IA Solutions to sheet 2 A^*

Exercise 1

We can formally write a "chip" as an ordered pair $c \in \{1,2,3,4,5\} \times \{b,w,e\}$ (this means we consider the empty space as a chip). A state s is then a set of five chips such that exactly one chip has each of the values $\{1,2,3,4,5\}$ in the first coordinate, and two chips have b and two w in the second coordinate. By this notation the start state is

$$\{(1,b),(2,b),(3,w),(4,w),(5,e)\}$$

There are multiple admissible heuristiques. Perhaps the simplest is to count how many black chips are out of position: i.e. how many black chips have some white chip to their right.

$$h_b(s) = |\ \{\ (i,b) \in s : \exists j > i \text{ such that } (j,w) \in s\ \}\ |$$

Symetrically, one can count how many white chips have some black chip to their left.

$$h_w(s) = |\ \{\ (i,w) \in s : \exists j < i \text{ such that } (j,b) \in s\ \}\ |$$

A better (larger) heuristic than both of these is to sum these two values.

$$h(s) = h_b(s) + h_w(s)$$

We show that h is monotone (this implies it is admissible by Exercise 3). Formally, this means that for any solution s^*, $h(s^*) = 0$, and that the cost of moving from s to s' satisfies

$$c(s,s') \geq h(s) - h(s')$$

The first part of the condition is satisfied by the definition. For the second, we consider each of the possible moves: firstly, moves of cost one do not change the value of the heuristic, so $h(s) - h(s') = 0 < 1$.

For moves from s to s' of cost two, only jumping a black over a white or jumping a white over a black actually change the state. If we jump a black, all the other blacks and all the whites remain in the same position, thus the number of blacks that have some white chip to their left can only decrease by one.

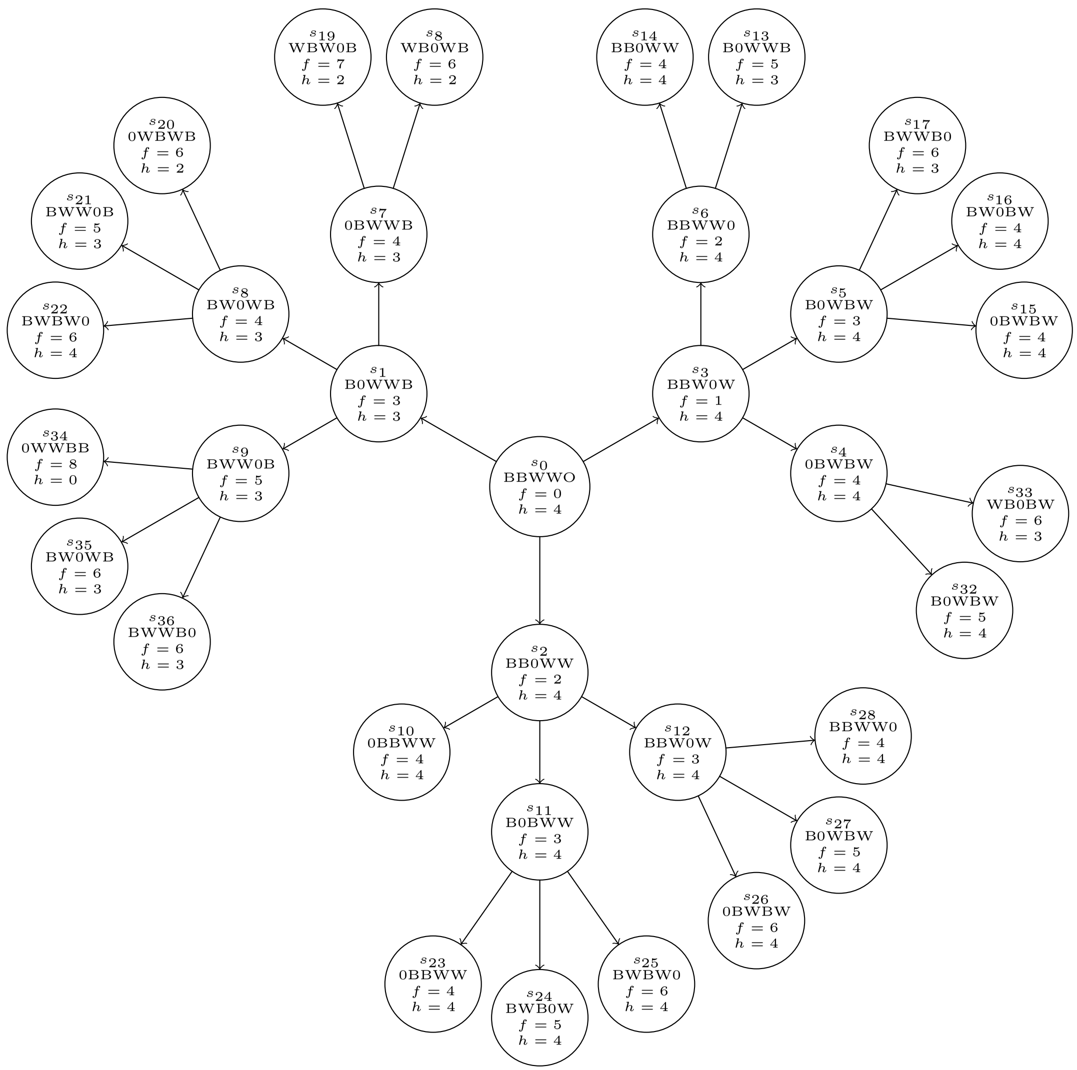
$$h_b(s) - 1 \geq h_b(s')$$

Similarly, only the white chip that was jumped over can change from having a black chip to its left to no longer having a black chip to its left, thus this value can also only decrease by one.

$$h_w(s) - 1 \geq h_w(s')$$

Overall the maximum that the heuristic can decrease is two, the same as the cost. The case for jumping a white is symmetric.

There are similar possibilities for moves of cost three; the maximum possible decrease of the heuristic is three.



States are written with Bs, Ws and 0 indicating black chips, white chips and the empty space. So the starting state is s_0=BBWW0. For a state s with path

$$s_0 = s_{i_0} \to s_{i_1} \to \cdots \to s_{i_k} = s$$

the cost function f is

$$f(s) = \sum_{j=1}^{k} c(s_{i_{j-1}}, s_{i_j})$$

We start with s_0 uncovered and in the first step expand and cover this node. step. When expanding a node, the order in which we uncover children depends upon how far to the left the moved chip is. The index of a state gives the order in which it was uncovered. Uncovered states are expanded in order of the minimal value of $f(s) + h(s)$ (ties broken by the index). When expanding a state, we do not include its parent as a child (the h value will be the same, and f larger). The order of expansions of nodes, and their values $f(s) + h(s)$ are:

s_0	4	s_7	7
s_3	5	s_8	7
s_1	6	s_{11}	7
s_2	6	s_{12}	7
s_6	6	s_4	8
s_5	7	s_9	8

After this point for all uncovered nodes s' we have $f(s') + h(s') \geq 8$. As h is admissible, any shortest possible solution thus has a cost of at least 8. As s_{34} (the first discovered solution) has cost 8 we know

$$s_0 \to s_1 \to s_9 \to s_{34}$$

gives a minimal cost path to the solution.

Exercise 2

1. (c) - there are n vehicles, and n^2 places for each, thus roughly $(n^2)^n = n^{2n}$ possiblities (this overcounts states where vehicles end up in the same squares).

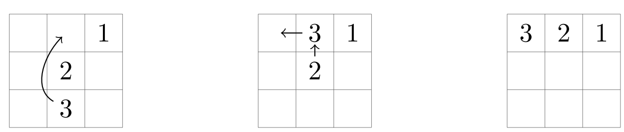
2. (c) - each vehicle has roughly 5 actions that it can perform, *independent* of what the other vehicles do.

3. The block distance or Manhattan distance, which here amounts to

$$h_i = |\, n - i + 1 - x_i \,| + n - y_i$$

4. (c).

   First, we show (a) and (b) are not admissible: as the value for (a) is always greater than or equal to (b), finding a counterexample for (b) suffices for both. So we look for a path from a state to the solution in less than $\max\{h_1, \ldots, h_n\}$ steps. Note for this purpose we can start from *any* state. In the first of the following grids, $\max\{h_1, \ldots, h_n\} = h_3 = 3$, but the solution is two moves away.



   We now argue that (c) is admissible. Imagine that we have a path from some start state to the solution. First, note that in moving from the state to the solution, the sum of the distances travelled by *all* the vehicles is at least

$$\sum_{i=1}^{n} h_i \geq n.\min\{h_1, \ldots, h_n\}$$

   In one step, the sum of the distances travelled by all the vehicles is at most n. This is because although an individual vehicle can travel two squares, to do so it must jump over a stationary vehicle, and only one vehicle can jump over such a stationary vehicle. So in any state reached after less than $\min\{h_1, \ldots, h_n\}$ steps the sum of the distances travelled by all vehicles is less than $n.\min\{h_1, \ldots, h_n\}$, which means that the state is not a solution. Thus the heuristic is admissible as the solution cannot be reached in fewer than $\min\{h_1, \ldots, h_n\}$ steps.

Exercise 3

We denote by s^* the solution, and by $c(s, s')$ the cost of moving from s to s'.

(1) A heuristic is monotone if $h(s^*) = 0$ and

$$c(s, s') \geq h(s) - h(s')$$

(2) A heuristic h is admissible if for any path to the solution

$$s = s_0 \to s_1 \to \cdots \to s_n = s^*$$

we have

$$\sum_{i=1}^{n} c(s_{i-1}, s_i) \geq h(s)$$

We show (1) ⇒ (2). First, note if there is no path to the solution then there is nothing to prove. So suppose that there is some *minimal cost* path, we proceed by induction on the *length* of the path. (Note the length of a path is different from the cost of a path.) Base case: $n = 0$ is direct from the definition. Suppose true for paths of length $n-1$ (the inductive hypothesis, IH). Take a state s with a path

$$s = s_0 \to s_1 \to s_2 \to \cdots \to s_n = s^*$$

Here $s_1 \to s_2 \to \cdots \to s_n = s^*$ must be a minimal cost path of length $n - 1$ from s_1 to s_n. Thus we have

$$\begin{aligned}
h(s) &\leq c(s_0, s_1) + h(s_1) && \text{by (1)} \\
&\leq c(s_0, s_1) + \sum_{i=1}^{n-1} c(s_i, s_{i+1}) && \text{by (IH)} \\
&= \sum_{i=1}^{n} c(s_{i-1}, s_i)
\end{aligned}$$