Carpool fairness in social networks

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The carpool problem
The carpool problem

Suppose that $n$ people, tired of spending their time and money in gasoline lines, decide to form a carpool... We want a scheduling algorithm that will be perceived as fair by all the members – [Fagin-Williams 1982]

- every day a subset of $n$ people share a ride
- who should drive?
Objective: everyone drives his fair share

- let $\sigma_1, \ldots, \sigma_T$ be the subsets (requests)
- suppose that driver $i$ has driven $m_{i,T}$ times
- his fair share is $f_{i,T} = \sum_{t: i \in \sigma_t} 1/|\sigma_t|$ 
- and his unfairness $|m_{i,T} - f_{i,T}|$
- **objective:** minimize $\max_i |m_{i,T} - f_{i,T}|$
- **not in this talk:** one-sided unfairness $(m_{i,T} - f_{i,T})$
Offline vs online

- We consider online algorithms
- An **online algorithm** selects the drivers based only on the past
- The **adversary** selects the sequence of requests, including its length, in advance
- The adversary knows the algorithm, but not the outcome of its random choices (**oblivious adversary**)
Offline algorithms

- There exists an offline algorithm, which
  - can maintain unfairness at most 1
  - for every driver
  - at every time step
for simplicity, we consider requests of size 2
Groups of size 2 suffice

• Every day only two people share a ride
• This is without significant loss of generality
  ▪ the general carpool problem reduces to groups of size 2
  ▪ the reduction changes the unfairness only by a factor of 2
• Similar problem: online edge orientation of a given graph to minimize (absolute) difference between outdegree and indegree
Basic online algorithms

**Random**
A random member drives. Unbounded unfairness (proportional to $O(\sqrt{T})$, where $T =$ number of requests)

**Local Greedy**
In every pair of drivers they drive alternatively. Randomize the first time. Unfairness $O(\sqrt{n \log n})$

**Global Greedy**
The driver with minimum unfairness drives; in case of a tie, select randomly
- **Conjecture:** Global Greedy has randomized unfairness $\Theta(\log n)$
History

- Fagin and Williams 1982: introduced the problem and the Global Greedy algorithm
- Ajtai, Aspnes, Naor, Rabani, Schulman, and Waarts [AANRSW '96]:
  - reduced the problem to groups of size 2
  - deterministic lower bound $O(n)$
  - randomized algorithms:
    - upper bound $\Theta(\sqrt{n \log n})$ by the Local Greedy algorithm
    - lower bound $\Omega(\sqrt[3]{\log n})$ (every algorithm)
This talk: general graphs

- a social network graph
- the request sequence contains only edges of the graph
## Overview

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Clique [AANRSW '96]

- deterministic: $\Theta(n)$
  - Adversary requests only pairs of drivers with the same unfairness
  - At the end (after $\Theta(n^3)$ steps), all drivers have distinct unfairness
  - Therefore one of them has unfairness outside the interval $[-n/2 + 1, n/2 - 1]$
• randomized:
  ▪ lower bound $O(\sqrt[3]{\log n})$ (based on the deterministic lower bound)
  ▪ upper bound $\Theta(\sqrt{n \log n})$ (Local Greedy)
• random sequences: Global Greedy has unfairness $\Theta(\log \log n)$
Deterministic algorithms
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Lower bound for clique

- Tight bound $\Theta(n)$ [AANRSW '96]
- For other graphs? stars?
• one-sided fairness? not in this talk
Lower bound on star and arbitrary graphs

- Fix a deterministic algorithm for the star of \( d \) leaves
- Reachable states \( \vec{x} = (x_1, \ldots, x_d) \), where \( x_i \) the unfairness of leaf \( i \)
- Define \( \phi(\vec{x}) = x_1 + 2x_2 + \cdots + 2^{d-1}x_d \)
- **Claim:** If \( \vec{x} \) minimizes \( \phi(\vec{x}) \), then \( \vec{x} + \vec{1} \) is also reachable
- \( |\vec{x} + \vec{1}|_1 - |\vec{x}|_1 = d \)
- Therefore either \( \vec{x} \) or \( \vec{x} + \vec{1} \) has root unfairness \( \lceil d/2 \rceil \)
Matching upper bound

**Theorem:** The deterministic unfairness of graphs of degree $d$ is exactly $\lfloor d/2 \rfloor$.

- Fix an almost balanced orientation (outdegree and indegree differ by at most 1)
- For every oriented edge $(i, j)$, service the odd requests with $i$ and even requests with $j$
Random sequences
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Global Greedy on the line

all pairs of requests are selected uniformly at random

• Global Greedy has expected unfairness $O(\log \log n)$ for the clique [AANRSW'96]
• We show that for sparse networks, Global Greedy does much worse
• In particular, for the line, Global Greedy has expected unfairness $\Omega((\log n / \log \log n)^{1/3})$

• Proof
  
  ▪ There exists a sequence $s_n^*$ of $n^3$ requests for which Global Greedy with adversarial tie breaking has unfairness $\Omega(n)$
  
  ▪ Break the line into $k/n$ segments of length $k$
  
  ▪ Consider a random sequence of length $k^3$ in one of the segments

  ▪ What is the probability that
    ○ no request falls into the boundaries and the random sequence is the bad sequence $s_k^*$? : $1/k^{k^3}$
    ○ Global Greedy breaks the ties as in the worst-case : $1/2^{k^3}$
The probability that the unfairness in every segment is less than $k$ is $(1 - 1/(2k)^{k^3})^{n/k}$, which is constant when we select $k = (\log n/ \log \log n)^{1/3}$. It follows that, with constant probability, a random sequence of length $k^3 \cdot n/k$ has unfairness $\Omega((\log n/ \log \log n)^{1/3})$. 
Random sequences on planar graphs

- Algorithm for stars
  - leaves have unfairness -1, 0, 1
  - when at 0, they help the unfairness of the root
- It has constant unfairness
- Extension to planar graphs
  - partition the edges of the graph into stars
  - every node belongs to at most 6 stars
  - run the algorithm for each star
  - the expected unfairness of every node is constant
  - the maximum unfairness among all nodes is $O(\log n)$
Randomized algorithms
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Static algorithms

Static randomized online algorithm :: there exists a probability distribution $\pi$ over the set of states

- the algorithm starts in $\pi$
- it remains in $\pi$ after every possible request.
- **Theorem:** The unfairness of every static algorithm is $\Omega(\sqrt{d})$, where $d$ is the degree of the social graph
• Upper bound (on stars)
  ▪ Balanced Local Greedy algorithm
    ○ Fix a balanced orientation
    ○ Every oriented edge \((i, j)\) alternates between unoriented and oriented according to fixed orientation starting at a random state
• Lower bound (on stars)
  1. Characterize the stationary distributions of the Markov chain
  2. Express the question as linear program
  3. Solve the linear program
• Static characterization
• for every \( x_{-i} \): \( \sum_{k \in \mathbb{Z}} \pi(k, \tilde{x}_{-i})(-1)^k = 0 \)
• eliminate variables \( \pi(\tilde{x}) \) when \( \tilde{x} \) has at least one 0
• the value of the linear program is at least equal to

\[
\min_{y_i^*} E \left[ \left| \sum_i y_i^* X_i \right| \right]
\]

- \( X_i \)'s are 0-1 unbiased binomial random variables
• this is minimized when half of \( y_i^* \)'s are 1 and half are \(-1\)
• exactly as in the Local Greedy algorithm
• Complete characterization
• what distributions $\pi$ are stationary distributions of Markov chains on the line (when $p_{x,x} = \text{const}$)?
  ▪ answer: $\pi(k) - \pi(k + 1) + \pi(k + 2) - \cdots \geq 0$, for every $k$
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