Separation models for elementary closures

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Reminder Chvátal-Gomory cuts.

We consider the pure integer linear programming problem

$$\min\{c^T x : Ax \leq b, x \geq 0, x \in \mathbb{Z}\}$$

and the two sets:

$$P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$$

$$S = conv\{x \in \mathbb{Z}^n_+ : Ax \leq b\} = conv(P \cap \mathbb{Z}^n)$$

A Chvátal-Gomory (CG) cut: $\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$ where $u \in \mathbb{R}^m_+$.
The **Chvátal closure** of P:

$$P^1 := \{x \geq 0 : Ax \leq b, \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{R}^m_+\}. \quad (1)$$

Chvátal-Gomory separation

Separation problem

Given any point $\hat{x} \in P$ find (if any) a CG cut $\alpha^T x \leq \alpha_0$ that is violated by $\hat{x}$, i.e., find $u \in \mathbb{R}^m_+$ such that $\lfloor u^T A \rfloor \hat{x} > \lfloor u^T b \rfloor$, or prove that no such u exists.

Optimizing (or separating) over the first Chvátal closure is NP-hard [Eisenbrand, 1999].

MIP model

The following MIP solves the separation problem [Fischetti and Lodi, 2005]

$$
\begin{aligned}
\max\ & \alpha^T \hat{x} - \alpha_0 \\
& \alpha^T \le u^T A,\ \ u \ge 0 \\
& \alpha_0 + 1 - \epsilon \ge u^T b \qquad \text{(CG-MIP)} \\
& \alpha,\ \ \alpha_0 \quad \text{integer}
\end{aligned}
$$

Validity of (CG-MIP) follows from the fact that $\alpha^T x \le \alpha_0$ is a CG cut if and only if:

1. (α, α_0) is integral,
2. $\alpha^T x \le \alpha_0 + 1 - \epsilon$ is valid for P.

Chvátal-Gomory cuts for mixed-integer sets [B., Cornuéjols, Dash, Fischetti and L. 2007]

We now consider the mixed-integer set:

$$S = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$$

and its relaxation:

$$P = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$$

The **projection** of P onto the space of x variables is:

$$\begin{aligned} P_x &= \{x \in \mathbb{R}^n_+ : \exists\, y \in \mathbb{R}^p_+ \text{ s.t. } Ax + Gy \leq b\} \\ &= \{x \in \mathbb{R}^n_+ : u^k A \leq u^k b,\ k = 1, \ldots, K\} \\ &= \{x \in \mathbb{R}^n_+ : \bar{A}x \leq \bar{b}\} \end{aligned}$$

where $u^1, \ldots, u^K$ are the (finitely many) extreme rays of the projection cone $\{u \in \mathbb{R}^m_+ : u^T C \geq 0^T\}$.

Projected Chvátal-Gomory cuts

We define a **projected Chvátal-Gomory (pro-CG) cut** as a CG cut derived from the system $\bar{A}x \le \bar{b}$, $x \ge 0$, i.e., an inequality of the form $\lfloor w^T \bar{A} \rfloor x \le \lfloor w^T \bar{b} \rfloor$ for some $w \ge 0$.

Any row of $\bar{A}x \le \bar{b}$ can be obtained as a positive combination of the rows of $Ax \le b$ with multipliers $\bar{u} \ge 0$ such that $\bar{u}^T C \ge 0^T$
$\Rightarrow$ a pro-CG cuts are of the form:

$$\lfloor u^T A \rfloor x \le \lfloor u^T b \rfloor \qquad \text{for any } u \ge 0 \text{ such that } u^T C \ge 0^T. \qquad (2)$$

Separation of pro-CG cuts

Separation by a **simple extension** of the CG separating MIP

$$\begin{aligned} \max\ & \alpha^T \hat{x} - \alpha_0 \\ & \alpha^T \leq u^T A \\ & 0^T \leq u^T C \\ & \alpha_0 + 1 - \epsilon \geq u^T b \\ & u \geq 0 \\ & \alpha \in \mathbb{Z}^n,\ \alpha_0 \in \mathbb{Z} \end{aligned}$$

Split cuts separation

We now consider a mixed-integer set of the form:

$$S = \{x \in \mathbb{R}^n : Ax \leq b, x_i \in \mathbb{Z}, i \in I\}$$

and its relaxation:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

(bounds (if any) are included in $Ax \leq b$).
For any $(\pi, \pi_0) \in \mathbb{Z}^n$ such that $\pi_i = 0$, $\forall i \notin I$:

$$P^{(\pi,\pi_0)} =$$
$$\text{conv}\left(\left(P \cap \left\{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\right\}\right) \cup \left(P \cap \left\{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\right\}\right)\right)$$

The **split closure** is:

$$P_{split} = \cap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}:\pi_i=0,\forall i\notin I} P^{(\pi,\pi_0)}.$$

and is a polyhedron [Cook, Kannan and Schrijver, 1990].
Separating over it is NP-Hard [Caprara and Letchford 2003].

Split closure separation

$$
\begin{aligned}
\max\ & \alpha^T \hat{x} - \beta \\
\text{s.t.}\ & \\
& u^T A + u_0 \pi = \alpha \\
& v^T A - v_0 \pi = \alpha \\
& u^T b + u_0 \pi_0 \leq \beta \\
& u^T b - v_0 (\pi_0 + 1) \leq \beta \\
& u_0 + v_0 = 1 \\
& u, v \in \mathbb{R}^m_+,\ u_0, v_0 \geq 0 \\
& \pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z} \\
& \pi_i = 0, i \notin I
\end{aligned}
\tag{CGLP}
$$

Simplifications to the model [Balas and Saxena 2005]

Using $u_0 + v_0 = 1$ one can simplify the model to:

$$
\begin{aligned}
\min\ & u^T(A\hat{x} - b) + u_0(\pi^T\hat{x} - \pi_0) \\
& u^T A - v^T A + \pi = 0 \\
& u^T b - v^T b + \pi_0 = u_0 - 1 \\
& 0 < u_0 < 1, u, v \geq 0 \\
& \pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z} \\
& \pi_i = 0, i \notin I
\end{aligned}
$$

- For any fixed value of parameter u_0 the model becomes a regular MIP (w.l.o.g. $u_0 \in (0, 1/2]$).
- Balas and Saxena considered a **heuristic list** of possible values for parameter u_0, say $(0.05, 0.1, 0.2, 0.3, 0.4, 0.5)$ and then enriched it, on the fly, by inserting new heuristic points.

Experimental strength of the closures

- The strengthe of the closures, namely CG, pro-CG and split (or MIR) closures, has been evaluated by running a cutting plane algorithm for a large (sometimes huge) computing time.
- **Goal of the investigation**: show the tightness of the closures, rather than investigating the practical relevance of the separation MIPping idea when used within a practical MIP solver.
- Tightness of the closures for MIPlib 3.0 instances, in terms of "**percentage of gap closed**", computed as

$$100 - \frac{100(opt\_value(P_I) - opt\_value(P^1))}{(opt\_value(P_I) - opt\_value(P))}$$

Strength of the closures

		Split closure	CG closure
% Gap closed	Average	71.71	62.59
% Gap closed	98-100	9 instances	9 instances
% Gap closed	75-98	4 instances	2 instances
% Gap closed	25-75	6 instances	7 instances
% Gap closed	< 25	6 instances	7 instances

Table: Percentage of gap closed for 25 *pure* integer linear programs in the MIPlib 3.0.

		Split closure	pro-CG closure
% Gap closed	Average	84.34	36.38
% Gap closed	98-100	16 instances	3 instances
% Gap closed	75-98	10 instances	3 instances
% Gap closed	25-75	2 instances	11 instances
% Gap closed	< 25	5 instances	17 instances

Table: Percentage of gap closed for 33 *mixed* integer linear programs in the MIPlib 3.0.

Nice features of rank-1 split cuts

Pure IP Instance	# Int Variables	Mean Support Size
nw04	87482	2.084
air05	7195	8.210
seymour	1372	5.263
misc03	159	3.771
p0033	33	4.847

Mixed IP Instance	# Int Variables	Mean Support Size
qnet1_o	1417	6.690
gesa2_o	720	4.937
arki001	538	3.146
vpm1	168	4.503
pp08aCUTS	64	3.850

Figure: Split cut properties (from Balas and Saxena, 2005)