Cutting plane methods for integer and combinatorial optimization

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Part I

Cutting Planes for Mixed-Integer Linear Programs

1. Mixed-Integer Linear Programming (MILP): notation.
   The Linear Programming (LP) relaxation.
   Strengthening the LP relaxation by cutting planes.
   How much cuts are important in the MILP software?

2. Cutting Planes for MILPs.
   Families of cutting planes and their relationships.

3. Advanced topics.
   Closures and separation.

3. Elementary Closures [Chvátal 1973]

Definition

Consider again the special case of S where $I = \{1, \ldots, n\}$, we define the Chvátal elementary (or first) closure as

$$P(S) = \{x \in \mathbb{R}^n \ : \ uAx \leq \lfloor ub \rfloor, \ u \in \mathbb{R}^m_+, \ uA \in \mathbb{Z}^n\}.$$

Proposition

$$S \subseteq P(S) \subseteq P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

Iterative application

Such a derivation can be iterative applied:

$$P^2(S) = P(P(S))$$

$$\vdots$$

$$P^k(S) = P(P^{k-1}(S))$$

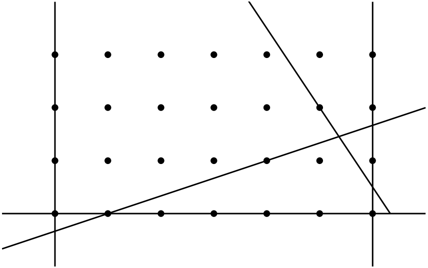
3. Chvátal-Gomory Theorem

Theorem

If A and b have rational coefficients, then any inequality $\alpha x \leq \beta$ valid for S can be obtained by applying the Chvátal procedure a fixed number of times, i.e., $\text{conv}(S) = P^k(S)$ for fixed k.

Example

$$\begin{aligned} \max\ & 9x_1 + 5x_2 & \\ & x_1 \leq 6 & u_1 \\ & -x_1 + 3x_2 \leq -1 & u_2 \\ & 3x_1 + 2x_2 \leq 19 & u_3 \\ & x \in \mathbb{Z}_+^2 & \end{aligned}$$



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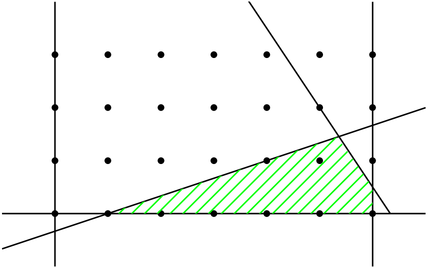

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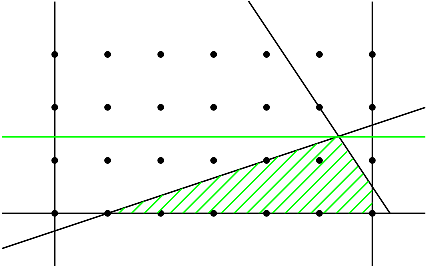
$$u_1 = 0, u_2 = 3/11, u_3 = 1/11$$



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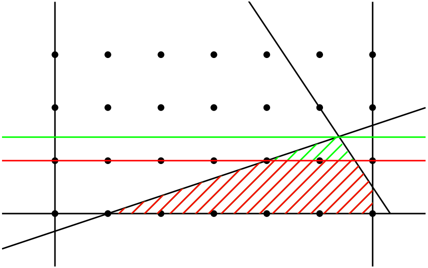

$u_1 = 0, u_2 = 3/11, u_3 = 1/11 \Rightarrow x_1 \leq 1$ (u_4)

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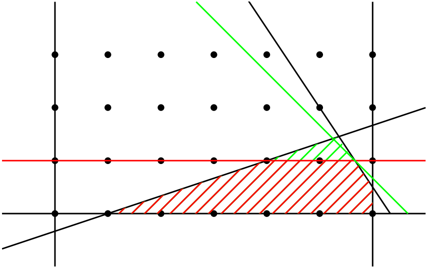

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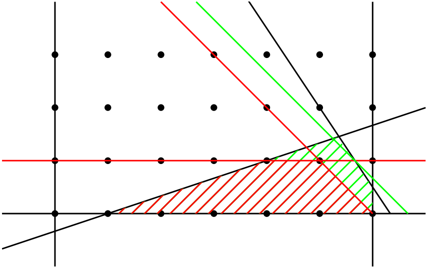

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$u_1 = u_2 = 0, u_3 = 1/3, u_4 = 1/3 \Rightarrow x_1 + x_2 \leq 6$

3. The cutting plane algorithm

Algorithm

1. $\mathcal{C} \leftarrow \emptyset$
2. Solve the LP relaxation
   $\max\{cx : Ax \leq b,\ \alpha x \leq \beta,\ \forall \alpha, \beta \in \mathcal{C},\ x \geq 0\}$. Let x^* the optimal solution.
3. If $x^* \in \mathbb{Z}^n$, then **STOP**.
4. Solve the associated separation problem to find $\alpha x \leq \beta$ such that $\alpha x^* > \beta$ while $\alpha x \leq \beta$ for all $x \in S$.
5. Add α, β to $\mathcal{C}$ and go to 2.

We assume A having integer coefficients.

3. Solving the LP relaxation

Complexity

[Eisenbrand 1999] proved that finding a Chvátal-Gomory cut separating an arbitrary x^* is $\mathcal{NP}$-hard.
However, in the special case where x^* is a vertex of the LP relaxation the separation is trivial.
This is indeed the case if the LP relaxation is solved by the simplex algorithm:
at the optimal solution, we have a basis and the so-called "tableau". A row of the tableau has the form

$$x'_i + \sum_{j \notin B} \overline{a}_{ij} x'_j = \overline{a}_{i0}$$

The optimal solution x^* has $x_i^{*,\prime} = 0$ for $i \notin B$ and $x_i^{*,\prime} = \overline{a}_{i0}$ for $i \in B$.

3. Gomory's Algorithm

We assume A having integer coefficients.

1. $\mathcal{C} \leftarrow \emptyset$
2. Solve through the simplex algorithm the LP relaxation $\max\{cx : Ax \leq b,\ \alpha x \leq \beta,\ \forall \alpha, \beta \in \mathcal{C},\ x \geq 0\}$.
   Let x^* be the optimal solution and B the optimal basis.
3. If $x^* \in \mathbb{Z}^n$, then **STOP**.
4. Select a row of the simplex tableau

$$x'_i + \sum_{j \notin B} \overline{a}_{ij} x'_j = \overline{a}_{i0}$$

   such that $\overline{a}_{i0} \notin \mathbb{Z}$.
5. Derive and add the Gomory cut

$$x'_i + \sum_{j \notin B} \lfloor \overline{a}_{ij} \rfloor x'_j \leq \lfloor \overline{a}_{i0} \rfloor \text{ and got to 2.}$$

3. Finiteness of Gomory's Algorithm

Lexicographic simplex

A solution x^* is lexicographically optima if:

- it is optimal;
- it is maximal in the lexicographic order: any other solution $\overline{x}$ is such that

$$x_1^* > \overline{x}_1 \text{ or } (x_1^* = \overline{x}_1 \text{ and } x_2^* > \overline{x}_2) \text{ or } \dots$$

$$\text{or } (x_i^* = \overline{x}_i \text{ for } i = 1, \dots, n-1 \text{ and } x_n^* > \overline{x}_n)$$

Theorem [Gomory 1958]

The algorithm converges within a finite number of iterations if the lexicographically optimal solution is used at each iteration.

3. Other closures

Split/MIG/MIR closure

Exactly in the same way as for the Chvátal closure one can define the Split Closure.
In particular the elementary split closure as the mixed-integer set composed by the original problem plus all split cuts which can be derived only using the original set of constraints.

Complexity

[Caprara & Letchford 2001] proved that, if x^* is arbitrary, then the separation of split cuts is $\mathcal{NP}$-hard as well.
However, as for Chvátal-Gomory cuts, MIG separation is trivial if x^* is a vertex of the LP relaxation.

3. Closures and polyhedra

$P^{(i)}$ is a rational polyhedron

In other words, A and b have integral entries.

The Elementary Chvátal closure is a polyhedron

In other words, although infinitely many rank-1 inequality exist, only a finite number of them is enough to define the Elementary Chvátal closure.
One can prove that it is enough to restrict in the CG derivation to $u < \mathbf{1}$.
Thus, $\{uA \in \mathbb{R}^n \ : \ 0 \leq u < \mathbf{1}\}$ is bounded, which implies $\{uA \in \mathbb{Z}^n \ : \ 0 \leq u < \mathbf{1}\}$ is finite.
Therefore, only a finite number of CGs are enough.

The Elementary Split closure is a polyhedron

Much more complicated to prove [Cook, Kannan & Schrijver 1990, Andersen, Cornuéjols & Li 2005, Dash, Günlük & Lodi 2010]