum_{j \in N} r^j x_j \qquad \forall i \in B.$$

- for each $j \in N$:
  - if $\exists \lambda_j \geq 0$ such that $f + \lambda_j r^j$ is on the boundary of S, let $\psi^j = \frac{1}{\lambda_j}$.
  - otherwise $\psi^j = 0$
- The cut $\sum_{j \in N}^k \psi^j x_j \geq 1$ is valid.

Remark on Intersection cuts

A bigger convex set S yields to a better cut:



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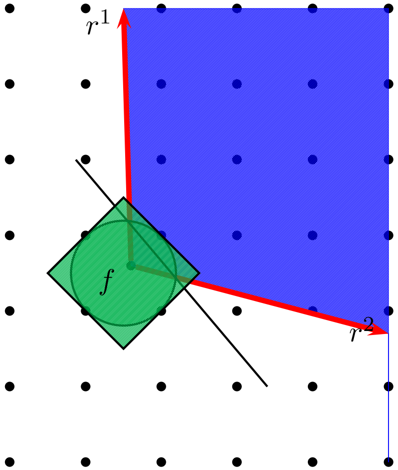

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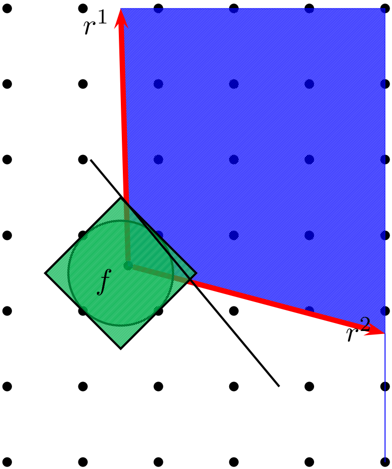
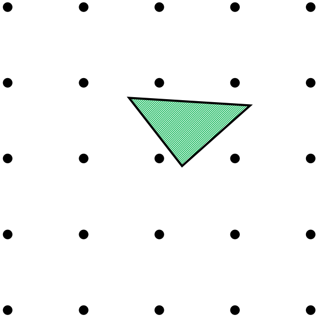


Maximal Lattice free sets

Set that does not contain any integral point in its interior and maximal by inclusion.



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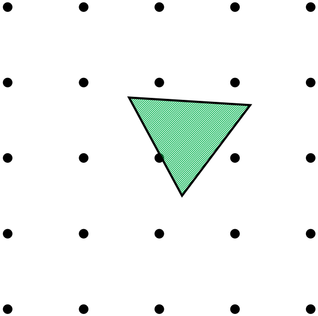

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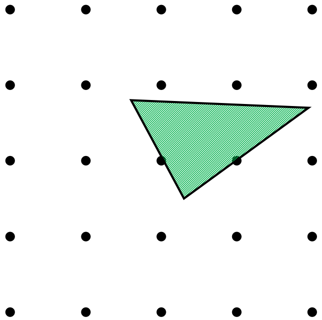

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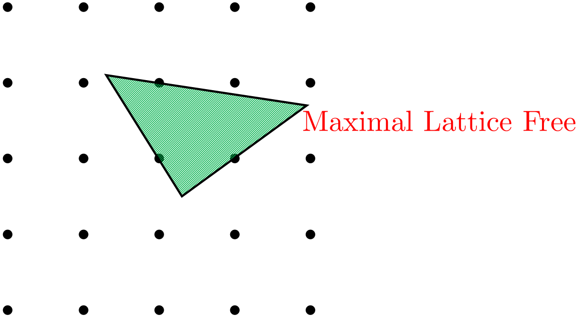

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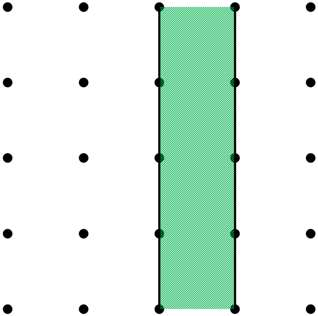

GMI Split (Maximal Lattice Free)

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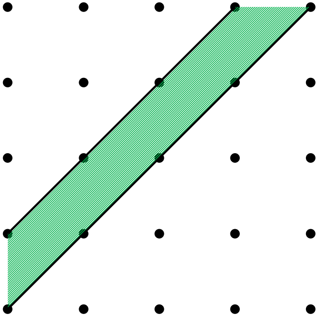

Split (Maximal Lattice Free)

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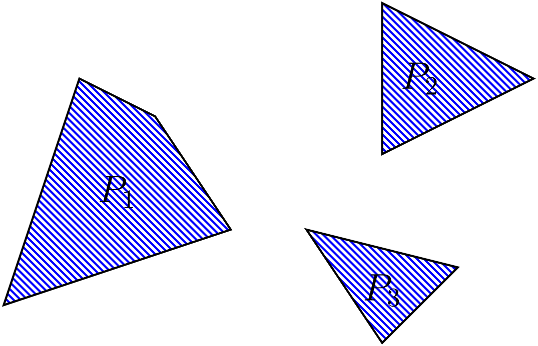
Unions of polyhedra

Given k polyhedra $P_1, \ldots, P_k$, we are interested in

$$\operatorname{conv}\left(\cup_{i=1}^{k} P_i\right)$$



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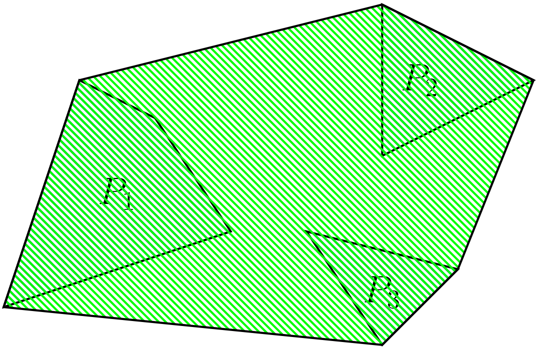


$\operatorname{conv}(P_1 \cup P_2 \cup P_3)$

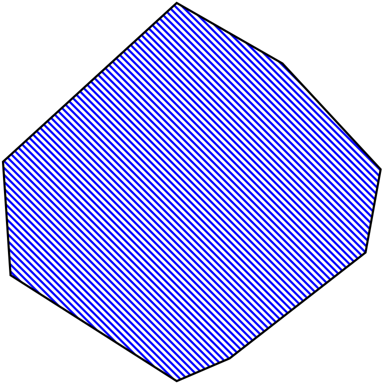
Use in MILP

Continuous Relaxation

$$P = \{x : Ax + Gy \leq b\}$$

Disjunction

$$x_i \leq k \vee x_i \geq k + 1$$



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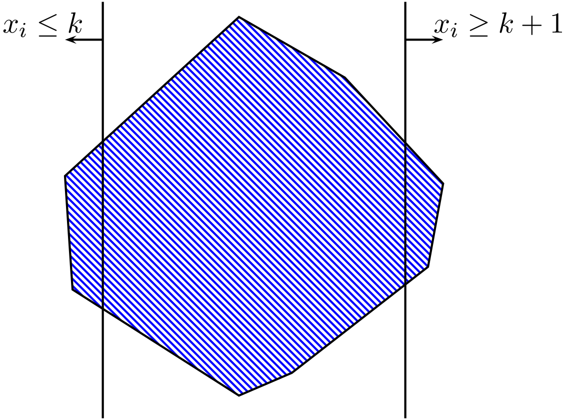

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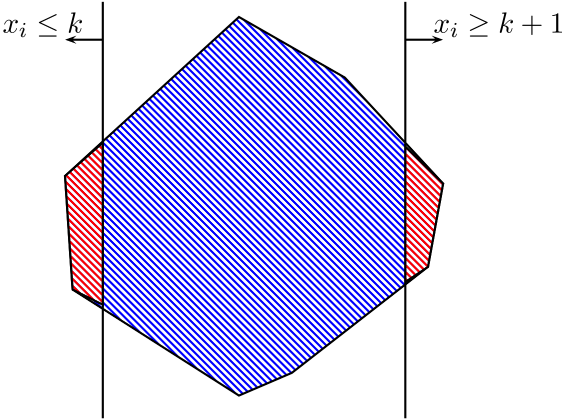

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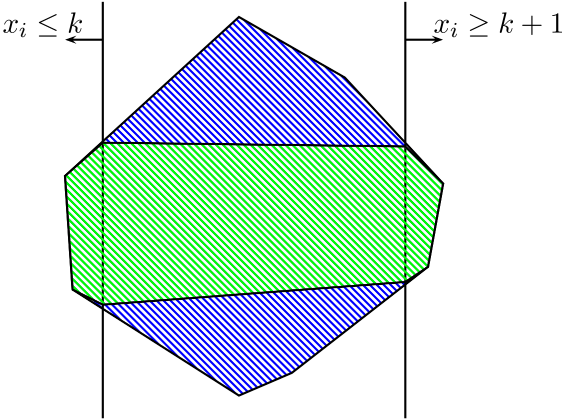
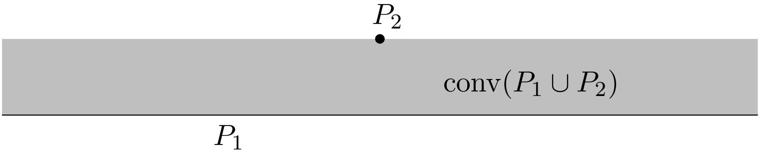


Theorem on unions of polyhedra

Remark

$\text{conv}(\cup_{i=1}^k P_i)$ may not be a closed set: let $P_1 = \{x \in \mathbb{R}^2 : x_2 = 0\}$ $P_2 = \{x \in \mathbb{R}^2 : x_1 = 1, x_2 = 1\}$ then $\text{conv}(P_1 \cup P_2) = \{x \in \mathbb{R}^2 : x_2 < 1\} \cup P_2$.



Union of polyhedra Theorem

Let $P_i = Q_i + C_i$ be nonempty polyhedra for $i = 1, \ldots, k$. Then $Q = \text{conv}(\cup_{i=1}^k Q_i)$ is a polytope, $C = \text{cone}(\cup_{i=1}^k C_i)$ is a polyhedral cone and

$$\text{clconv}(\cup_{i=1}^k P_i) = Q + C$$

Sketch of proof (leave out the case $\cup_{i=1}^k P_i = \emptyset$)

(i) $Q_i = \text{conv}(V_i)$ therefore $Q = \text{conv}(\cup_{i=1}^k V_i)$ is a polytope.
(ii) $C_i = \text{cone}(R_i)$ therefore $C = \text{cone}(\cup_{i=1}^k R_i)$ is a polyhedral cone.
(iii) to show: $\text{clconv}(\cup_{i=1}^k P_i) \subseteq Q + C$, $\text{conv}(\cup_{i=1}^k P_i) \subseteq Q + C$ is sufficient.
Let $x \in \text{conv}(\cup_{i=1}^k P_i)$:

$$x = \sum_{i=1}^k y_i z^i \text{ with } y_i \geq 0, \ \sum_{i=1}^k y_i = 1 \text{ and } z^i \in P^i.$$

Then $z^i = w^i + r^i$ with $w^i \in Q_i$ and $r^i \in C_i$. Thus

$$x = \sum_{i=1}^k y_i w^i + \sum_{i=1}^k y_i r^i \in Q + C.$$

Sketch of proof (II)

(iv) $Q + C \subseteq \text{clconv}(\cup_{i=1}^k P_i)$.

Let $x \in Q + C$. Then:

$x = \sum_{i=1}^k y_i w^i + \sum_{i=1}^k r^i$, with $w^i \in Q^i$, $y_i \geq 0$, $x^i \in C_i$ and $\sum_{i=1}^k y_i = 1$. Let $I = \{i : y_i > 0\}$.

Define:

$$x^\epsilon = \sum_{i \in I} \left(y_i - \epsilon \frac{k}{|I|} \right) w^i + \sum_{i=1}^k \epsilon \left(w^i + \frac{1}{\epsilon} r^i \right)$$

For $\epsilon > 0$ small enough $\epsilon \frac{k}{|I|} \geq 0$ and $x^\epsilon \in \text{conv}(\cup_{i=1}^k P_i)$.

Furthermore $\lim_{\epsilon \to 0} x^\epsilon = x$. □

Balas Theorem

Let $P_i = \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ be polyhedra for $i = 1, \ldots, k$, then

$$\mathrm{proj}_x(Y) = Q + C$$

with

$$Y = \begin{cases} A_i x^i \leq b_i y_i & \text{for } i = 1, \ldots, k \\ \sum_{i=1}^{k} x^i = x & \\ \sum_{i=1}^{k} y_i = 1 & \\ y_i \geq 0 & \text{for } i = 1, \ldots, k \end{cases}$$

Furthermore if $\cup P_i = \emptyset$ or $C_j \subseteq \mathrm{conv}(\cup_{i:P_i \neq \emptyset} C_i)$:

$$\mathrm{proj}_x(Y) = \mathrm{cl}conv(\cup_{i=1}^{k} P_i)$$

Sketch of the proof (leave out the $\emptyset$ case)

(i) $\text{proj}_x(Y) \subseteq Q + C$.
Let $(x, x^1, y_1, \ldots, x^k, y_k) \in Y$. For i such that $y_i > 0$ $\frac{x^i}{y_i} \in P_i$.
For i such that $y_i = 0$, $x^i \in C_i$.
(ii) $Q + C \subseteq \text{proj}_x(Y)$
Let $x \in Q + C$.

$$x = \sum_{i=1}^{k} y_i z^i + \sum_{i=1}^{k} r^i \text{ with } y_i \geq 0,\ \sum_{i=1}^{k} y_i = 1,\ z^i \in Q^i \text{ and } r^i \in C_i.$$

For i such that $y^i > 0$, let $x^i = y^i z^i + r^i$. For i such that $y^i = 0$, let $x^i = r^i$. One can check that $(x, x^1, y_1, \ldots, x^k, y_k) \in Y$. $\square$

Separation of disjunctive cut

Let

$$P_D = \text{cl}conv(\cup_{i=1}^k P_i).$$

Separation Problem

Given $\hat{x} \in \mathbb{R}^n$, find (α, β) in $\mathbb{R}^{n+1}$ such that $\alpha^T x \le \beta$ is valid for P_D and $\alpha^T x > \beta$ or show that $\hat{x} \in P_D$.

Find a solution to:

$$A_i x^i \le b_i y_i \text{ for } i = 1, \ldots, k$$

$$\sum_{i=1}^k x^i = \hat{x}$$

$$\sum_{i=1}^k y_i = 1$$

$$y_i \ge 0 \text{ for } i = 1, \ldots, k$$

By Farkas Lemma, this system has a solution if and only if:
$\exists\ \alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$,
$u^1, \ldots, u^k \in \mathbb{R}^m_+$ such that:

$$u^{i^T} A^i = \alpha \ i = 1, \ldots, k$$

$$u^{i^T} b \le \beta \ i = 1, \ldots, k$$

$$\alpha^T \hat{x} > \beta$$

Application to Mixed Integer Programming: split cuts

Consider the mixed-integer set:

$$S = \{x \in \mathbb{R}^n_+ : Ax \leq b, x_i \in \mathbb{Z}, i \in I\}$$

and its relaxation:

$$P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$$

Given $(\pi, \pi_0) \in \mathbb{Z}^n$ such that $\pi_i = 0, \forall i \notin I$, we consider the folowing set:

$$P^{(\pi,\pi_0)} =$$
$$\text{conv}\left(\left(P \cap \left\{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\right\}\right) \cup \left(P \cap \left\{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\right\}\right)\right)$$

Proposition

$$S \subseteq P^{(\pi,\pi_0)} \subseteq P$$

Valid inequalities for $P^{(\pi,\pi_0)}$ are called split cuts [Cook, Kannan and Schrijver, 1990].

Separating Split using Linear Programming

$$P^{(\pi,\pi_0)} = \text{conv}\left(\left(P \cap \left\{\pi^T x \le \pi_0\right\}\right) \cup \left(P \cap \left\{\pi^T x \ge \pi_0 + 1\right\}\right)\right)$$

Proposition [Balas 73]

Let $\hat{x} \in P$, $x \in P^{(\pi,\pi_0)}$ if and only if the optimum of the following LP is non-positive.

$$\begin{aligned}
\max\ & \alpha^T \hat{x} - \beta \\
\text{s.t.}\ & \\
& u^T A + u_0 \pi \ge \alpha \\
& v^T A - v_0 \pi \ge \alpha \\
& u^T b + u_0 \pi_0 \le \beta \\
& u^T b - v_0(\pi_0 + 1) \le \beta \\
& u, v \in \mathbb{R}^m_+,\ u_0, v_0 \ge 0
\end{aligned} \qquad \text{(CGLP)}$$

The cut generation LP

- If $\hat{x}$ is a vertex of P such that $\pi_0 < \pi^T x < \pi_{0+1}$, always a cut.
- If it has a positive solution, it is unbounded.
- Usually, impose a normalization constraint to bound it:
    1. $u_0 + v_0 = 1$
    2. $\sum_{i=1}^{m}(u_i + v_i) + u_0 + v_0 = 1$
- If $\pi = e_i$: lift-and-project cut [Balas Ceria Cornuejols 93].
- For splits, and lift-and-project, can actually be solved in the LP tableau [Balas Perregaard 2003].

Balas Jeroslow Strengthening

Let $(\alpha, \beta, u, v, u_0, v_0)$ be a feasible solution of (CGLP) with $u_0 > 0$ and $v_0 > 0$. Define $\hat{m}_i = \frac{v^T A^i - u^T A^i}{u_0 + v_0}$ and

$$\tilde{\alpha}_i = \begin{cases} \max\{u^T A^i + u_0 \lceil \hat{m}_i \rceil, v^T A^i - v_0 \lfloor \hat{m}_i \rfloor\} & \text{if } i \in I \\ \min\{u^T A^i, v^T A^i\} & \text{otherwise.} \end{cases}$$

Proof idea

For any $m \in \mathbb{Z}^n$ such that $m_i = 0$ for $i \notin I$ and $m_i \in \mathbb{Z}$ for $i \in I$, the following disjunction is valid:

$$\left(\pi^T x + m^T x \leq \pi_0\right) \vee \left(\pi^T x + m^T x \geq \pi_0\right)$$

For u, v, u_0, v_0 fixed we can find m which give the best cut coefficients for $i \in I$:

$$\tilde{\alpha}_i = \max_{m_j \in \mathbb{Z}} \{\min\{u^T A^i + u_0 m_i, v^T A^i - v_0 m_i\}\}$$