The Uncapacitated Asymmetric Traveling Salesman Problem with Multiple Stacks

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JFRO - March 2012

Joint work with Sylvie Borne and Roland Grappe

Agenda

1. General results
   - Introduction
   - Polyhedral results
2. Focus on two stacks
   - Formulation
   - Valid inequalities

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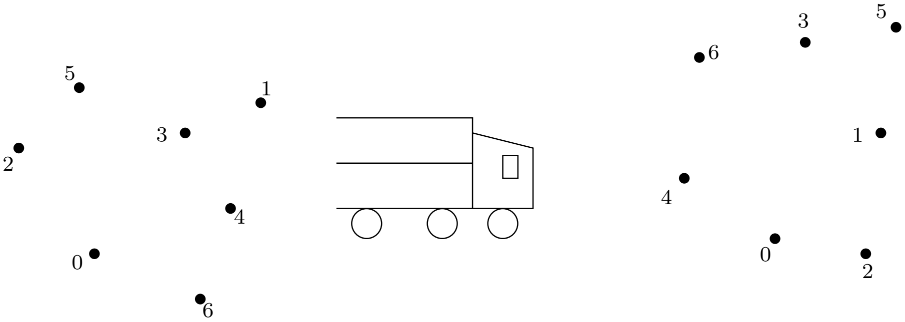
Example



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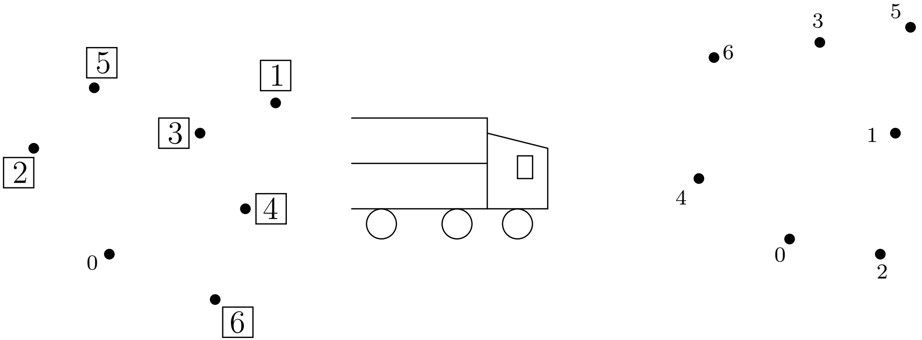

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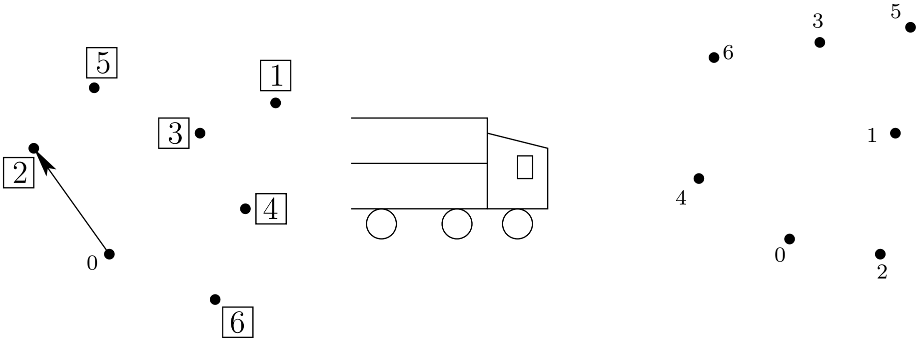

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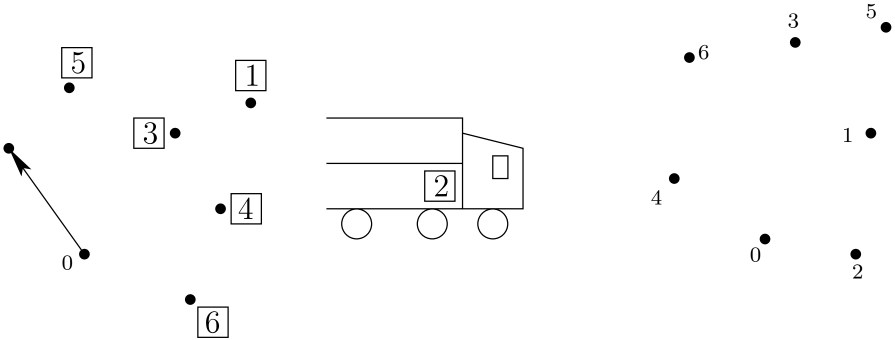

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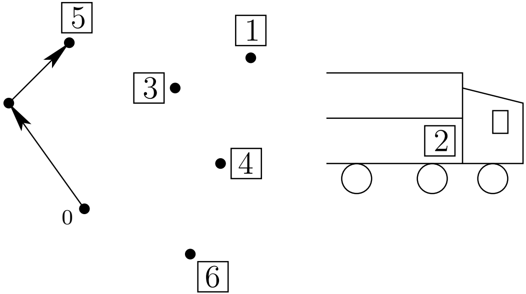

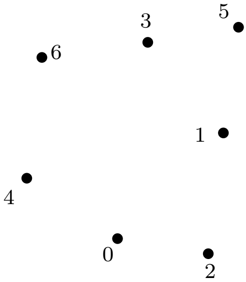

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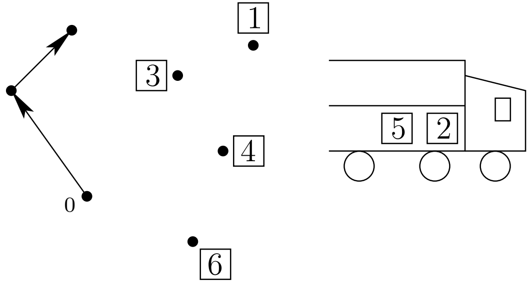

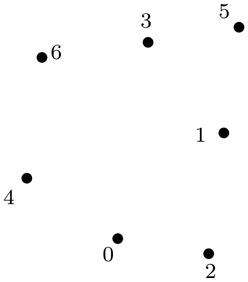

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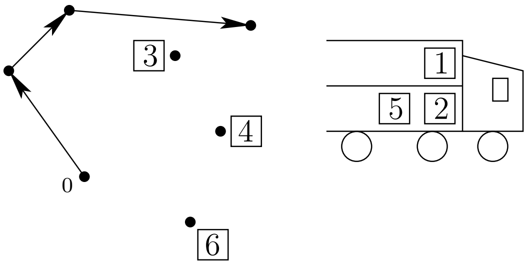

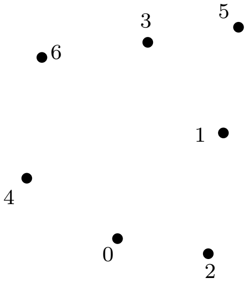

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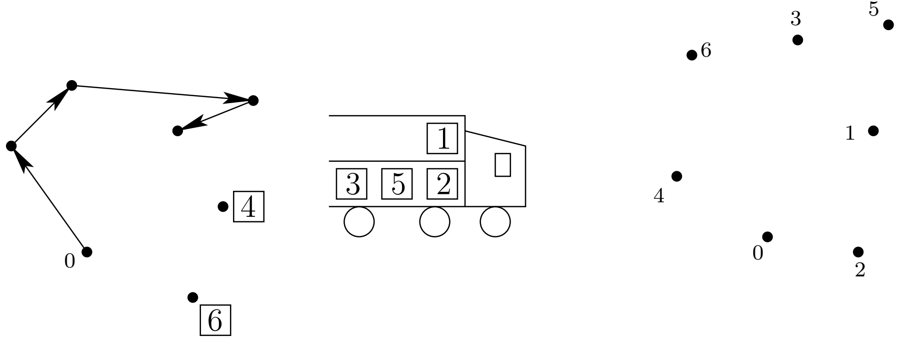

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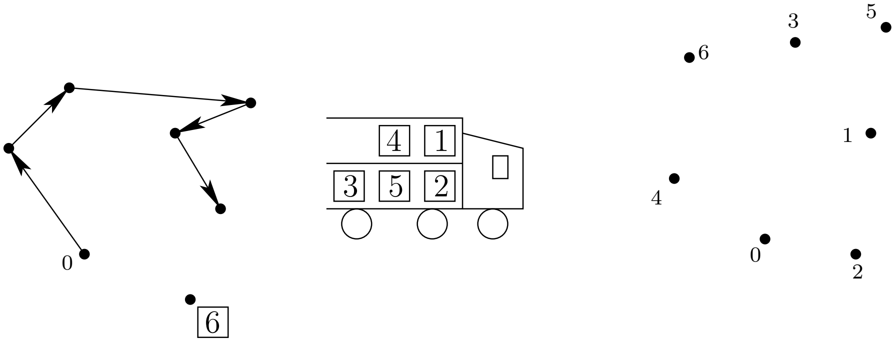

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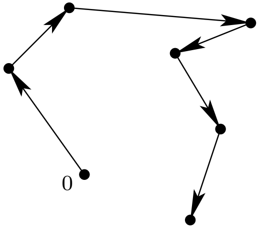

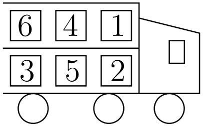

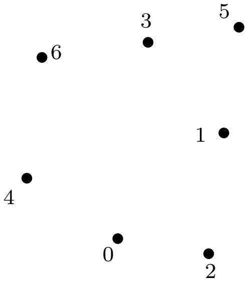

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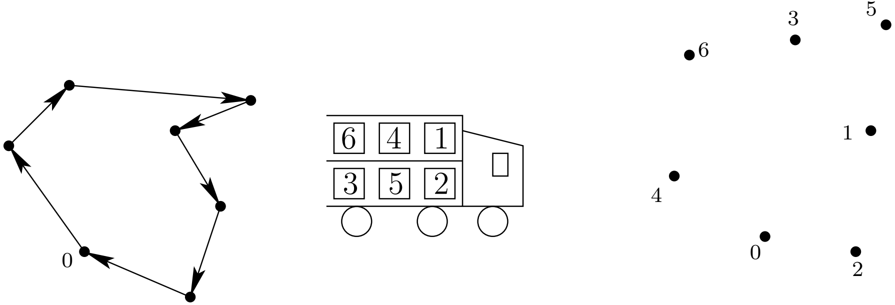

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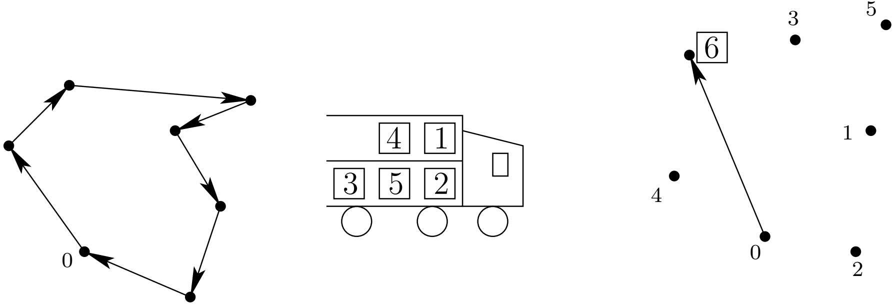

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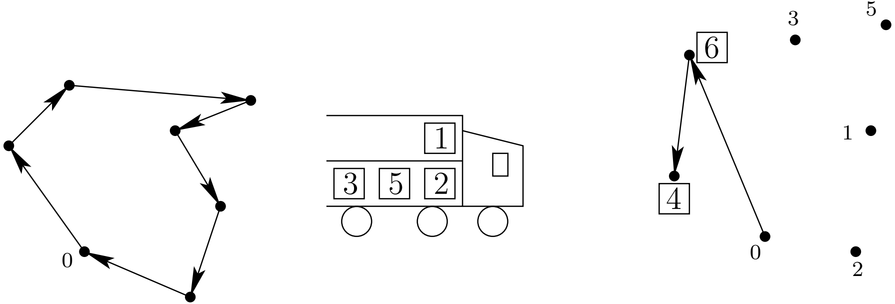

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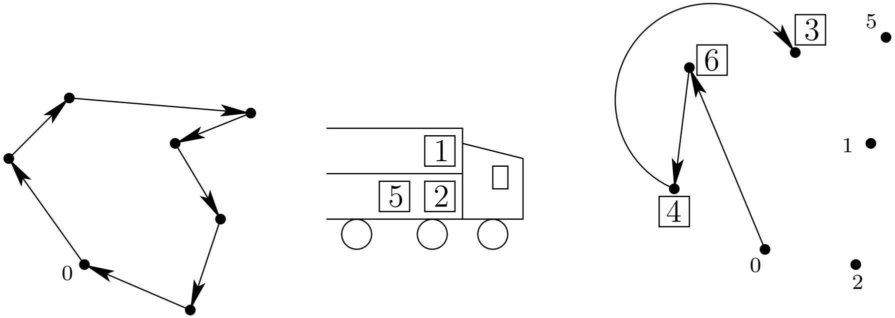

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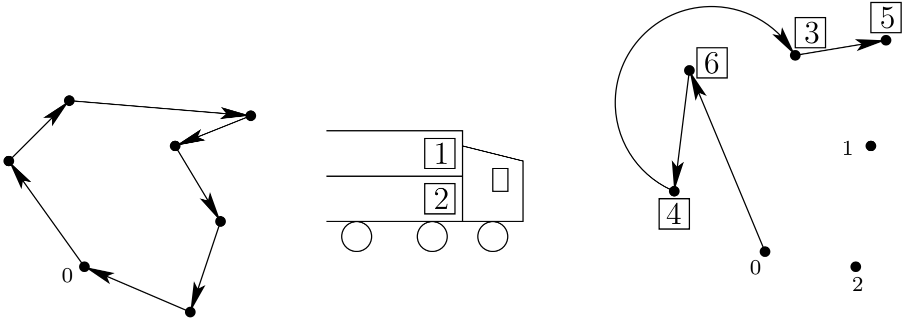

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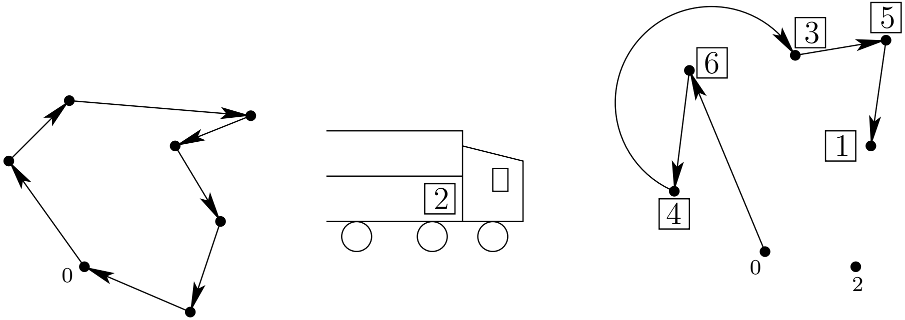

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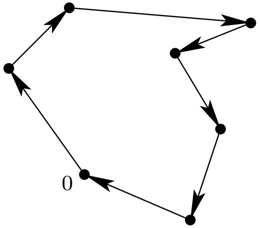

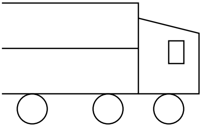

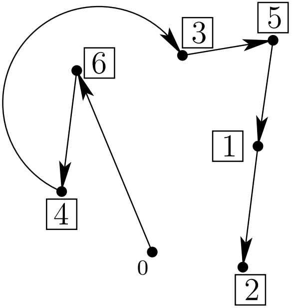

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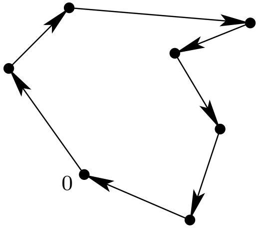

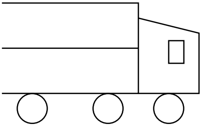

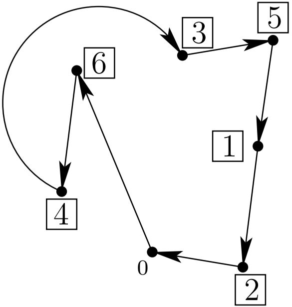

Definition

Input

- Complete digraph $D = (V, A)$ with $V = \{0, \ldots, n-1\}$
- Arc costs vectors c^1 and c^2
- k: number of uncapacitated stacks

Problem

Find two hamiltonian circuits C^1 and C^2 s.t.

- There exists a loading plan into k stacks
- $c^1(C^1) + c^2(C^2)$ is minimum

Remark

- $k = 1$: reduces to compute one ATSP
- $k \geq n - 1$: reduces to compute two ATSPs

Consistency

C^1 and C^2 *k-consistent* $\Leftrightarrow$ there exists a loading plan into k stacks

Proposition (Bonomo et al., Toulouse et al., Casazza et al.)

C^1 and C^2 are k-consistent iff no $k+1$ vertices of $V \setminus \{0\}$ form an increasing sequence for both circuits.

Proof: $(\Rightarrow)$

easy.

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Proof: $(\Leftarrow)$

- $i \prec j$ if i precedes j in C^1 and C^2 for $i \neq j \in V \setminus \{0\}$.
- $G = (V \setminus \{0\}, E)$, $E = \{ij : i \prec j \text{ or } j \prec i\}$.
- Increasing sequence $\Leftrightarrow$ clique in G.
- Size of a clique in G is at most k.
- G is perfect $\Rightarrow \chi(G) \leq k$.
- Each color (stable set) corresponds to a stack.

Consistency

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C^1 and C^2 are k-consistent iff no $k+1$ vertices of $V \setminus \{0\}$ form an increasing sequence for both circuits.

Remark

Checking consistency can be done in polynomial time.

State of the art

Consistency with stack capacity (Bonomo et al.)

- NP-complete in general
- Polynomial for fixed k

From stacks to ATSPs (Toulouse et al., Casazza et al.)

- NP-complete in general
- Polynomial for fixed k (dynamic programming)

Approximation (Toulouse)

- Uncapacitated: 1/2 approx for max STSP2S
- Capacitated: 1/2 - ϵ differential approx

State of the art

Local searches (Petersen et al., Felipe et al., Côté et al.)

- VNS
- LNS

Results up to $n = 67$ (3 stacks)

Exact Algorithms

- Different ILP (Petersen et al., Alba et al.): B&B, B&C
- k best TSPs (Lusby et al.)
- B&B for 2 stacks (Carrabs et al.)

Results up to $n = 14$ (2 stacks)

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Polyhedral results

Lemma

C hamiltonian circuit. $\mathcal{S}$ set of circuits k-consistent with C.
If $k \geq 2$, then $dim(conv(\mathcal{S})) = dim(ATSP_n)$.

$\overline{Id}_n = 0, n-1, n-2, \ldots, 1$

Proof:

- W.l.o.g., $C = \overline{Id}_n$. Set $d_n = dim(ATSP_n)$.
- $dim(conv(\mathcal{S})) \leq d_n$.
- Since $\mathcal{P}_{2,n} \subseteq \mathcal{P}_{k,n}$, find $d_n + 1$ affinely independant circuits 2-consistent with $\overline{Id}_n$.

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Proof: (Induction)

- True for $n \leq 4$.
- Hypothesis: $C_1, \ldots, C_{d_n+1}$ a.i. 2-consistent with $\overline{Id}_n$.
- (C_i, n) 2-consistent with $\overline{Id}_{n+1}$ for $i = 1, \ldots, d_n + 1$.

$\Rightarrow$ $d_n + 1$ a.i. circuits 2-consistent with $\overline{Id}_{n+1}$.

Remark: Each of them contains the arc $(n, 0)$.

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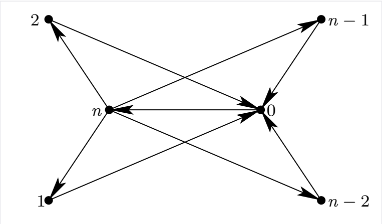
Adding new a.i. circuits:



Unused arcs

Polyhedral results

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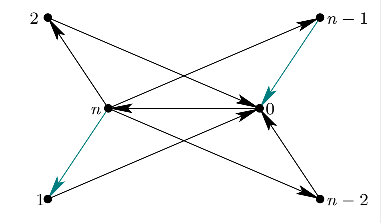
Adding new a.i. circuits:

- $0, 2, 3, \ldots, n-2, n, 1, n-1$



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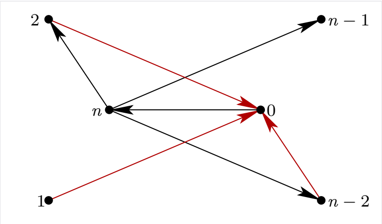
Adding new a.i. circuits:

- $0, 2, 3, \ldots, n-2, n, 1, n-1$
- $0, i+1, i+2, \ldots, n, 1, 2, \ldots, i,$ for $i = 1, 2, \ldots, n-2$



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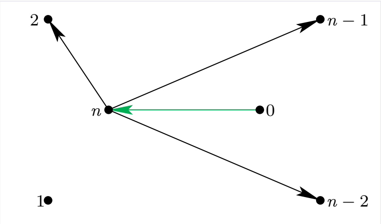

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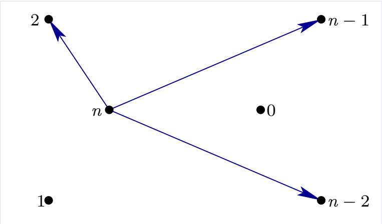

Unused arcs

Polyhedral results

Theorem (Borne, Grappe, L.)

Given $k \geq 2$, $dim(\mathcal{P}_{k,n}) = 2d_n$.

Proof:

- $C_1, \dots, C_{d_n+1}$ a.i. hamiltonian circuits.
- $H_1, \dots, H_{d_n+1}$ a.i. circuits 2-consistent with C_{d_n+1}.

$$\begin{pmatrix} C_1 \\ \overline{C}_1 \end{pmatrix} \cdots \quad \begin{pmatrix} C_{d_n} \\ \overline{C}_{d_n} \end{pmatrix} \quad \begin{pmatrix} C_{d_n+1} \\ H_1 \end{pmatrix} \cdots \quad \begin{pmatrix} C_{d_n+1} \\ H_{d_n+1} \end{pmatrix}$$

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$$\begin{cases} \lambda_1 \begin{pmatrix} C_1 \\ \\ \overline{C}_1 \end{pmatrix} \cdots + \lambda_{d_n} \begin{pmatrix} C_{d_n} \\ \\ \overline{C}_{d_n} \end{pmatrix} + \mu_1 \begin{pmatrix} C_{d_n+1} \\ \\ H_1 \end{pmatrix} \cdots + \mu_{d_n+1} \begin{pmatrix} C_{d_n+1} \\ \\ H_{d_n+1} \end{pmatrix} = 0 \\ \displaystyle\sum_{i=1}^{d_n} \lambda_i + \sum_{i=1}^{d_n+1} \mu_i = 0 \end{cases}$$

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Polyhedral results

Theorem (Borne, Grappe, L.)

Given $k \geq 2$, every facet of $ATSP_n$ defines a facet of $\mathcal{P}_{k,n}$.

Proof:

- $C_1, \dots, C_{d_n}$ a.i. hamiltonian circuits of a facet F of $ATSP_n$.
- $H_1, \dots, H_{d_n+1}$ a.i. circuits 2-consistent with C_{d_n}.

$$\begin{pmatrix} C_1 \\ \\ \overline{C}_1 \end{pmatrix} \cdots \begin{pmatrix} C_{d_n-1} \\ \\ \overline{C}_{d_n-1} \end{pmatrix} \begin{pmatrix} C_{d_n} \\ \\ H_1 \end{pmatrix} \cdots \begin{pmatrix} C_{d_n} \\ \\ H_{d_n+1} \end{pmatrix} \text{ a.i. and belong to } F'.$$

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Formulation

Variables

$$x_{ij}^h = \begin{cases} 1 & \text{if } (i,j) \text{ belongs to } C^h, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\ h = 1,2, \forall\ (i,j) \in A.$$

Linear ATSP Constraints

$$\sum_{j \in V \setminus \{i\}} x_{ij}^h = 1 \quad \forall\ i \in V, \forall\ h = 1,2, \tag{1}$$

$$\sum_{i \in V \setminus \{j\}} x_{ij}^h = 1 \quad \forall\ j \in V, \forall\ h = 1,2, \tag{2}$$

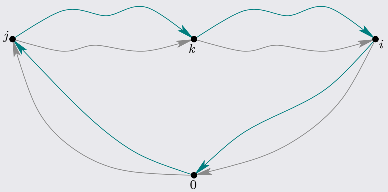
$$\sum_{a \in \delta^+(W)} x_a^h \geq 1 \quad \forall\ \emptyset \subset W \subset V, \forall\ h = 1,2, \tag{3}$$

$$0 \leq x_a^h \leq 1 \quad \forall\ a \in A, \forall\ h = 1,2. \tag{4}$$

Formulation

$$C^1 \text{ and } C^2 \text{ 2-consistent} \Leftrightarrow \nexists\ i,j,k \text{ with } i \prec j \prec k$$

Forbidden structure



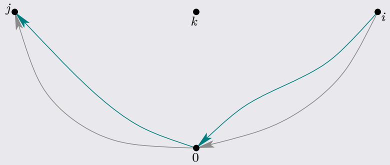
Consistency constraints

$$\sum_{h=1,2} \sum_{a \in P^h} x_a^h \leq |P^1| + |P^2| - 1 \quad \begin{array}{l} \forall\ i \neq j \neq k \neq i \in V \setminus \{0\}, \\ \forall\ P^1, P^2 \in \mathcal{P}_{ij}^0(D \setminus \{k\}). \end{array} \qquad (5)$$

Formulation

$$C^1 \text{ and } C^2 \text{ 2-consistent} \Leftrightarrow \nexists\, i, j, k \text{ with } i \prec j \prec k$$

Forbidden structure



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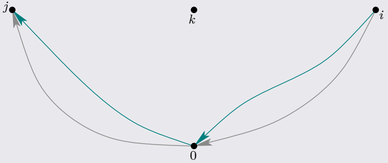

Consistency constraints

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Theorem (Borne, Grappe, L.)

$\mathcal{P}_{2,n} = conv(\{(x^1, x^2) \in \{0,1\}^A \times \{0,1\}^A : (x^1, x^2) \text{ satisfies (1)-(5)}\})$

Linear relaxation

Theorem (Borne, Grappe, L.)

The linear relaxation is polynomial-time solvable.

Proof:

- Constraints (1),(2),(4): polynomial number
- Constraints (3): polynomial number of minimum cuts

Linear relaxation

Theorem (Borne, Grappe, L.)

The linear relaxation is polynomial-time solvable.

Proof:

Consistency constraints ($\tilde{x} = 1 - \bar{x}$)

$$\sum_{h=1,2} \sum_{a \in P^h} \tilde{x}_a^h \geq 1 \qquad \begin{array}{l} \forall\ i \neq j \neq k \neq i \in V \setminus \{0\}, \\ \forall\ P^1, P^2 \in \mathcal{P}_{ij}^0(D \setminus \{k\}). \end{array}$$

- For fixed i, j, k:
  Find a minimum $i0j$-path P^h of $D \setminus \{k\}$ for $h = 1, 2$.

Linear relaxation

Theorem (Borne, Grappe, L.)

The linear relaxation is polynomial-time solvable.

Proof:

- For fixed i, j, k and fixed h:
  Compute in $D \setminus \{k\}$:
  - Q_1: minimum $i0$-path
  - Q_2: minimum $0j$-path

  If $\tilde{x}^h((Q_1, Q_2)) < 1$, then (Q_1, Q_2) is a $i0j$-path.

$\Rightarrow$ Computation of 2 minimum paths.

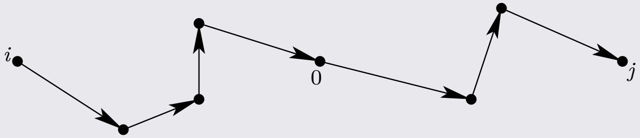
$\Rightarrow$ Polynomial separation for consistency inequalities (5).

Agenda

1. General results
   - Introduction
   - Polyhedral results
2. Focus on two stacks
   - Formulation
   - Valid inequalities

Strenghtening the consistency constraints (Alba et al.)

Adding arcs in each path P^h



Strenghtening the consistency constraints (Alba et al.)

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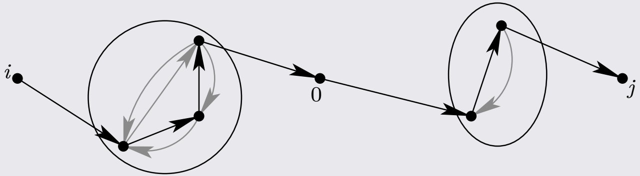

Strenghtening the consistency constraints (Alba et al.)

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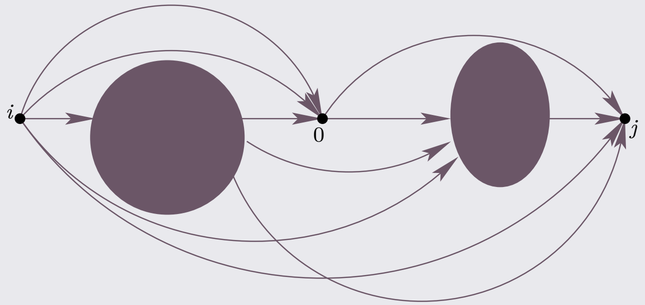
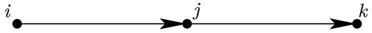


New inequalities

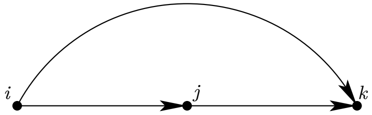


P_3-subgraph inequalities

$$x^1(B) + x^2(B) \leq 3$$

B: Set of arcs in the figure.

New inequalities



P_3-subgraph inequalities

$$x^1(B) + x^2(B) \leq 3$$

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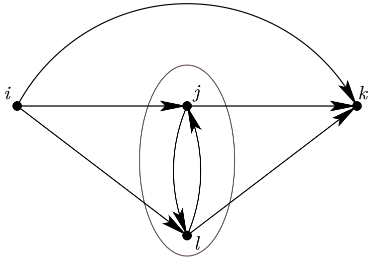
New inequalities

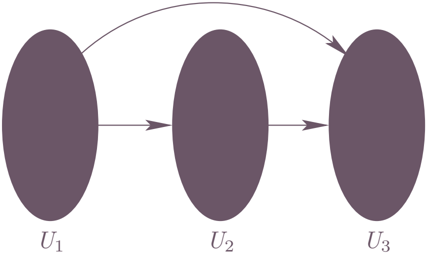


P_3-subgraph inequalities

$$x^1(B) + x^2(B) \leq 5$$

B: Set of arcs in the figure.

New inequalities

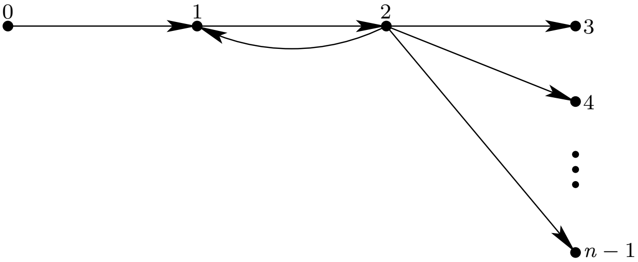


P_3-subgraph inequalities

$$x^1(B) + x^2(B) \leq 2(|U_1| + |U_2| + |U_3| - 1) - 1$$

B: Set of arcs in the figure.

New inequalities

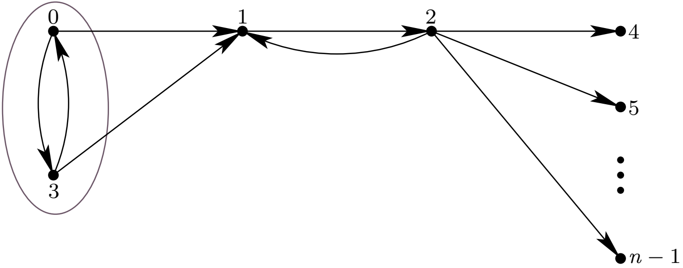


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 4$$

B: Set of arcs in the figure.

New inequalities

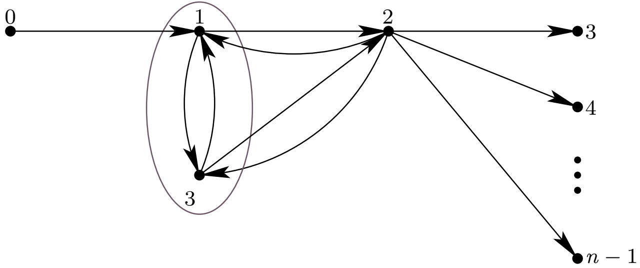


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 6$$

B: Set of arcs in the figure.

New inequalities

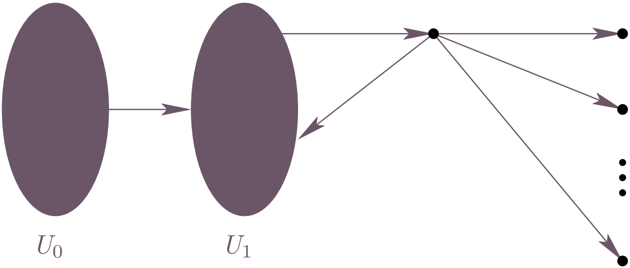


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 6$$

B: Set of arcs in the figure.

New inequalities

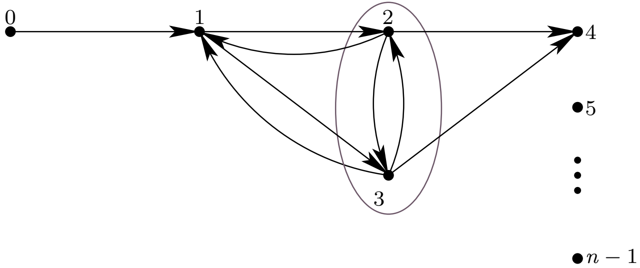


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 2(|U_0| + |U_1| + 1) - 2$$

B: Set of arcs in the figure.

New inequalities

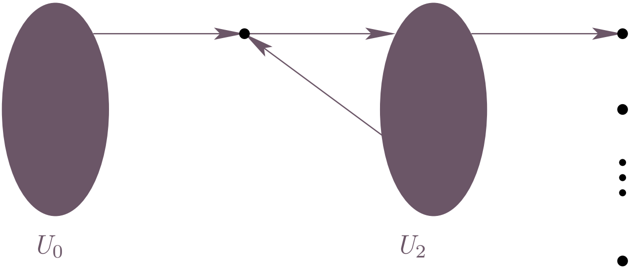


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 6$$

B: Set of arcs in the figure.

New inequalities

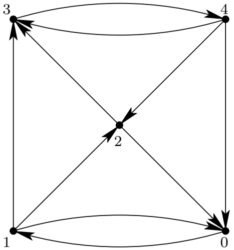


P_4-subgraph inequalities

$$x^1(B) + x^2(B) \leq 2(|U_0| + |U_2| + 1) - 2$$

B: Set of arcs in the figure.

New inequalities



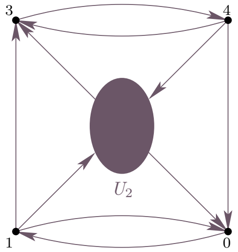
- B: Set of arcs in the figure
  $U = \{0, 1, 2, 3, 4\}$

If $C^h \cap B$ is a path covering U:
$1 \prec_{C^h} 3 \prec_{C^h} 4$

W_5-subgraph inequalities

$$x^1(B) + x^2(B) \leq 7$$

New inequalities



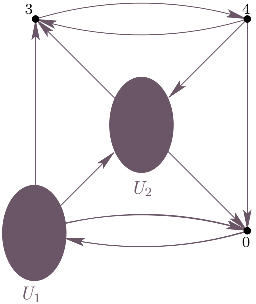
- B: Set of arcs in the figure
  $U = \{0, 1, 3, 4\} \cup U_2$

If $C^h \cap B$ is a path covering U:
$1 \prec_{C^h} 3 \prec_{C^h} 4$

W_5-subgraph inequalities

$$x^1(B) + x^2(B) \leq 2(|U_2| + 3) - 1$$

New inequalities



- B: Set of arcs in the figure
  $U = \{0, 3, 4\} \cup U_1 \cup U_2$

If $C^h \cap B$ is a path covering U:
either $U_1 \prec_{C^h} 3 \prec_{C^h} 4$
or there exists $v_1 \in U_1$ s.t. $v_1 \prec_{C^h} 3 \prec_{C^h} 4 \prec_{C^h} V \setminus U$

W_5-subgraph inequalities

$$x^1(B) + x^2(B) \leq 2(|U_1| + |U_2| + 2) - 1$$

Conclusion & Perspectives

Conclusion

- Polyhedral results
- Formulation for 2 stacks
- Valid inequalities

Perspectives

- Separation algorithms
- Taking into account stack capacities
- Adding extra variables (?)

Thank you for your attention