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Restricting the domain allows for weaker independence

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Abstract Arrows classical axiom of independence of irrelevant alternatives may be more descriptively thought of as binary independence. This can then be weakened to ternary independence, quaternary independence, etc. It is known that under the full domain these are not real weakenings as they all collapse into binary independence (except for independence over the whole set of alternatives which is trivially satisfied). Here we investigate whether this still happens under restricted domains. We show that for different domains these different levels of independence may or may not be equivalent. We specify when and to what extent different versions of independence collapse into the same condition.

1 Introduction

Independence of irrelevant alternatives (IIA) is a central axiom of Arrow's (1950) celebrated impossibility. In general, axioms of this type require that whenever individuals' preferences over a given set of alternatives remain the same, then so should the social preference. Arrow's original formulation of IIA requires independence over every subset of alternatives. However, there is an alternative formulation which requires independence only for subsets of cardinality two. Although this formulation is superficially weaker, it is straightforward to observe that independence over pairs implies independence over any set, and thus that the two formulations are equivalent.

Blau (1971) showed further that independence over larger subsets implies independence over smaller subsets. Thus we observe what we call the *Blau equivalence*: roughly expressed, this states that all versions of independence have the same strength. The Blau equivalence implies that Arrow's result cannot be escaped by

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weakening independence to subsets of some fixed cardinality greater than two. In turn we call this stronger version of Arrow's result the *Blau impossibility*.

The Blau equivalence rests upon the full domain assumption. In this paper we consider the effects of imposing domain restrictions.[1] Section 2 presents the setting formally, including a formal definition of the condition requiring that independence over sets of cardinality k is satisfied. Section 3 shows that under restricted domains, unlike under the full domain, increasing k may effectively weaken independence—hence the Blau equivalence may fail. Section 4 characterises the domains for which nominally different versions of independence have effectively different strengths, thus also effectively providing a characterization for the Blau equivalence. In Section 5 we develop the idea of what we call the Blau partition of a domain. We show that there are many possible such partitions; thus that the Blau equivalence can fail in many different ways. We develop a particular domain in Section 6, making sure during its construction that it is also *Arrovian*, i.e. subject to Arrow's impossibility. We define a SWF on this domain that satisfies a weaker version of independence alongside Arrow's other axioms. This verifies that on some domains it is possible to escape Arrow's result through weakening independence. Section 7 makes some final remarks.

2 Definitions

Let $N = \{1,\ldots,n\}$ be a set of agents, for some integer $n \geq 2$. Let A be a set of alternatives with $m = |A| \geq 3$. We will refer to particular alternatives as $a_1, a_2, \ldots, a_m$, and arbitrary alternatives as $x, y, z, \ldots$ We represent each agent's preference over the alternatives as a complete preorder. A social welfare function (SWF) aggregates the preferences of all the agents into a single complete preorder.

Denote by $\mathscr{W}(A)$ be the set of complete and transitive binary relations over A. We write $R \in \mathscr{W}(A)$ for an arbitrary complete preorder. We use P to denote the strict component of R, so xPy iff $\neg yRx$. We say that a complete preorder R is a linear order iff it is also antisymmetric.

Let the non-empty set $D \subseteq \mathscr{W}(A)$ denote the domain. A specific agent i's preference corresponds to some $R_i \in D$. A collection of n such preferences, one for each agent, is called a preference profile, or simply a profile, and is written $\underline{R} \in D^N$. We write $(\underline{R}_{-i}, R'_i)$ for the profile where agent i has preference R'_i and all other agents have the same preferences as in $\underline{R}$. A SWF defined over D maps a profile to a social preference, formally it is a function $f : D^N \to \mathscr{W}(A)$. We use $f^*(\underline{R})$ to denote the strict component of $f(\underline{R})$.

Given a set $X \subseteq A$, the restriction of a preference R to X is

$$R|_X = \{(x,y) \in R : x,y \in X\}.$$

We use the same notation to apply restrictions to domains (sets of preferences) and preference profiles. Formally, for $X \subseteq A$, $D \subseteq \mathscr{W}(A)$ and $\underline{R} \in D^N$ we write

$$D|_X = \{R|_X : R \in D\} \quad \text{and} \quad \underline{R}|_X = (R_i|_X)_{i \in N}.$$

[1] Restricting the domain is a typical approach to impossibilities in social choice theory. Gaertner (2001; 2002) has produced elaborate surveys of the literature.

We say two preferences R and R' *coincide* on X iff $R|_X = R'|_X$. Similarly, preference profiles, and domains, coincide on X iff their restrictions to X are identical. A preference R over X *extends* a preference R' over $Y \subseteq X$ iff these coincide on Y.

Most of this paper is concerned with axiomatics; general properties of SWFs. The following are well-known examples of such properties. A SWF f is *dictatorial* iff there is some $i \in N$ such that xP_iy implies $x[f^*(\underline{R})]y$. A SWF f is *Pareto optimal* iff $xP_iy, \forall i \in N$ implies $x[f^*(\underline{R})]y$. These two axioms, alongside IIA which we formally define below, are the traditional conditions for the impossibility result of Arrow (1950). Along the lines of Blau (1971), we consider a generalized notion of what it means for a SWF to be "independent". A SWF is k-IND, for an integer $2 \leq k \leq m$, if for every $X \subseteq A$ with $|X| = k$, for any two profiles $\underline{R}, \underline{R}' \in D^N$ that coincide on X, we have $f(\underline{R})|_X = f(\underline{R}')|_X$. As discussed in the introduction, IIA is sometimes expressed as 2-IND and sometimes as the conjunction of all k-IND for $k = 2, \ldots, m$; these formulations are equivalent.

3 A counterexample on restricted domains

One may question the sense of defining k-IND separately from 2-IND. The fact that independence over smaller sets implies independence over larger sets seems to have become common knowledge soon after Arrow's original presentation of IIA. Further, this "upwards" implication holds over any domain.

Theorem 1 *For any SWF f over any domain $D \subseteq \mathscr{W}(A)$, if f satisfies k-IND then it also satisfies l-IND, for any $2 \leq k < l \leq m$.*

Proof Take two profiles $\underline{R}$ and $\underline{R}'$ such that $\underline{R}|_Y = \underline{R}'|_Y$ for some set Y of cardinality l. We want to show that $f(\underline{R})|_Y = f(\underline{R}')|_Y$. For each distinct pair $\{x,y\} \subset Y$, there is a set X_{xy} such that $\{x,y\} \subseteq X_{xy} \subset Y$ and $|X_{xy}| = k$.[2] Clearly $\underline{R}|_{X_{xy}} = \underline{R}'|_{X_{xy}}$, thus by k-IND $f(\underline{R})|_{X_{xy}} = f(\underline{R}')|_{X_{xy}}$. This implies $f(\underline{R})|_{\{x,y\}} = f(\underline{R}')|_{\{x,y\}}$. As this is the case for every pair $x, y \in Y$, $f(\underline{R})|_Y = f(\underline{R}')|_Y$ as required. □

This observation dates back to May (1954), though the fact that it applies in all domains is not explicitly noted.

In 1971 Blau proved the inverse "downwards" implication, thus showing that nominally weaker versions of independence imply nominally stronger versions. The Blau equivalence amounts to the combination of the upwards and downwards implications.

Theorem 2 (Paraphrase of Blau's (1971) "Theorem 4") *For any SWF f over the full domain $D = \mathscr{W}(A)$, if f satisfies l-IND then it also satisfies k-IND, for any $2 \leq k < l < m$.*

Proof It suffices to show that when f satisfies l-IND it satisfies $(l-1)$-IND. Take f that satisfies l-IND. Take two arbitrary profiles, and suppose there is a set X of size $l-1$ such that the two profiles coincide on this set. We suppose that the two profiles

[2] Note $\subset$ is used for proper set inclusion; $X \subset Y$ implies $|X| < |Y|$.

are identical except for one voter, they can thus be written $(\underline{R}_{-i}, R_i)$ and $(\underline{R}_{-i}, R_i')$. Extending to the case where multiple voters have different preferences only involves iteration of the following. Consider two alternatives $x, y \in A$ such that $x \neq y$, $x \notin X$ and $y \notin X$ (these exist as $|X| \leq m-2$). Take a preference $R'' \in \mathscr{W}(A)$ such that this extends both $R_i|_{X\cup\{x\}}$ and $R_i'|_{X\cup\{y\}}$. Equivalently, $R''|_{X\cup\{x\}} = R_i|_{X\cup\{x\}}$ and $R''|_{X\cup\{y\}} = R_i'|_{X\cup\{y\}}$. There may be multiple possible R_i''s, but there is always at least one. Now by l-IND we have $f(\underline{R}, R_i)|_{X\cup\{x\}} = f(\underline{R}, R'')|_{X\cup\{x\}}$ and $f(\underline{R}, R'')|_{X\cup\{y\}} = f(\underline{R}, R_i')|_{X\cup\{y\}}$. Thus clearly

$$f(\underline{R}, R_i)|_X = f(\underline{R}, R'')|_X = f(\underline{R}, R_i')|_X \ .$$

□

However, Blau's proof for the downward direction uses two conditions not present for the upward direction: first, it supposes the full domain; and second it requires a strict inequality $l < m$. The second of these two is necessary because m-IND is trivially satisfied by any SWF. We will be more interested in what happens when we relax the assumption of a full domain, which is used by Blau in the following manner: Blau's proof takes two preferences and "connects" them through a third. This third preference is guaranteed to exist on the full domain. When we consider restricted domains, this is no longer always the case. In fact in the following domain this third "connecting" preference *never* exists, for $l = 3$.

Example 1 Fix $m = 4$. Let D be the domain containing the following six linear orders, where higher alternatives are preferred to lower ones.

R^{I}	R^{II}	R^{III}	R^{IV}	R^{V}	R^{VI}
a_2	a_3	a_3	a_2	a_1	a_4
a_1	a_1	a_4	a_4	a_4	a_1
a_3	a_2	a_2	a_3	a_2	a_3
a_4	a_4	a_1	a_1	a_3	a_2

Every preference in this domain is a linear order. Further, every triple has six different possible orderings:

$$\forall x, y, z \in A \text{ such that } x,\ y, \text{ and } z \text{ are distinct,} \quad \left| D|_{\{x,y,z\}} \right| = 6.$$

Thus in a setting where preferences are restricted to linear orders, all triples are free.[3] It is a standard result, established by Blau (1957) that if all triples are free Arrow's impossibility holds. Thus any Pareto optimal SWF on this domain that satisfies 2-IND is dictatorial.

Note that given any linear order over three alternatives, only one preference in the domain extends this. No preference in the domain extends a non-linear complete preorder of three alternatives. That is to say, for $X \subset A$ with $|X| = 3$ and $R, R' \in D$, if $R|_X = R'|_X$ then $R = R'$. This implies that any SWF over this domain satisfies 3-IND trivially. For example, ranking by Borda scores provides a non-dictatorial and Pareto optimal SWF. By the above paragraph this cannot satisfy 2-IND; more generally this shows that 3-IND does not imply 2-IND.

[3] Note here, as is standard in the literature, "triple" means a subset of alternatives of size three. A triple is said to be free iff every possible ordering over this triple exists within the domain.

The above example implies that k-IND and l-IND for $k \neq l$ need not be equivalent. The next section first considers k-IND for some fixed value of k, before returning to the issue of comparing k-IND to l-IND for $k \neq l$.

4 The Blau equivalence: when are different versions of independence equivalent?

Given a particular SWF defined on a particular domain, k-IND may or may not be satisfied. In this section we give a necessary and sufficient condition for this property. This will allow us to determine the domains for which k-IND implies l-IND, for arbitrary values of k and l. This generalizes the question as posed by Blau, who showed that on the full domain, if a SWF satisfies k-IND then it also satisfies l-IND for $k,l < m$. We have already seen in Section 3 that there are domains where this general result does not hold. This section will allow us to determine when it does.

Definition 1 For a domain D, integer $k = 2,\dots,m$ and two alternatives $x,y \in A$, two preferences $R,R' \in D$ are (k,x,y)-adjacent iff there is a set $X \subseteq A$ such that $|X| = k$, $\{x,y\} \subseteq X$ and $R|_X = R'|_X$.

Just considering adjacency does not suffice; recall that in Blau's method uses a third profile to "connect" two profiles that were not directly adjacent. For domains that are less populated than the full domain, it may be necessary to use even more profiles to form this connection.

Definition 2 For a domain D, integer $k = 2,\dots,m$ and two alternatives $x,y \in A$, two preferences $R,R' \in D$ are (k,x,y)-reachable iff there is a sequence of preferences $R = S_0,S_1,\dots,S_t = R'$ such that S_s and S_{s+1} are (k,x,y)-adjacent. If $R,R' \in D$ are (k,x,y)-reachable we write $R \sim^k_{xy} R'$.

Remark 1 For all k,x,y, the relation $\sim^k_{xy}$ is an equivalence relation on D. Note that for $k < l$ the relation $\sim^l_{xy}$ refines $\sim^k_{xy}$. In particular, the equivalence classes of $\sim^m_{xy}$ are the singleton subsets of D.

Lemma 1 *A social welfare function f on D satisfies k-IND iff for any pair of alternatives $x,y \in A$ and for any agent i and pair of profiles $\underline{R}$ and $(\underline{R}_{-i},R'_i)$, if R_i and R'_i are (k,x,y)-reachable then $f(\underline{R})|_{\{x,y\}} = f(\underline{R}_{-i},R'_i)|_{\{x,y\}}$.*

Proof (If.) Take a SWF f and suppose that the right hand side (RHS) above holds. Take X such that $|X| = k$ and $\underline{R},\underline{R}' \in D$ such that $\underline{R}|_X = \underline{R}'|_X$. Now, for all $i \in N$, we have $R_i|_X = R'_i|_X$; thus for all $\{x,y\} \subseteq X$, for all $i \in N$, R_i,R'_i are (k,x,y)-adjacent, thus (k,x,y)-reachable. Thus by the RHS for all $\{x,y\} \subseteq X$ we have $f(\underline{R})|_{\{x,y\}} = f(\underline{R}')|_{\{x,y\}}$, so clearly $f(\underline{R})|_X = f(\underline{R}')|_X$ as required.

(Only if.) Suppose that k-IND is satisfied. Suppose R_i,R'_i are (k,x,y)-reachable. Thus there is a list $R_i = S_0,S_1,\dots,S_t = R'_i$ with each $S_i \in D$ such that for each $s < t$, there is a $X_s \supseteq \{x,y\}$ with $|X_s| = k$ and $S_s|_{X_s} = S_{s+1}|_{X_s}$. Thus by k-IND $f(\underline{R}_{-i},S_s)|_{X_s} = f(\underline{R}_{-i},S_{s+1})|_{X_s}$, so in particular $f(\underline{R}_{-i},S_s)|_{\{x,y\}} = f(\underline{R}_{-i},S_{s+1})|_{\{x,y\}}$. Thus

$$f(\underline{R})|_{\{x,y\}} = f(\underline{R}_{-i},S_1)|_{\{x,y\}} = \cdots = f(\underline{R}_{-i},R')|_{\{x,y\}}$$

as required. □

This characterization will allow us to determine when different versions of independence imply each other. First, let us formally define this generalized concept.

Definition 3 For $2 \leq k,l \leq m$, a domain D is (k,l)-equivalent iff (any SWF on D is k-IND iff it is l-IND).

Our first theorem in this section provides a necessary and sufficient condition on domains determining when (k,l)-equivalence holds.

Theorem 3 *A domain D is (k,l)-equivalent iff for all $x,y \in A$, $\sim_{xy}^k = \sim_{xy}^l$.*

Proof (If.) Suppose f satisfies l-IND. We use the if direction of Lemma 1 to show that f also satisfies k-IND: take arbitrary $x,y \in A$, agent $i \in N$ and pair of profiles $\underline{R}$ and $(\underline{R}_{-i},R_i')$ such that R_i,R_i' are (k,x,y)-reachable. By the RHS, these are also (l,x,y)-reachable. Thus as f satisfies l-IND, by the only if direction of Lemma 1 we have $f(\underline{R})|_{\{x,y\}} = f(\underline{R}_{-i},R_i')|_{\{x,y\}}$ as required.

(Only if.) If $k = l$ this is trivially satisfied, so without loss of generality assume $k < l$. We proceed by contraposition: suppose the negation of the RHS. To accord with Remark 1, $\sim_{xy}^l$ must properly refine $\sim_{xy}^k$. Thus there are alternatives $x,y \in A$, agent $i \in N$, and $S,S' \in D$ such that $S \sim_{xy}^k S'$ but not $S \sim_{xy}^l S'$. We construct an f that satisfies l-IND but violates k-IND. First, let $\mathscr{S} = \{R \in D : R \sim_{xy}^k S\}$ and $\mathscr{S}' = D \backslash \mathscr{S}$. Define f such that it

1. returns some fixed ordering over all pairs $\{z,w\}$ where $z,w \neq x,y$,
2. sets $x[f(\underline{R})]z$ and $y[f(\underline{R})]z$ for all $z \neq x$, and
3. returns $x[f(\underline{R})]y$ iff $\forall i \in N$, $R_i \in \mathscr{S}$.

We claim that f satisfies l-IND but not k-IND. First let us demonstrate the violation of k-IND: for $\underline{R}$ where $\forall i \in N$, $R_i = S$, $f(\underline{R})|_{\{x,y\}} \neq f(\underline{R}_{-1},S')|_{\{x,y\}}$ but S,S' are (k,x,y)-reachable. Now we show the satisfaction of l-IND. First note for all pairs except $\{x,y\}$, all return the same ordering over these pairs, thus the condition of Lemma 1 is trivially satisfied. It remains to check for the pair $\{x,y\}$. Take an arbitrary agent i and pair of profiles $\underline{R}$ and $(\underline{R}_{-i},R_i')$ such that R_i and R_i' are (k,x,y)-reachable. If for all $j \in N$, $R_j \in \mathscr{S}$, then as $R_i \in \mathscr{S}$ and $R_i \sim_{xy}^k R_i'$ we also have $R_i' \in \mathscr{S}$, thus $f(\underline{R})|_{\{x,y\}} = f(\underline{R}_{-i},R_i')|_{\{x,y\}}$. So suppose for some $j \in N$, $R_j \notin \mathscr{S}$. If $j \neq i$ then we still have $R_j \notin \mathscr{S}$ within $(\underline{R}_{-i},R_i')$, whereas if $j = i$ then $R_i' \notin \mathscr{S}$, thus $f(\underline{R})|_{\{x,y\}} = f(\underline{R}_{-i},R_i')|_{\{x,y\}}$. □

5 Violating the Blau equivalence: what Blau partitions are possible?

As the name suggests, for a given domain D, (k,l)-equivalence is an equivalence relation on the integers $\{2,\ldots,m\}$. Given that we can determine when the different versions of independence imply one another, we now ask: what is the structure of these for a given domain?

Definition 4 For a given domain, the Blau partition is the partition of $\{2,\ldots,m\}$ determined by (k,l)-equivalence. That is, given a domain D, integers k and l with $k < l$ are in the same equivalence class of the Blau partition iff D is (k,l)-equivalent.

Definition 5 For $p, q \in \mathbb{N}$, an integer interval is a set $\{x \in \mathbb{N} : p \leq x \leq q\}$. An interval partition is a partition whose equivalence classes are all integer intervals.

We use the following notation for integer intervals and interval partitions. An interval partition of $\{2, \ldots, m\}$ can be concisely expressed by the suprema of its equivalence classes. Thus for the interval partition

$$\{\{2, \ldots, q_1\}, \{q_1 + 1, \ldots, q_2\}, \ldots, \{q_{t-1} + 1, \ldots, q_t\}\}$$
$$\text{where } q_1 < q_2 < \cdots < q_t = m$$

we write $[\![q_1, q_2, \ldots, q_t]\!]$.

We now show that Blau partitions do not contain any gaps, and thus must be composed of intervals.

Proposition 1 *Every Blau partition is an interval partition.*

Proof Suppose that we have (k, l)-equivalence. Take p such that $k < p < l$. It suffices to show that we have (k, p)-equivalence. By Theorem 1, if a SWF is k-IND, then it is p-IND. Similarly, if the SWF is p-IND it is also l-IND, thus by (k, l)-equivalence it is k-IND as required. □

For example, the Blau partition under the full domain is $[\![m-1, m]\!]$. The Blau partition under the domain of Example 1 is $[\![2, 4]\!]$.

Remark 2 A domain satisfies the Blau equivalence if and only if it has Blau partition $[\![m-1, m]\!]$.

One way of violating the Blau equivalence is if the Blau partition only contains a single equivalence class. This only occurs for extremely restricted domains.

Proposition 2 *The Blau partition contains one equivalence class if and only if $D \subseteq \{R, R^{-1}\}$ for some linear order R, where $xR^{-1}y$ if and only if xRy.*

Proof (If.) For any SWF, the property of k-IND is trivially satisfied for all $k = 2, \ldots, m$.
(Only if.) Contraposition: there must be $R, R' \in D$ such that $R|_{\{x,y\}} = R'|_{\{x,y\}}$ for some x, y. We describe a SWF f that satisfies m-IND but not 2-IND. Indeed, as m-IND is trivially satisfied for any SWF, setting $xf(\underline{R})y$ if and only if no agents in $\underline{R}$ have linear order R suffices (arbitrarily fix the ordering over the other candidates). □

Of course, more interesting violations are also possible. We show in the next theorem that any interval partition that contains m as a singleton is the Blau partition under some domain. Indeed, for an arbitrary such interval partition we provide an explicit construction of the required domain.

Theorem 4 *For any interval partition of $\{2, \ldots, m\}$ which contains $\{m\}$ as an equivalence class, there is a domain D such that the Blau partition under this domain is this interval partition.*

Proof We use some additional notation. Recall we label the alternatives $a_1, \ldots, a_m$. Let $R^{\downarrow}$ be the linear order over A such that $a_i R a_j$ iff $i < j$. Write $R^{\downarrow k}$ for the ordering that has $a_i P a_j$ for $i < j$, $i \neq k, j \neq k$ and $a_i P a_k$ for all $i \neq k$. Informally, $R^{\downarrow k}$ starts with $R^{\downarrow}$ and sends the kth element to the bottom, thus $R^{\downarrow} = R^{\downarrow m}$.

Take a set $\{q_1, \ldots, q_k\} = K \subseteq \{2, \ldots, m\}$. Let $D^* = \{R^{\downarrow k} : k \in K\}$. We will show that the Blau partition of the domain D^* is $[\![q_1, \ldots, q_k]\!]$ if $m-1, m \in K$. By transitivity, it suffices to show that

1. if $k \in K$, then $(k, k+1)$-equivalence does not hold, and
2. if $k \notin K$, then $(k, k+1)$-equivalence holds.

Let us deal with these in turn.

1. Here $R^{\downarrow k} \in D^*$. Consider $a_1 P a_2 \ldots P a_k \in \mathscr{W}(\{a_1, \ldots, a_k\})$. Both $R^{\downarrow}$ and $R^{\downarrow k}$ are extensions of this within D^*, which implies that $R^{\downarrow} \sim^k_{a_1 a_l} R^{\downarrow k}$. Now, all preferences $R' \neq R^{\downarrow k}$ in the domain have $a_k R' a_i$ for the $m-k$ alternatives a_i where $i > k$. This implies that no linear order in the domain coincides with $R^{\downarrow k}$ on a superset of $\{a_{k-1} a_k\}$ of size $k+1$. This implies that $R^{\downarrow} \not\sim^{k+1}_{a_1 a_k} R^{\downarrow k}$. As $\sim^k_{a_1 a_k} \neq \sim^{k+1}_{a_1 a_k}$, by Theorem 3 we do not have $(k, k+1)$-equivalence.
2. Here $R^{\downarrow k} \notin D^*$. We first show the following lemma.

   Lemma 2 *Take integers k', k'' such that $2 \leq k' < k'' \leq m$. Suppose for any $S, S' \in D$ and set $X \subset A$ such that $S|_X = S'|_X$ and $|X| = k'$, there are $Y, Y' \supset X$ and $S'' \in D$ such that $S|_Y = S''|_Y$, $S''|_{Y'} = S'|_{Y'}$ and $|Y| = |Y'| = k''$. Then we have (k', k'')-equivalence.*

   Proof (of Lemma 2) To see this, take arbitrary $\{x, y\} \subset A$ and $R, R' \in D$ such that $R \sim^{k'}_{xy} R'$. We want to show that $R \sim^{k''}_{xy} R'$. So, suppose there is a list $R = S_1, S_2, \ldots S_t = R'$ such that for each $i = 1, \ldots, t-1$ there is a set $X_i \supset \{x, y\}$, with $|X| = k'$, and such that $S_i|_{X_i} = S_{i+1}|_{X_i}$. We want to find a list $R = S'_1, S'_2, \ldots S'_{t'} = R'$ such that for each $i = 1, \ldots, t'-1$ there is a set $Y_i \supset \{x, y\}$, with $|Y| = k''$, and such that $S'_i|_{Y_i} = S'_{i+1}|_{Y_i}$. We expand the first list, adding in a single new linear order between each adjacent pair. First, for $1 \leq i \leq t$, set $S_i = S'_{2i-1}$, with in particular $S_t = S'_{2t-1} = S'_{t'}$. Then, for $1 \leq i < t$, take the two linear orders required by the condition as $S = S_i$ and $S' = S_{i+1}$, and define each $S'_{2i} = S''$. ■

   We now show that the condition of Lemma 2 is satisfied for $k' = k$ and $k'' = k+1$. Consider an arbitrary set $X = \{a_{s_1}, \ldots, a_{s_k}\}$, with $s_i > s_j$ if $i > j$ (of cardinality k) and an arbitrary ordering S over this set.
   Suppose $S \neq R^{\downarrow}|_X$, then there is at most one extension of S in D^*, so there is nothing to prove. So suppose $S = R^{\downarrow}|_X$, and consider R and R' in the domain that extend S. Note these must be of the form $R^{\downarrow p}$ and $R^{\downarrow q}$ for $p, q \neq s_1, s_2, \ldots, s_{k-1}$, as otherwise $S \neq R^{\downarrow}|_X$. Without loss of generality suppose $p < q$.
   Case 1. If $p, q \neq s_k$, $R^{\downarrow p}|_{X \cup |q|} = R^{\downarrow}|_{X \cup |q|}$ and $R^{\downarrow q}|_{X \cup |p|} = R^{\downarrow}|_{X \cup |p|}$ as required.
   Case 2. Suppose $p = s_k$.
   Subcase a. Suppose $s_k < m$. Then there is $r \notin K$ such that $r > s_k$. As $s_k > k$, there is some $r' \notin K$ such that $r' < s_k$. We have $R^{\downarrow p}|_{X \cup \{r'\}} = R^{\downarrow}|_{X \cup \{r'\}}$ and $R^{\downarrow q}|_{X \cup \{r\}} = R^{\downarrow}|_{X \cup \{r\}}$ as required.

Subcase b. Suppose $s_k = m$. As $m-1, m \in K$, $k < m-1$, thus there are distinct $r, r' \notin K$ such that $r, r' < s_k$. Without loss of generality suppose $r \neq p$ and $r' \neq q$, we then have $R^{\downarrow p}|_{X\cup\{r\}} = R^{\downarrow}|_{X\cup\{r\}}$ and $R^{\downarrow q}|_{X\cup\{r'\}} = R^{\downarrow}|_{X\cup\{r'\}}$ as required. □

6 Blau's impossibility: do k-IND SWFs exist on Arrovian domains?

We now have a better idea of when l-IND implies k-IND for $l > k$. The fact that for some domains the Blau equivalence fails opens up the possibility that on a domain where the Arrovian impossibility holds, i.e. a domain where 2-IND and PO imply dictatoriality, we may nonetheless have a SWF that satisfies k-IND, PO and non-dictatoriality with $2 < k < m$. This may be thought of as a *successful* weakening of independence; such a weakening overcomes what we call Blau's impossibility.

In fact, we have already seen a somewhat trivial successful weakening of independence. For the domain of Example 1 ranking by Borda scores satisfies 3-IND, but as every triple is free this domain is Arrovian. This example is trivial in that 3 belongs to the "top" equivalence class in the Blau partition, thus any SWF on this domain satisfies 3-IND. We now describe an example of a SWF that is not trivial in this sense, defined on the union of single-peaked and single-dipped domains. Of course, single-peakedness is well known as an escape from the Arrovian impossibility.[4]

Definition 6 A domain is *single-peaked* if there is a linear order $\succ$ on A, called the *axis*, such that ($x \succ y \succ z$ or $z \succ y \succ x$) implies (xRy implies yRz).

Note that single-peakedness by itself does not define a particular domain; it is instead a property that a domain might satisfy. For a fixed axis $\succ$ there is a unique largest (by cardinality and inclusion) single-peaked domain that contains all other single-peaked domains with the same axis.

Definition 7 A *maximal single-peaked domain* is a largest single-peaked domain with respect to some given axis.

Maximal single-peaked domains are examples of domains that satisfy the Blau equivalence but where the Arrovian impossibility does not apply.

Proposition 3 *For $m > 2$, any maximal single-peaked domain satisfies the Blau equivalence, i.e. it has Blau partition $[\![m-1, m]\!]$.*

Proof As for $m > 2$ a maximal single-peaked domain has more than two elements, by Proposition 2 we know that there is not one single partition. Thus it suffices to show $(l, l+1)$-equivalence for $l = 2, \ldots, m-2$. We proceed by showing that the condition of Lemma 2 holds. Thus suppose that $S|_X = S'|_X$ for $S, S' \in D$ and $X \subset A$ with $|X| = l$. Note as $|X| = l \leq m-2$ there are $a, b \in A \backslash X$. For $x \in A$, $Y \subseteq A$ and $R \in D$, define

$$\mu(x, Y, R) = |\{y \in Y \mid x \succ y, yRx\}| - |\{y \in Y \mid y \succ x, yRx\}| \quad .$$

[4] Black (1948) provides the classic reference concerning single peaked domains.

This may be thought of as a measure of "how far anti-clockwise" x is in R with respect to some set of alternatives Y. Now, there is a $S'' \in D$ such that $S''|_X = S|_X$, $\mu(a,X,S) = \mu(a,X,S'')$ and $\mu(b,X,S') = \mu(b,X,S'')$. Then, as required, $S|_{X\cup\{a\}} = S''|_{X\cup\{a\}}$ and $S'|_{X\cup\{b\}} = S''|_{X\cup\{b\}}$. □

By removing preferences from the single-peaked domain we can create a domain to which neither the Blau equivalence nor the Arrovian impossibility implies. For instance, consider the domain D_{p} with the following rankings:[5]

$R^{\mathsf{p}1}$	$R^{\mathsf{p}2}$	$R^{\mathsf{p}3}$	$R^{\mathsf{p}4}$	$R^{\mathsf{p}5}$	$R^{\mathsf{p}6}$	$R^{\mathsf{p}7}$	$R^{\mathsf{p}8}$
a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5
a_2	a_1	a_3	a_2	a_4	a_3	a_5	a_4
a_3	a_3	a_1	a_4	a_2	a_5	a_3	a_3
a_4	a_4	a_4	a_1	a_5	a_2	a_2	a_2
a_5	a_5	a_5	a_5	a_1	a_1	a_1	a_1

Let us draw attention to the pair (a_1,a_2). Because $R^{\mathsf{p}6}$ and $R^{\mathsf{p}7}$—and also $R^{\mathsf{p}7}$ and $R^{\mathsf{p}8}$—are $(4,a_1,a_2)$-adjacent, therefore

$$\{5\} \text{ is a member of the Blau partition of } D_{\mathsf{p}}. \tag{$\star$}$$

However, none of $R^{\mathsf{p}6}$, $R^{\mathsf{p}7}$ or $R^{\mathsf{p}8}$ are $(4,a_1,a_2)$-adjacent to $R^{\mathsf{p}4}$, as the rankings, firstly, of a_1 and a_5 and, secondly, of $_2$ and a_4 are different in each pair of preferences. It can similarly be seen that $R^{\mathsf{p}5}$ is not $(4,a_1,a_2)$-adjacent to any of the other rankings in D_{p}. Therefore it is not $(4,a_1,a_2)$-reachable to any other ranking, although it is certainly $(2,a_1,a_2)$-reachable to other rankings. Thus 2 and 4 are in different sets of the Blau partition for this domain, so

$$D_{\mathsf{p}} \text{ does not satisfy the Blau equivalence.} \tag{$\dagger$}$$

The inverse of a single-peaked domain is a single dipped domain. Without going into too many details, the single dipped version of the above domain is D_{d}:

$R^{\mathsf{d}1}$	$R^{\mathsf{d}2}$	$R^{\mathsf{d}3}$	$R^{\mathsf{d}4}$	$R^{\mathsf{d}5}$	$R^{\mathsf{d}6}$	$R^{\mathsf{d}7}$	$R^{\mathsf{d}8}$
a_1	a_1	a_1	a_1	a_5	a_5	a_5	a_5
a_2	a_2	a_2	a_5	a_1	a_4	a_4	a_4
a_3	a_3	a_5	a_2	a_4	a_1	a_3	a_3
a_4	a_5	a_3	a_4	a_2	a_3	a_1	a_2
a_5	a_4	a_4	a_3	a_3	a_2	a_2	a_1

We now consider the union domain $D_{\mathsf{u}} = D_{\mathsf{p}} \cup D_{\mathsf{d}}$. This satisfies the free triple condition and thus is subject to the Arrovian impossibility: there is no PO, 2-IND and non-dictatorial SWF defined on D_{u}. As the only rankings in the domain with a_2 preferred to a_1 are in D_{p} (because $R^{\mathsf{d}8} = R^{\mathsf{p}8}$), the statements $(\star)$ and $(\dagger)$ equally apply

5 Preferences were not removed arbitrarily to create this domain. For the remaining preferences, note that the alternatives on either side of the "peak" are balanced; that these alternatives are interspersed as far as possible. Cf. the idea of *equidistantly single-peaked* domains described by Ozdemir and Sanver (2007).

if we substitute D_{u} for D_{p}. This opens up the possibility for a successful weakening of independence. Define $D_{\mathsf{L}} = \{D^{\mathsf{p}i}, D^{\mathsf{d}i} : \text{ for } i = 1,2,3,4\}$ and $D_{\mathsf{R}} = \{D^{\mathsf{p}i}, D^{\mathsf{d}i} : \text{ for } i = 5,6,7,8\}$.

Definition 8 *Selective pairwise majority* is the function on D_{u} that proceeds as follows: divide the voters into those who have preferences in D_{L} and those who have preferences in D_{R}, then perform *pairwise majority*[6] on the larger of these two groups. Break ties arbitrarily in a consistent and deterministic manner.

Theorem 5 *Selective pairwise majority is non-dictatorial SWF on D_{u} that satisfies PO and 4-IND.*

Proof First note that D_{L} and D_{R} are each *single-crossing domains*[7] (see Figure 1), thus this function does actually return a total preorder as required. As pairwise majority on D_{L}—or indeed on D_{R}—is non-dictatorial, so to is selective pairwise majority. If all agents prefer x to y, then a majority also do so for whichever of D_{L} or D_{R} has more voters; selective pairwise majority is PO.

To finish we show that selective pairwise majority is 4-IND. Take arbitrary $R \in D_{\mathsf{L}}$ and $R' \in D_{\mathsf{R}}$. Note that $a_1 R a_5$ but $a_5 R' a_1$; and also $a_2 R a_4$ but $a_4 R' a_2$. As there are only five alternatives, R and R' never coincide on a set of four alternatives. Thus R and R' are not (k,x,y)-adjacent for any pair of alternatives x,y. As R and R' were chosen arbitrarily, they are not (k,x,y)-reachable either.[8] So for a R'' such that $R \sim^4_{xy} R''$, $R'' \in D_{\mathsf{L}}$, by the definition of the SWF substituting this R'' for R in a profile will not change the outcome. The same point applies if we substitute R' in the place of R. Thus by Lemma 1 we have 4-IND. $\square$

7 Final remarks

We have focused on the Blau (1971) equivalence, which concerns the impossibility of weakening binary independence by considering independence over sets of higher cardinality. We started by observing that this equivalence may vanish under domain restrictions, i.e., for some domains k-IND diverges from 2-IND for $2 < k < m$. Next, we provided, for any given domain D and any given value of k, a necessary and sufficient condition for a SWF to satisfy k-IND. We used this result to identify a necessary and sufficient condition which renders a domain D (k,l)-equivalent—a result which enables the determination of cases where the Blau equivalence holds. The Blau equivalence itself was defined in terms of Blau partitions; we also saw that

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6 Pairwise majority determines the ranking over pairs of alternatives based upon which is preferred by more voters.

7 A domain is single-crossing if its *preferences* can be listed $R^1, R^2, \ldots, R^t$—or *placed on a line*—such that, for all x and y, if xR^1y and yR^sx, then xR^iy for $s \leq i \leq t$. Gans and Smart (1996) provide some economic applications of this property. Rothstein (1991) shows that for any profile on a single-crossing domain there is a *representative voter* whose (strict) preferences coincide with the (strict) majority relation (though note he does not use the term single-crossing.

8 For all alternatives $x \neq y$, the set of equivalence classes of $\sim^{xy}_4$ on D_{u} is the following refinement of $\{D_{\mathsf{L}}, D_{\mathsf{R}}\}$: $\{\{R^{\mathsf{p}1}, R^{\mathsf{p}2}, R^{\mathsf{p}3}, R^{\mathsf{d}2}, R^{\mathsf{p}3}\}, \{R^{\mathsf{p}6}, R^{\mathsf{p}7}, R^{\mathsf{p}8}, R^{\mathsf{d}6}, R^{\mathsf{p}7}\}, \{R^{\mathsf{p}4}\}, \{R^{\mathsf{p}5}\}, \{R^{\mathsf{d}4}\}, \{R^{\mathsf{d}5}\}\}$.

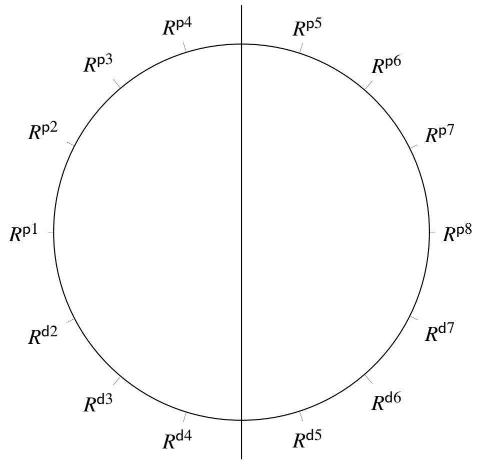

Fig. 1 The single-crossing line for D_{L} is the semicircle to the left; that for D_{R} is the semicircle to the right.

for almost any such partition there is a domain that exemplifies it. Finally, in the section preceding this one we turned attention to defining a particular five candidate Arrovian domain that violates the Blau equivalence, and a non-dictatorial SWF on this domain that is PO and 4-IND.

What results can we draw from our work? There is no logical dependence between the Blau equivalence and the Arrovian impossibility. We have seen domains where both hold (the full domain), where just the Blau equivalence holds (any maximal single-peaked domain), where just the Arrovian impossibility holds (the domain of Example 1 and the domain D_{u} in Section 6), and also where neither hold (the domains D_{p} and D_{d} in Section 6).

Let us restrict our attention to Arrovian domains. Here the Blau equivalence implies that the impossibility holds even for weaker versions of independence. Hence, when the Blau equivalence fails, a potential escape from Arrow's impossibility arises: there may be domains where Arrow's result holds, i.e. where 2-IND, Pareto optimality and non-dictatoriality are logically incompatible; but also where there exist non-dictatorial SWFs that satisfy Pareto optimality and some version of k-IND.

As was noted by Blau (1971), it is not interesting to weaken independence to m-IND, because this is trivially satisfied by any SWF. For example, on any domain, ranking by Borda scores is non-dictatorial and satisfies m-IND and PO, though this escape from Arrow's impossibility is somewhat unsatisfying. We consider this SWF to be similarly unsatisfying on the domain of Example 1. Although on this four candidate domain ranking by Borda scores satisfies 3-IND, so too will any other SWF: 3 and 4 are in the same equivalence class of the Blau partition of this domain. Blau's impossibility, when elaborated in line with the above strand of thought, properly states: there is no non-dicatorial, PO and k-IND SWF, for any k not in the equivalence class of the Blau partition that contains m. Thus Blau's impossibility holds on Arrovian domains where there are only two equivalence classes in the Blau partition, though

Section 5 shows that there are many domains with more than two such equivalence classes.

Section 6 gives an example of a non-trivial weakening of independence: a non-dictatorial SWF defined on a five candidate Arrovian domain that satisfies PO and 4-IND, where 4 is in a different equivalence class to 5 in the Blau partition. Thus Blau's impossibility is extensionally a stronger result than Arrow's impossibility; it applies to fewer domains. Of course, the domain and SWF we describe are (somewhat) designed to provide the necessary example. However, it is interesting to note the way that what started with a single-peaked and single-dipped domains changed into a single-crossing condition. These are all Condorcet domains, where the pairwise majority relation is transitive, but we see no particular reason why successful weakenings should involve domains of this type. We have found at least one other (less simply expressed) example on the domain D_{u}; we conjecture that there are many different types of successful weakenings.

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