Extended Formulations for Combinatorial Optimization

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ISCO 2016 Summer School
Day 2 (afternoon)

Warm-up

Extended Formulation of the Stable Set Polytope of a Comparability Graph

$P = (V, \preccurlyeq)$ partially ordered set (poset)

$G = G(P)$ comparability graph of P (always perfect)

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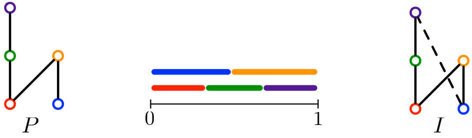
- $\forall v \in V$: (y_{v^-}, y_{v^+}) open interval $\subseteq (0,1)$
- $v \preccurlyeq w \implies y_{v^+} \leqslant y_{w^-}$



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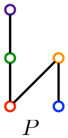
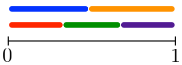
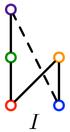






$$\text{STAB}(G(P)) = \Big\{ x \in \mathbb{R}^V \;|\exists \{(y_{v^-}, y_{v^+})\}_{v \in V} \text{ consistent with } P \\ \text{s.t. } x_v = y_{v^+} - y_{v^-} \quad \forall v \in V \Big\}$$

$f : A \times B \to \{0, 1\}$ Boolean function ($\equiv$ binary matrix)

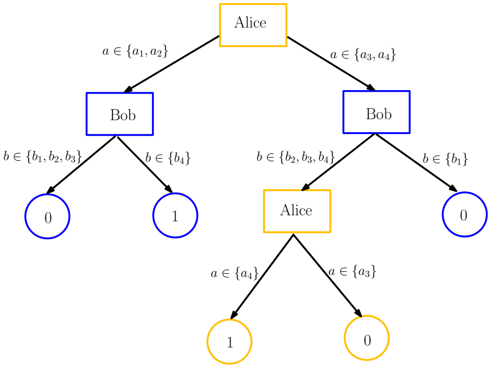
Two players:

- Alice knows $a \in A$
- Bob knows $b \in B$

want to **compute** $f(a, b)$ **by exchanging bits**

Goal: Minimize complexity := #bits exchanged

	b_1	b_2	b_3	b_4
a_1	0	0	0	1
a_2	0	0	0	1
a_3	0	0	0	0
a_4	0	1	1	1



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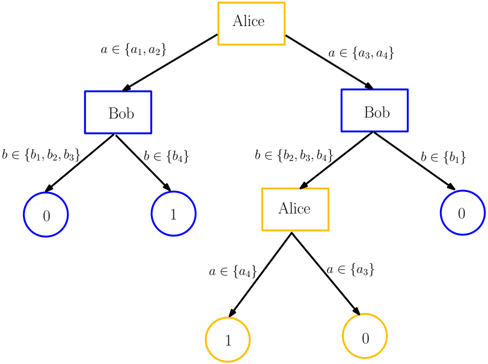

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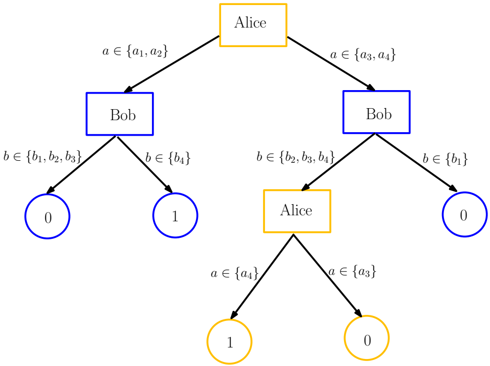

Observation

$\exists$ complexity c protocol for computing $f \implies \mathrm{rk}_+(f) \leqslant 2^c$

G graph with n vertices

$A = \{a \in \{0,1\}^n \mid a \text{ encodes a clique in } G\}$

$B = \{b \in \{0,1\}^n \mid b \text{ encodes a stable set in } G\}$

$$f(a,b) = \begin{cases} 1 & \text{if } a, b \text{ are disjoint} \\ 0 & \text{if } a, b \text{ intersect} \end{cases}$$

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Theorem (Yannakakis '91)

$\exists\ O(\log^2 n)$-*complexity protocol for* $f = f(G)$

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Corollary (Yannakakis '91)

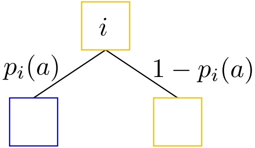
$\forall$ *perfect graphs* G: $\quad xc(STAB(G)) = 2^{O(\log^2 n)} = n^{O(\log n)}$

Extended Formulations as Communication Protocols

Randomized Protocols Computing a Function in Expectation

The main differences:

- Alice and Bob can use (private) random bits to make choices



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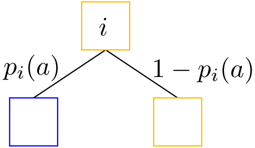

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Theorem (Faenza, F, Grappe & Tiwary '11)

If $c = c(f)$ is the minimum complexity of a randomized communication protocol with nonnegative outputs computing f in expectation, then

$$rk_+(f) = \Theta(2^c)$$

Write $M = TU$, where

$$\left|\begin{array}{l} T \in \mathbb{R}_+^{m\times r} \text{ row-stochastic (w.l.o.g.)} \\ U \in \mathbb{R}_+^{r\times n} \\ r \leqslant \mathsf{rk}_+(M) + 1 \end{array}\right.$$

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Protocol:

- Alice gets row index i, Bob gets column index j
- Alice picks random column index $k \in [r]$ w.p. T_{ik}, sends it to Bob
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Expected value on input (i, j): $\displaystyle\sum_{k=1}^{r} T_{ik} U_{kj} = M_{ij}$

Complexity: $\log \mathsf{rk}_+(M) + O(1)$

Three equivalent ways to look at EFs:

1. A linear system $Ex + Fy = g$, $y \geqslant 0$ with $y \in \mathbb{R}^r$

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1. A linear system $Ex + Fy = g$, $y \geqslant 0$ with $y \in \mathbb{R}^r$
2. A rank-r nonnegative factorization $S = TU$ of slack matrix S
3. A $\log r$-complexity randomized protocol computing S in expectation

Monotone Boolean Functions, Monotone Circuits

Definition (monotone Boolean function)

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is said to be *monotone* if

$$\forall x, y \in \{0,1\}^n \quad : \quad x \leqslant y \implies f(x) \leqslant f(y)$$

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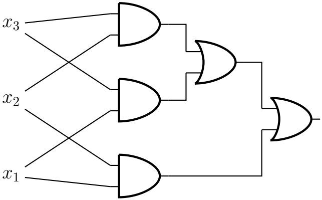
Examples:

- *Weighted threshold function*. Given $s_1, \ldots, s_n, D \in \mathbb{R}_+$:

  $$f(x) = 1 \iff \sum_{i=1}^{n} s_i x_i \geqslant D$$

- Any function f computed by a **monotone circuit** with n inputs



Definition (monotone Karchmer-Wigderson game)

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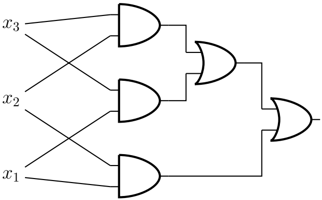
Proof. $(\leqslant)$



□

- $P = P(f) := \mathsf{conv}(f^{-1}(1))$
- $Q = Q(f) := \{x \mid \forall a \in f^{-1}(0) : \sum_{i:a_i=0} x_i \geqslant 1\}$

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Observation (Hrubeš, rediscovered by Göös '16)

$\exists$ depth-D monotone circuit computing $f \implies \exists$ complexity-D protocol for the corresponding monotone KW game $\implies \exists$ size-$n2^D$ EF of the pair (P, Q)

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Proof.

$$S_{ab} = \left(\sum_{i:a_i=0} b_i\right) - 1 = \sum_{i:a_i=0,\ i \neq i^*} b_i = (a_1, \ldots, a_{i^*-1}, a_{i^*+1}, \ldots, a_n) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_{i^*-1} \\ b_{i^*+1} \\ \vdots \\ b_n \end{pmatrix}$$

$\square$

Extended Formulations as Communication Protocols

Consequences of the KW/EF connection

Theorem (Raz-Wigderson '90)

Every monotone circuit that decides if a given n-vertex graph has a perfect matching, has depth $\Omega(n)$.

follows in a *black-box* way from

Theorem (Rothvoss'14)

The matching polytope $P_{\mathrm{match}}(K_n)$ has extension complexity $2^{\Omega(n)}$.

Theorem (matroids from the bases)

A collection $\mathcal{B}$ of subsets of a finite set E form the bases of a matroid if and only if

1. *$\mathcal{B}$ is nonempty*
2. *$\forall A, B \in \mathcal{B}$ and $a \in A \smallsetminus B$, $\exists b \in B \smallsetminus A$ s.t. $A \smallsetminus \{a\} \cup \{b\} \in \mathcal{B}$*

Definition (sparse paving matroid)

Let $\mathcal{N}$ be a collection of subsets of E s.t.

1. all the sets in $\mathcal{N}$ have the same size r
2. no two sets in $\mathcal{N}$ have Hamming distance $\leqslant 2$

Then $\mathcal{B} := \binom{E}{r} \setminus \mathcal{N}$ form the bases of a matroid, called a *sparse paving matroid*.

Exercise

Prove that there are at least $2^{2^{\Omega(n)}}$ sparse paving matroids.

Observation (Göös)

Facet-defining rank inequalities for sparse paving matroid of rank r are all of the form

$$\sum_{e \in F} x_e \leqslant r - 1 = |F| - 1 \iff \sum_{e \in F} (1 - x_e) \geqslant 1$$

where $F \in \mathcal{B}$ or

$$\sum_{e \in E} x_e \leqslant r$$

Get from this:

- every depth-D monotone circuit for the function $f : \{0,1\}^E \to \{0,1\}$ such that $f(x) = 1$ iff $\overline{x}$ is independent implies a size-$O(n2^D)$ extended formulation for the matroid polytope of a sparse paving matroid.
- f is a *slice* function, so the same applies to non-monotone circuits $\implies$ huge obstacle for lower bounds

Given *sizes* $s_1, \ldots, s_n \in \mathbb{R}_+$, *demand* $D \in \mathbb{R}_+$:

- Weighted threshold function: $f(x) = 1 \iff \sum_{i=1}^n s_i x_i \geqslant D$
- *Min-knapsack polytope*: $P := \mathrm{conv}\,\{x \in \{0,1\}^n \mid \sum_{i=1}^n s_i x_i \geqslant D\}$

Question

Is there a size-$n^{O(1)}$ $O(1)$-apx EF for min-knapsack?

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Theorem (Bazzi, F, Huang, Svensson '16)

$\forall \varepsilon > 0$, $\exists$ *size-*$(1/\varepsilon)^{O(1)} n^{O(\log n)}$ $(2+\varepsilon)$*-apx EF for min-knapsack*

Extended Formulations as Communication Protocols

Min-knapsack

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Starting point:

- Hrubeš-Göös connection
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Theorem (Carr et al. '06)

The (exponentially many) knapsack-cover inequalities provide a 2-apx of min-knapsack

Our approach:

- relax right-hand side to $\alpha D(a)$ where $\alpha = 2/(2+\varepsilon) \approx 1$
- coarse $+$ fine approximation
- try to find a "big" item i^* such that $a_{i^*} = 0$, $b_{i^*} = 1$
- use several threshold functions
- make things constructive (given an objective function!)

Extended Formulations as Communication Protocols

Making things constructive

Consider for p, q large:

- p inequalities: $A_i x \geqslant b_i$ $(i \in [p])$
- q solutions: s_j $(j \in [q])$ of this system

Data defines slack matrix $M \in \mathbb{R}_+^{p \times q}$ by $M_{ij} := A_i s_j - b_i$

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Question

How can we pick the equations? How can we use the extended formulation in an algorithm?

In case the non-negative factorization comes from a communication protocol, can write:

$$
\begin{aligned}
A_i x - b_i &= \sum_{\ell \text{ leaf}} q_i(\ell) \cdot y_\ell \quad \forall i \in [p] \\
y_\ell &\geqslant 0 \quad \forall \ell \text{ leaf}
\end{aligned}
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where $q_i(u) :=$ probability of reaching node u of the protocol tree on any input pair of the form $(i, *)$

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Lemma

Let $\Delta := \max\{-\log(q_i(\ell)) \mid i \in [p],\ \ell \text{ leaf},\ q_i(\ell) > 0\}$ and let h denote the height of the protocol tree. For any fixed $i \in [p]$, one can write down the corresponding equation in $O(2^h \Delta \log \Delta \log \log \Delta)$ time and $O(2^h \Delta)$ space.

This leads to a **cutting plane algorithm**. Given costs $c \in \mathbb{R}^n$

1. Initialize $I \subseteq [p]$
2. Solve
$$\begin{aligned} \min c^\intercal x \\ \text{s.t. } A_i x - b_i &= \textstyle\sum_{\ell \text{ leaf}} p_i(\ell) \cdot y_\ell \quad && \forall i \in I \\ y_\ell &\geqslant 0 && \forall \ell \text{ leaf} \end{aligned}$$
3. Get optimum solution x^*
4. Check if x^* violates any constraint $A_i x \geqslant b_i$ $(i \in [p])$
5. If yes, add i to I and repeat. If no, **STOP**.

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Assuming that the separation can be done in time $T(n)$, can solve the LP in time

$$O\left((n+r)(T(n) + r\Delta \log \Delta \log\log \Delta + n^{O(1)}(n+r)\Delta)\right)$$

Announcement

It is time for a break!

More Lower-Bounding techniques

Overview

Techniques to prove **lower bounds** on non-negative rank:

More Lower-Bounding techniques

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Techniques to prove **lower bounds** on non-negative rank:

1. rectangle covering lower bound

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More Lower-Bounding techniques

Overview

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More Lower-Bounding techniques

Hyperplane separation lower bound

Consider any matrix $S \in \mathbb{R}_{+}^{k \times \ell}$

Assume that **weights** $W \in \mathbb{R}^{k \times \ell}$ ($-$'s allowed) satisfy:

$$\langle W, X \rangle \leqslant \delta \cdot ||X||_{\infty} \quad \forall X \in \mathbb{R}_{+}^{k \times \ell} \text{ that is rank-1}$$

Then if $S = \sum_{i=1}^{r} X_i$ where $X_i \in \mathbb{R}_{+}^{k \times \ell}$ are rank-1:

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$$\Longrightarrow \boxed{r \geqslant \frac{\langle W, S \rangle}{\delta \cdot ||S||_\infty}}$$

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Choose optimum δ:

$$\begin{aligned}\delta &:= \max\{\langle W, X\rangle \mid X \text{ rank-1}, X \in [0,1]^{k\times\ell}\\ &= \max\{\sum_{i,j} W_{ij} x_i y_j \mid x \in [0,1]^k, y \in [0,1]^\ell\}\\ &= \max\{\sum_{i,j} W_{ij} x_i y_j \mid x \in \{0,1\}^k, y \in \{0,1\}^\ell\}\\ &= \max\{\langle W, R\rangle \mid R \text{ rectangle}\}\end{aligned}$$

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The **hyperplane separation bound** was used to prove:

- UDISJ(n) still has super-polynomial non-negative rank even if add same number ρ everywhere, as long as $\rho \leqslant n^{1/2-\varepsilon}$ (Braun, F, Pokutta & Steurer)
- $\mathsf{xc}(P_{\mathrm{match}}(K_n)) = 2^{\Theta(n)}$ (Rothvoss'14)

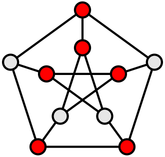
More Lower Bounding Techniques

A Sample Result

$G = (V, E)$ n-vertex graph

Definition

$U \subseteq V$ is a *vertex cover* if every edge has at least one endpoint in U



Basic IP for VC:

$$\begin{array}{lll} \min & \sum_{i \in V} c_i x_i & \\ \text{s.t.} & x_i + x_j \geqslant 1 & \forall ij \in E \\ & x_i \in \{0,1\} & \forall i \in V \end{array}$$

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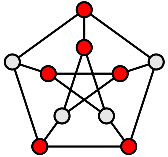

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1. **Explicit constraints**: e.g., if G contains odd cycle C then

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2. **Implicit constraints** (lift-and-project): e.g., Sherali-Adams hierarchy strengthens current relaxation $Ax \geqslant b$ by
   - Multiplying each inequality of $Ax \geqslant b$ by $\prod_{i \in I} x_i \prod_{i \in J} (1 - x_i)$ for each pair (I, J) of disjoint subsets of $[n]$ with $|I| + |J| = r$
   - Expand using $x_i^2 = x_i$
   - Replace each monomial $\prod_{i \in S} x_i$ by a variable y_S where $S \subseteq [n]$, $|S| \leqslant r + 1$
   - Add the constraint $y_\emptyset = 1$
   - Project by letting $x_i := y_{\{i\}}$

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1. *Adding all inequalities with support at most* ϵn *leaves a gap* $\geqslant 2 - \epsilon$
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Many follow-up works on **lift-and-project hierarchies**:

- STT'07: $o(n)$ rounds of LS do not help
- GMPT'07: $O(\sqrt{\log n / \log\log n})$ rounds of LS$_+$ do not help
- CMM'09: n^δ rounds of SA do not help (for small $\delta > 0$)

We generalize the results on hierarchies and prove that

every polynomial size LP for VC **has integrality gap** 2

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$\mathrm{VC}(G)$ for **fixed** G

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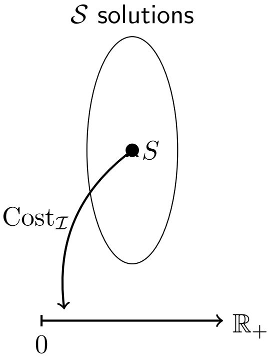

Setting up an LP relaxation:

1. pick d, system $Ax \geqslant b$ in $\mathbb{R}^d$
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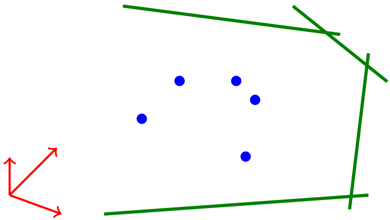

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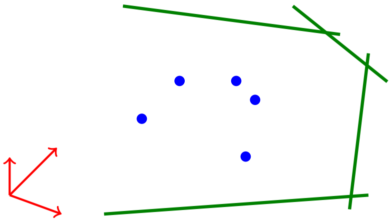

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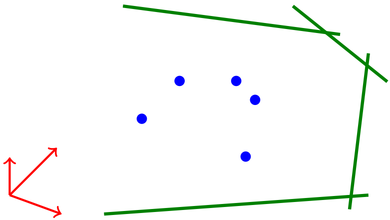

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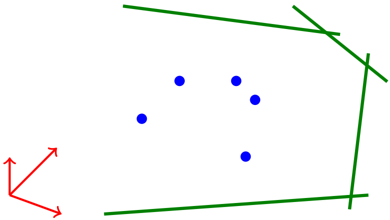

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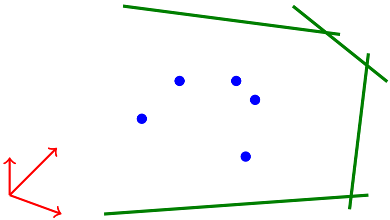

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Similar for **max** problems, e.g., $\mathrm{IS}(G)$ = max independent set problem on G

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$\forall \epsilon > 0$, $\forall n_0$, $\exists$ *graph* G *with* $n \geqslant n_0$ *vertices such that*

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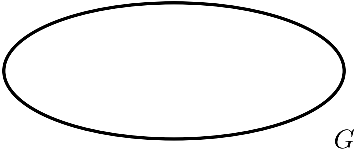

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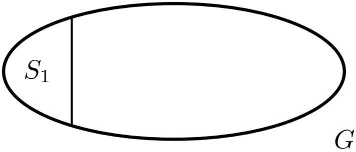

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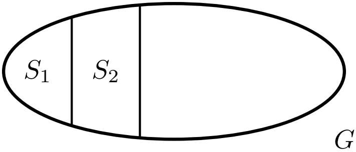

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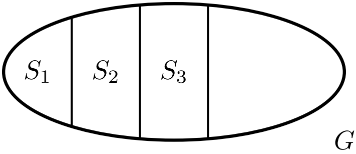

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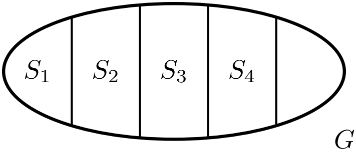
\text{s.t.} & 0 \leqslant x_i \leqslant y_j \leqslant 1 \quad \forall i \in S_j \\
& \sum_j y_j \leqslant \lfloor \sqrt{2n} \rfloor
\end{array}
$$



Remark

We just care about the *existence or non-existence* of LP relaxations:

- **do not bound** the time it takes to write down the LP (important)
- **do not bound** the coefficients! (not so important)

From FJ'14:

$\exists$ $O(n)$**-size** $O(\sqrt{n})$**-apx LP relaxation for IS**(G) **for any graph** G

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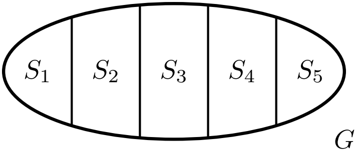

Constraint satisfaction problems

Parameters of the problem:

- number of variables n
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- **Solutions:** assignments $x \in [R]^n$ of values to the vars $x_1, \ldots, x_n$
- **Instances:** sets $\mathcal{I} := \{P_1, \ldots, P_m\}$ of Boolean predicates $P_i : [R]^n \to \{0,1\} : x \mapsto P_i(x)$, each depending on k variables
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Example (Max-Cut(n))

- arity $k = 2$
- domain size $R = 2$ and domain $\{0, 1\}$
- predicates of the form $P(x) = x_i \oplus x_j$

Recent results

BPZ'14 prove super-polynomial lower bounds for

- $(1.5 - \epsilon)$-apx LP relaxations of $\mathrm{VC}(G)$
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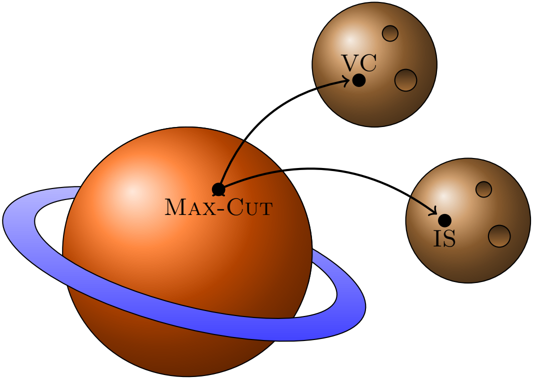
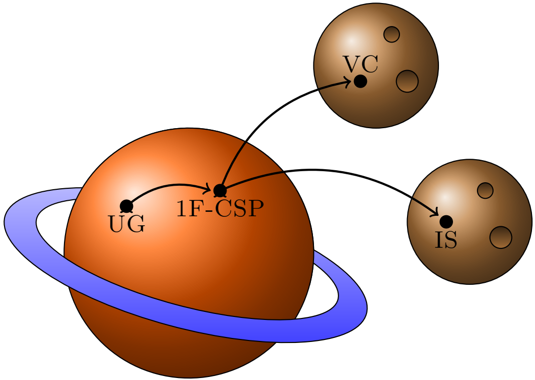


Our proof



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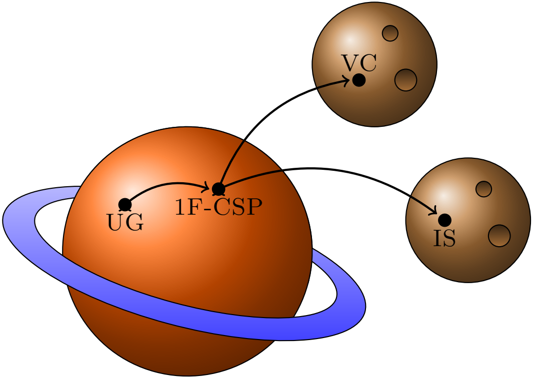


We use:

- a different starting point: UG (unique games)
- CMM'09: UG fools $O(n^{\delta})$ rounds of Sherali-Adams
- a different leaving point: $\mathrm{1F\text{-}CSP}$ (one free bit CSPs)
- CLRS'13: Sherali-Adams is "optimal" among all LPs for CSPs

One free bit CSPs

Definition (1F-CSP(n, k))

- **Solutions:** assignments $x \in \{0,1\}^n$
- **Instances:** sets $\mathcal{I} := \{P_1, \ldots, P_m\}$ of Boolean predicates $P_i : \{0,1\}^n \to \{0,1\} : x \mapsto P_i(x)$, each depending on k of the n variables, and each having exactly **two** satisfying assignment on these k bits
- **Objective function:** $\mathsf{Val}_{\mathcal{I}}(x) := \frac{1}{m} \sum_{i=1}^{m} P_i(x)$

Rem:

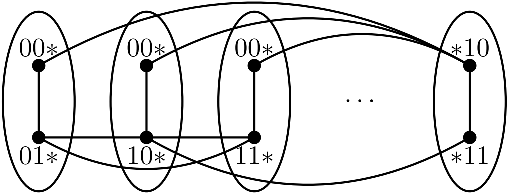
- $\textsc{Max-Cut}(n)$ is a special case for $k = 2$

FGLSS graphs

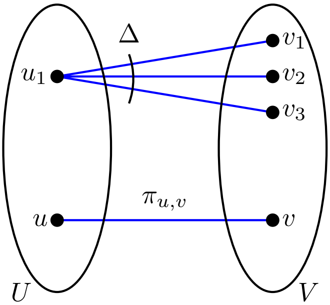
after Feige, Goldwasser, Lovász, Safra and Szegedy '96

$G^*(n,k)$ is the graph with:

- two vertices per one free bit predicate $P : \{0,1\}^S \to \{0,1\}$ where $S \subseteq [n]$, $|S| = k$
- one vertex for each partial assignment $\alpha \in \{0,1\}^S$ satisfying P
- edges between conflicting partial assignments



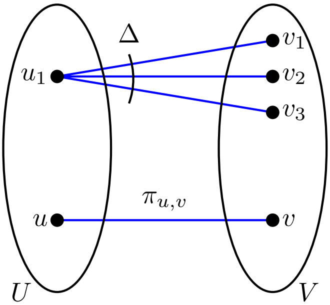
Unique games



Given:

- $G = (U, V, E)$ bipartite graph, Δ-regular
- domain $[R]$
- permutation $\pi_{u,v} : [R] \to [R]$ for each edge $uv \in E$

Unique games



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UG (unique games) is the $\textsc{Max-CSP}$ with

- arity $k = 2$
- one predicate per edge $uv \in E$ that is true iff $\pi_{u,v}(x_u) = x_v$

Unique games conjecture

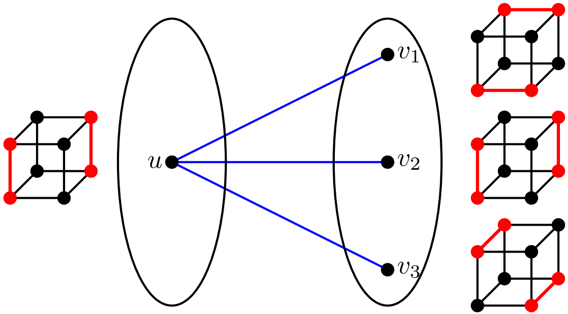
Conjecture (Khot'02)

For every $\epsilon, \delta > 0$, there exists a sufficiently large domain size $R = R(\epsilon, \delta)$ such that the following promise problem is NP-hard. Given a UG instance $(G, [R], (\pi_{uv})_{uv \in E})$, distinguish between the following two cases:

1. $\exists$ assignment satisfying at least a $(1 - \epsilon)$-fraction of the edges;
2. $\nexists$ assignment satisfying more than a δ-fraction of the edges.

The Bansal-Khot PCP and 1F-CSP's

For every bipartite, Δ-regular UG instance $(G, [R], (\pi_{uv})_{uv \in E})$ we obtain a $1\mathrm{F}\text{-}\mathrm{CSP}$ instance by reinterpreting the PCP due to BK'09



Reductions

Consider:

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A *reduction from* Π_1 *to* Π_2 is defined by two maps

1. $\mathfrak{I}_1 \to \mathfrak{I}_2 : \mathcal{I}_1 \mapsto \mathcal{I}_2$
2. $\mathcal{S}_1 \to \mathcal{S}_2 : S_1 \mapsto S_2$

subject to:

$$\mathsf{Val}_{\mathcal{I}_1}(S_1) = \underbrace{\mu(\mathcal{I}_1)}_{\text{affine shift}} - \underbrace{\alpha(\mathcal{I}_1)}_{\geqslant 0} \cdot \mathrm{Cost}_{\mathcal{I}_2}(S_2) \qquad \forall \mathcal{I}_1 \in \mathfrak{I}_1, S_1 \in \mathcal{S}_1$$

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The reduction is *exact* if additionally

$$\mathsf{OPT}(\mathcal{I}_1) = \mu(\mathcal{I}_1) - \alpha(\mathcal{I}_1) \cdot \mathsf{OPT}(\mathcal{I}_2) \qquad \forall \mathcal{I}_1 \in \mathfrak{I}_1$$

Definition

For $c \geqslant s > 0$, an LP relaxation is a (c,s)-apx for a **max** problem $\Pi = (\mathcal{S}, \mathfrak{I})$ if

$$\forall \mathcal{I} \in \mathfrak{I} : \qquad \mathsf{OPT}(\mathcal{I}) \leqslant s \quad \Longrightarrow \quad \mathrm{LP}(\mathcal{I}) \leqslant c$$

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Rem:

- c is the *completeness*
- s is the *soundness*

Theorem (BPZ'14)

Let Π_1 be a max problem and let Π_2 be a min problem. Suppose $\exists$ an **exact** *reduction from Π_1 to Π_2 with constant affine shift $\mu(\mathcal{I}_1) = \mu$. Then, every ρ_2-apx LP relaxation for Π_2 gives a (c_1, s_1)-apx LP relaxation for Π_1, of the same size, where*

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Proof. Consider ρ_2-apx LP relaxation $Ax \geq b$ for Π_2 with realizations

- x^{S_2} for $S_2 \in \mathcal{S}_2$
- $f_{\mathcal{I}_2} : \mathbb{R}^d \to \mathbb{R}$ for $\mathcal{I}_2 \in \mathfrak{I}_2$

Use same LP $Ax \geq b$ for Π_1 with realizations

- $x^{S_1} := x^{S_2}$ with $S_1 \mapsto S_2$
- $f_{\mathcal{I}_1}(x) := \mu - \alpha(\mathcal{I}_1) f_{\mathcal{I}_2}(x)$ with $\mathcal{I}_1 \mapsto \mathcal{I}_2$

We claim this is a (c_1, s_1)-apx for Π_1

Proof (continued). Using:

$$\mathsf{OPT}(\mathcal{I}_2) \leqslant \rho_2 \, \mathrm{LP}(\mathcal{I}_2) \qquad \text{and} \qquad \mathsf{OPT}(\mathcal{I}_1) = \mu - \alpha(\mathcal{I}_1)\, \mathsf{OPT}(\mathcal{I}_2)$$

we have:

$$\mathrm{LP}(\mathcal{I}_1)$$

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