IA Solutions 5: Game Theory

Exercise 1: Pure Equilibria

For all the games except the bottom right, the Nash equilibria are HG and BD. For the game at the bottom right the Nash equilibria are HD and BG.

Exercise 2: Mixed Equilibria

Recall that a profile of strategies, one for each player, is a Nash equilibria if no player can get a higher payoff if they change their strategy. We want to check that this is the case for the profile $(\langle 3/4, 0, 1/4\rangle, \langle 0, 1/3, 2/3\rangle)$.

So let us see if the row player can get a higher payoff by changing their strategy. We fix the column player's strategy as it is stated in the question, and then consider what strategies are possible for the row player. In a mixed game, the possible strategies for the row player are probability distributions over their pure strategies. Let us write the probability that the row player plays H as p_H and the probability that the row player plays M as p_M. Note that the probability that the row player plays B is thus $1-p_H-p_M$, as these probabilities must add up to 1. Thus, the row players payoff for an arbitrary strategy (with the column players strategy fixed) is:

$$\begin{aligned} u_{\text{row}}(\langle p_H, p_M, p_B\rangle, \langle 0, 1/3, 2/3\rangle) &= 3 \times p_H \times 1/3 \ + \ 1 \times p_H \times 2/3 \\ &\quad + \ 1 \times p_M \times 1/3 \ + \ 2 \times p_M \times 2/3 \\ &\quad + \ 5 \times (1 - p_H - p_M) \times 1/3 \\ &= 5/3 \end{aligned}$$

We obtain the above by multiplying the probability that we end up in each state (HG, HC, etc) by the utility the row player receives in that state. Note that this is a constant. This means that no matter what strategy the row player uses, she will not do better off than the strategy given in the question.

We need to do the same for the column player, fixing the the row player's strategy. Doing this will show that the column player also has a constant payoff.

Exercise 3

The main problem here is expressing the question formally. Wheat is taken as a continuous commodity: that is, one can produce and sell fractions of wheat.

1. We want to find a dominant strategy, which is a strategy for a player such that no matter what the other players do, this strategy maximises the first player's payoff. First, note the situation is symmetric, so it suffices to find the dominant strategy for any one farmer. The payoff for a farmer k is

$$u_k = q_k \times e^{-Q} = q_k \times e^{-q_k} \times e^{\sum_{i \neq k} q_i}$$

If we fix arbitrary strategies for all players except k, the above value is maximised when $q_k \times e^{-q_k}$ is maximised. We can differentiate this to work out the maximum:

$$\frac{d}{dq_k} q_k \times e^{-q_k} = e^{-q_k} - q_k \times e^{-q_k}$$

which is zero only when $q_k = 1$. (You can also check this is actually a maximum. Or you can just read the maximum on the provided graph.)

If all the farmers produce this amount, each individual farmer has a profit of $u_k = 1 \times e^{-n} = {}^1\!/e^n$.

2. Here the total profit is precisely $\sum_k u_k = Q \times e^{-Q}$, which is maximised when the total amount of wheat produced is 1. This means that the total cost for all the wheat is ${}^1\!/e$. If the farmers each produce only ${}^1\!/n$, then they will each share equally the total profit, meaning each receives a payoff of ${}^1\!/ne$. Note that (for $n \geq 2$) this is more than if they all play their dominant strategy.

   Whether or not you think the farmers can agree to produce only this much depends upon your view on human nature.