Which dictatorial domains are superdictatorial? A complete characterisation for the Gibbard-Satterthwaite impossibility

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Abstract

A γ-dictatorial domain is one over which the Gibbard-Satterthwaite impossibility can be proven. A γ-dictatorial domain whose superdomains are all γ-dictatorial is qualified to be γ-superdictatorial. We provide a complete characterization of γ-superdictatorial product domains.

Keywords: Gibbard-Satterthwaite, Strategyproofness, Restricted domains, Superdictatoriality

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1. Posing the problem

The non-existence of interesting strategy-proof social choice functions, established by Gibbard (1973) and Satterthwaite (1975), depends on the preferences which individuals are allowed to have. In fact, there is a large literature which analyses the Gibbard-Satterthwaite (GS) impossibility under various domains of preferences – a key result of which is by Aswal et al. (2003) who provide a sufficient condition for a domain to exhibit the GS impossibility.[1]

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[1] See Gaertner (2001) for a comprehensive study of domain restrictions in social choice theory.

We present an analysis within this literature regarding a property called *superdictatoriality.* Write $\mathcal{L}(A)$ and $\mathcal{W}(A)$ for the set of linear and weak orders, respectively, over a given set of alternatives A. We assume $|A| \geq 3$. Any $R \in \mathcal{W}(A)$ represents a preference over A; we write P for the strict part of R. A *profile* $\boldsymbol{R} \in \mathcal{W}(A)^N$ is a vector of preferences, one for each voter $i \in N$, with $|N| \geq 2$. Given a profile $\boldsymbol{R}$ and preference $R'_i \in \mathcal{W}(A)$, we write $(\boldsymbol{R}_{-i}, R'_i)$ for the profile obtained by changing voter i's preferences from R_i to R'_i. A *social choice function* (SCF) is some $f\colon U \to A$, where the domain is some non-empty $U \subseteq \mathcal{W}(A)^N$. A domain U is a *product domain* if $U = \prod_{i \in N} \mathcal{D}_i$, where for each $i \in N$, $\emptyset \neq \mathcal{D}_i \subseteq \mathcal{W}(A)$. *Power domains* are the special case where all voters have the same set of admissible preferences, i.e. $\mathcal{D}_i = \mathcal{D}_j$ for all $i, j \in N$. *Linear order domains* only contain profiles with linear order preferences. Denote by $\mathscr{P}^*$ the set of power domains, $\mathscr{P}$ the set of product domains, $\mathscr{L}$ the set of linear order domains, and $\mathscr{U}$ the set of all possible domains; note $\mathscr{P}^* \subset \mathscr{P} \subset \mathscr{U}$. We write $\mathsf{top}(R) = \{x \in A : \text{ there is no } y \in A \text{ with } yPx\}$ for the top indifference class of $R \in \mathcal{W}(A)$. A SCF $f\colon U \to A$ is *dictatorial* if there exists $i \in N$ such that for all $\boldsymbol{R} \in U$, $f(\boldsymbol{R}) \in \mathsf{top}(R_i)$; f is *manipulable* if there is some $\boldsymbol{R} \in U$ and some $R'_i \in \mathcal{W}(A)$ with $(\boldsymbol{R}_{-i}, R'_i) \in U$ such that $f(\boldsymbol{R}_{-i}, R'_i) P_i f(\boldsymbol{R})$. An SCF is *strategyproof* if it is not manipulable. A domain U is γ*-dictatorial* if every strategyproof and onto SCF over U is dictatorial. Given a family of domains $\mathscr{D}$, a domain $U \in \mathscr{D}$ is γ*-superdictatorial on* $\mathscr{D}$ if every superdomain $U' \supseteq U$, $U' \in \mathscr{D}$ is γ-dictatorial.

The failure of γ-superdictatoriality amounts to the ability of overcoming the GS non-existence result by enlarging the domain over which SCFs are defined. This is a rather counterintuitive possibility. So superdictatoriality is a concept of interest precisely because it is not trivially satisfied. The first observation in this direction is made by Bordes and Le Breton (1990) who consider Arrow's result in economic environments where domains necessarily admit indifferences. They define superdictatoriality for the Arrovian impossibility and show that the power domain $\mathcal{L}(A)^N$ is not Arrovian superdictatorial. In other words, they show that proving Arrow's impossibility over $\mathcal{L}(A)^N$ does not automatically imply that every subdomain of $\mathcal{W}(A)^N$ that contains $\mathcal{L}(A)^N$ is dictatorial. This observation suggests caution in deriving conclusions about the Arrovian impossibility for economic environments based on results for voting environments.[2]

Coming back to the GS impossibility, Beja (1993) considers weak orders and shows that for any $\mathcal{D}$ where $\mathcal{L}(A) \subseteq \mathcal{D} \subseteq \mathcal{W}(A)$, D^N is γ-dictatorial, which establishes that $\mathcal{L}(A)^N$, as well as any superdomain of $\mathcal{L}(A)^N$, is γ-superdictatorial on $\mathscr{P}^*$.[3] Later, Sanver (2007) describes sets $\mathcal{D} \subset \mathcal{D}' \subset \mathcal{L}(A)$ where $\mathcal{D}^N$ is γ-dictatorial but $\mathcal{D}'^N$ is not, thereby establishing on $\mathscr{P}^* \cap \mathscr{L}$ the existence of γ-dictatorial domains that are not γ-superdictatorial. He goes on to provide a full characterization of γ-superdictatoriality on $\mathscr{P}^* \cap \mathscr{L}$. So the literature answers the γ-superdictatoriality question for $\mathcal{L}(A)^N$ and its superdomains on $\mathscr{P}^*$ and for $\mathcal{L}(A)^N$ and its subdomains on $\mathscr{P}^* \cap \mathscr{L}$. We complete the picture by characterising γ-superdictatoriality on product domains. This characterization represents a unification of the previous results through two corollaries: one strengthens Beja's (1993) result by providing a characterization of γ-superdictatoriality on $\mathscr{P}^*$; the other recreates Sanver's (2007) characterization of γ-superdictatoriality on $\mathscr{P}^* \cap \mathscr{L}$. We also argue that a meaningful characterization for arbitrary domains is beyond reach.

2. The result

A product domain $U \in \mathscr{P}$ is *regular* if for all $x \in A$ and all $i \in N$, there exists $P_i \in \mathcal{D}_i$ such that $\mathsf{top}(P_i) = \{x\}$.[4]

[2]Following Bordes and Le Breton (1990), Kelly (1994) gives a full characterization of the power domains between $\mathcal{L}(A)$ and $\mathcal{W}(A)$ over which the Arrovian impossibility applies. Also on this theme, Ozdemir and Sanver (2007) provide a sufficient condition for the superdictatoriality of linear order domains.

[3]Beja (1993) also considers superdictatoriality with respect to the Arrovian impossibility.

[4]We extend Sanver's (2007) definition of regularity from power domains to product domains. The condition (for power domains) is also known as "minimal richness" (Aswal et al., 2003).

Theorem 2.1. *A product domain $U \in \mathscr{P}$ is γ-dictatorial and regular if and only if U is γ-superdictatorial on $\mathscr{P}$.*

The following lemma concerns *unanimous* SCFs; those f such that for all $x \in A$ and $\boldsymbol{P} \in U$, if $\mathsf{top}(P_i) = \{x\}$ for all $i \in N$ then $f(\boldsymbol{P}) = x$.

Lemma 2.2. *Any onto and strategyproof SCF $f\colon U \to A$ is also unanimous.*

Proof. Suppose that $\mathsf{top}(P_i) = \{x\}$ for all $i \in N$. Take a profile $\boldsymbol{P'}$ such that $f(\boldsymbol{P'}) = x$ which exists as f is onto. Let $\boldsymbol{P}^0 = \boldsymbol{P'}$ and $\boldsymbol{P}^i = (\boldsymbol{P}^{i-1}_{-i}, P_i)$ for $i \in N$. Then for each i, $f(\boldsymbol{P}^i) = x$, as otherwise voter i could manipulate to $\boldsymbol{P}^{i-1}$. □

Proof of theorem. First we prove the "only if" part. Consider an arbitrary domain $U = \prod_{i \in N} \mathcal{D}_i$ that is regular and γ-dictatorial. Take a superdomain $U' = \prod_{i \in N} \mathcal{D}'_i$ that is only different for a single agent k, that is $D_k \subset D'_k$ and for all $i \neq k$, $D_i = D'_i$. We claim, and show below, that U' is γ-dictatorial. As U' is also obviously regular, we can reapply this claim to U' and a further superdomain that only differs from U' for a single agent. Repeated applications of the claim in this manner imply that any super domain $U'' = \prod_{i \in N} \mathcal{D}''_i$ of U with $D_i \subseteq D''_i$ for each i is dictatorial, as required.

We here show that U' is γ-dictatorial. Let $f'\colon U' \to A$ be an onto and strategyproof function, we want to show that f' is dictatorial. Let f be the restriction of f' to U, that is $f\colon U \to A$ such that $f(\boldsymbol{P}) = f'(\boldsymbol{P})$ for all $\boldsymbol{P} \in U$. By Lemma 2.2 f' is unanimous. As U is regular this implies that f is onto. Because f' is strategyproof it follows that f is strategyproof. As U is a γ-dictatorial domain it follows that f is dictatorial; we refer to the dictator as individual d. We prove that f' is dictatorial with the same dictator d. Let $\boldsymbol{P'}$ be a profile in U'. It suffices to show that $f'(\boldsymbol{P'}) \in \mathsf{top}(P'_d)$. We consider two cases.

1. Suppose that $d = k$. Take a profile $\boldsymbol{P}$ in U such that $P_i = P'_i$ for $i \neq k$, and such that $\mathsf{top}(P_k) = \{x\}$, where $x \in \mathsf{top}(P'_k)$, such a $P_k \in D_k$ exists by regularity. Because f is dictatorial with dictator $d = k$ we have $f(\boldsymbol{P}) = x$.

Thus $f'(\boldsymbol{P}) = x$. Thus $f'(\boldsymbol{P}') \in \mathsf{top}(P'_k)$, because otherwise agent k could manipulate to $\boldsymbol{P}$.

2. Suppose that $d \neq k$. Take $\boldsymbol{P}$ in U such that $P_i = P'_i$ for $i \neq k$, and such that $\mathsf{top}(P_k) = \{f'(\boldsymbol{P}')\}$. Write $y = f(\boldsymbol{P}) = f'(\boldsymbol{P})$ and $z = f'(\boldsymbol{P}')$. By strategyproofness yR_kz. As $\mathsf{top}(P_k) = \{z\}$, this implies that $z = y$. By the dictatoriality of d on f we have $y \in \mathsf{top}(P_d)$. By construction $\mathsf{top}(P_d) = \mathsf{top}(P'_d)$. By the established equalities $f'(\boldsymbol{P}') \in \mathsf{top}(P'_d)$ as required.

For the "if" part we prove the contrapositive. First, if $U \in \mathscr{P}$ is not γ-dictatorial then it is trivially not γ-superdictatorial on $\mathscr{P}$. So suppose $U = \prod_{i \in N} \mathcal{D}_i$ is not regular. We generalise the example given by Sanver (2007), thereby providing a product superdomain of U that is not γ-dictatorial as required. As U is not regular there is a voter, without loss of generality voter 1, and an alternative $a \in A$ such that there is no $P \in \mathcal{D}_1$ such that $\mathsf{top}(P) = \{a\}$. Let f be an SCF with dictator 1 such that (i) it is independent of the other preferences, i.e. $f(\boldsymbol{P}_{-1}, P_1) = f(\boldsymbol{P}'_{-1}, P_1)$ and (ii) it never selects a as a winner, i.e. if a is tied first in voter 1's ranking it selects a different alternative. Denote the range of f by $f(U)$ and let $X = A \backslash f(U)$ and $b \in f(U)$. Note $a \in X \neq \emptyset$. For each $x \in X$ consider a strict preference P^x such that xP^xbP^xy for all $y \neq x, b$. Denote by P^b any strict preference such that bP^by for all $y \neq b$. We add the strict preferences P^b and P^x for each $x \in X$ to each voter's domain; i.e. define $\mathcal{D}'_i = \mathcal{D}_i \cup \{P^b\} \cup \bigcup_{x \in X} \{P^x\}$. Now define $f' \colon \prod_{i \in N} \mathcal{D}'_i \to A$ by

$$f'(\boldsymbol{P}) = \begin{cases} f(\boldsymbol{P}'_{-1}, P_1) & \text{if } P_1 \in \mathcal{D}_1, \text{ where } \boldsymbol{P}'_{-1} \in \prod_{i \neq 1} \mathcal{D}_i, \\ b & \text{if } P_1 = P^b \\ \max_{P_2}\{x, b\} & \text{if } P_1 = P^x \text{ for some } x \in X. \end{cases}$$

Note the first possibility is unambiguous by (i) above. We now verify that f' is strategyproof, non-dictatorial and onto. For strategyproofness, note that if $P_1 \neq P^x$ for some $x \in X$, then no voter can manipulate as 1 acts as a dictator. If $P_1 = P^x$ then either x or b is returned and no voters other than 1 or 2 can affect this outcome. Voter 2 cannot manipulate as they can only change the choice between these two alternatives and their preferred is already returned. If x is returned voter 1 trivially cannot manipulate. If b is returned, voter 1 only prefers x, but as is no profile where $\boldsymbol{P} \in U$ where $f(\boldsymbol{P}) = x$ voter 1 cannot make x winning. This case also shows that voter 1 is not a dictator, and nor are any of the other voters: consider $f(P^b, P^a, P^a, \dots, P^a) = b$. It remains to show that f' is onto: for any $x \in X$, we have $x = f(P^x, P^x, \dots)$. For any $x \notin X$, there is some profile $\boldsymbol{P} \in U$ such that $x = f(\boldsymbol{P}) = f'(\boldsymbol{P})$. □

The "only if" part of Theorem 2.1 equally applies to any subsets of $\mathscr{P}$. Regarding the "if" part, we note that the counterexample constructed in the proof of Theorem 2.1 adds the same set of linear orders to every voter i's potential preferences $\mathcal{D}_i$. Thus if the original domain is in $\mathscr{P}^*$ so too is the constructed domain; this also applies to $\mathscr{L}$. Hence, the "if" part of Theorem 2.1 also applies to these subdomains.[5] Thus we have the following corollaries: Corollary 2.3 directly implies the results of Beja (1993); Corollary 2.4 recreates the characterization of Sanver (2007).

Corollary 2.3. *A power domain $U \in \mathscr{P}^*$ is γ-dictatorial and regular if and only if U is γ-superdictatorial on $\mathscr{P}^*$.*

Corollary 2.4. *A linear order power domain $U \in \mathscr{P}^* \cap \mathscr{L}$ is γ-dictatorial and regular if and only if U is γ-superdictatorial on $\mathscr{P}^* \cap \mathscr{L}$.*

[5]Also concerning the "if" part of Theorem 2.1, a referee has pointed our attention to similarities to the notion of "inseparable pairs" described by Kalai and Ritz (1980). In order for a pair of alternatives a and b to be inseparable, a must be ranked above b in some ordering, but whenever a is ranked above b there must be no alternatives ranked between a and b. Kalai and Ritz (1980) show that this is a sufficient condition for non-dictatoriality given the specific Arrovian framework they define. We are working in a choice theoretic framework; we thus modify the notion to that of "*top*-inseparable-pairs", i.e. a and b such that a is sometimes ranked top but whenever this is the case b is ranked immediately below it. We in effect show that if a (product) domain contains a top-inseparable-pair (for some voter) then it is not γ-dictatorial. This leads to the main result: if a domain is not regular, it is easy to create a superdomain that has a top-inseparable-pair, and thus the original domain is not γ-superdictatorial.

A meaningful characterization for arbitrary domains seems beyond reach. Manipulation is defined with respect to two profiles that disagree on the preference of a single voter; this fact can be readily exploited to construct examples of γ-dictatorial domains that fail γ-superdictatoriality on the family of arbitrary domains $\mathscr{U}$. We make the point by describing a (non-product) superdomain of $\mathcal{L}(A)^N$ that is not γ-dictatorial. For some $a \in A$, let P^a be the preference where a is at the top and all other alternatives are indifferent and P^* the preference of complete indifference. Consider $U = \mathcal{L}(A)^N \cup \{(P^a, P^*, P^*, \dots)\}$. Trivially no voter can manipulate from a profile in $\mathcal{L}(A)^N$ to $(P^a, P^*, P^*, \dots)$ or vice versa. Define $f\colon U \to A$ as a dictatorship for voter 1 if the profile is in $\mathcal{L}(A)^N$ and as some $b \neq a$ otherwise. This is non-dictatorial, strategyproof and onto.

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