Timed possibilistic logic

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Abstract. This paper is an attempt to cast both uncertainty and time in a logical framework. It generalizes possibilistic logic, previously developed by the authors, where each classical formula is associated with a weight which obeys the laws of possibility theory. In the possibilistic temporal logic we present here, each formula is associated with a time set (a fuzzy set in the more general case) which represents the set of instants where the formula is certainly true (more or less certainly true in the general case). When a particular instant is fixed we recover possibilistic logic. Timed possibilistic logic generalizes possibilistic logic also in the sense that we substitute the lattice structure of the set of the (fuzzy) subsets of the temporal scale to the lattice structure underlying the certainty weights in possibilistic logic. Thus many results from possibilistic logic can be straightforwardly generalized to timed possibilistic logic. Illustrative examples are given.

1. Introduction

Although temporal knowledge may be pervaded with imprecision and uncertainty as any kind of knowledge, there have been very few works trying to handle uncertainty in temporal reasoning ; among exceptions let us mention Kandrashina [26] who tried to characterize approximate equality and inequality relations between time points in an axiomatic manner, Fall [21] who propagates uncertainty and imprecision along the temporal axis using belief functions, Dutta [19], [20] who has more recently modeled the lack of knowledge about events by means of fuzzy sets of time intervals, and Dean and Kanazawa [8] who use a probabilistic model for representing the propensity of a formula to persist in being true. Also, Borillo and Gaume [6] allow for a non-graded treatment of incomplete information in a calculus of events.

This situation seems to be mainly due to the fact that the handling of time and the management of uncertainty are two distinct important issues, each of them raising its own specific problems whose solutions require a lot of research efforts. However a proposal has been made for the representation of imprecise or uncertain temporal knowledge in the framework of possibility theory (see [15]) where fuzzily-known dates, time intervals with ill-known bounds, uncertain precedence relations between events can be handled. Nevertheless the above-mentioned proposal was not formalized in a logical framework but was rather based on Zadeh's [31] approach to approximate reasoning using possibility distributions for the representation of fuzzy incomplete knowledge. In the meantime a so-called possibilistic logic (Dubois and Prade [13], Dubois, Lang and Prade [9], [11], [27]) has been developed for the treatment of uncertain formulas and has been shown to logically embed an important part of Zadeh's approximate reasoning machinery [10].

Logical formalisms for the treatment of time (e.g. Bestougeff and Ligozat [5]) can be classified in three categories :

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(i) classical logic approaches where time is handled as an ordinary logical variable and where the specificity of time is not acknowledged ;
(ii) modal logics where time indexes the worlds where a formula is true and modal operators capture the notions of past and future (e.g. Audureau, Enjalbert and Fariñas del Cerro, [3]) ;
(iii) reified logics (McDermott [28], Allen [2]) where for instance the time interval in which a formula holds for true is explicitly associated to the formula. This approach, which presents obvious advantages, is very commonly used in Artificial Intelligence (e.g. Shoham [29], Joubel and Raiman [25]).

What is proposed in this paper is a kind of reified temporal logic which can be viewed as a generalization of possibilistic logic, and which as such can deal both with time and uncertainty in a unified manner. A possibilistic formula is a pair composed of a classical logic formula and a certainty degree between 0 and 1 which is interpreted as a lower bound of a necessity measure and which, as such, is governed by the axiomatics of these measures. A *timed* formula is similarly a pair made of a classical logic formula and a set-representation of a time period in which the formula is *certainly* true ; the time period can be more generally a fuzzy set of time points. The degree of membership of a time-point is then viewed as the degree of certainty of the formula at this time point. The algebraic structures which underlie the second element of the pair have common properties in possibilistic logic and in timed possibilistic logic, namely it is a lattice structure. It enables us to generalize results of possibilistic logic to timed possibilistic logic. The word "timed" is used here as being more specific than "temporal" : temporal reasoning is more general than "timed logic" since the latter only handles "timed formulae" while the former is supposed to handle statements about time like precedence constraints.

The paper is organized as follows : Section 2 gives the necessary background on possibilistic logic ; Section 3 presents a timed logic where formulas are associated with ordinary time sets, while Section 4 considers a fuzzy timed logic. Examples illustrate the approach.

2. Possibilistic Logic

In this section we give the formal definitions and the main results of possibilistic logic (see [11] for a complete exposition).

2.1. Possibilistic Logic : Basic Ideas

From now on, $\mathcal{L}$ will denote a propositional or first-order logical language, consisting of closed well-formed formulae (in the classical sense). Let Ω be the set of interpretations for this language.

A *possibility distribution* on Ω is simply a function π from Ω to [0,1] ; π is said to be *normalized* if and only if $\exists\ \omega \in \Omega$ such that $\pi(\omega) = 1$; the quantity $SN(\pi) = 1 - \sup\{\pi(\omega) \mid \omega \in \Omega\}$ is called sub-normalization degree of π ; it is equal to 0 if π is normalized. π reflects the available knowledge and $\pi(\omega)$ estimates the extent to which it is possible that the interpretation ω corresponds to the one underlying the real world. Since Ω

is exhaustive, π should be normalized : at least one interpretation should be fully possible. A possibility distribution encodes a preferential ordering among the interpretations ; see [16].

A possibility distribution π on Ω gives birth to two functions from $\mathscr{L}$ to [0,1], called *possibility and necessity measures* [30],[14] and denoted respectively by Π and N, defined by :

$$\Pi : \mathscr{L} \to [0,1] \qquad (\forall\varphi \in \mathscr{L}) \quad \Pi(\varphi) = \sup\{\pi(\omega) \mid \omega \models \varphi\}$$

$$N : \mathscr{L} \to [0,1] \qquad (\forall\varphi \in \mathscr{L}) \quad N(\varphi) = 1 - \Pi(\neg\varphi) = \inf\{1 - \pi(\omega) \mid \omega \models \neg\varphi\}.$$

While the possibility estimate $\Pi(\varphi)$ measures the extent to which φ is compatible with the available knowledge (describing what we know about the real world), the dual necessity (or certainty) measure $N(\varphi)$ estimates to what extent φ is entailed by the available knowledge.

A possibility measure Π satisfies the following properties :

(i) $\Pi(\bot) = 0$; $\Pi(\top) = 1$

(ii) $\forall\varphi, \forall\psi, \Pi(\varphi \vee \psi) = \max(\Pi(\varphi),\Pi(\psi))$[1]

where $\bot$ and $\top$ denote contradiction and tautology respectively. We emphasize that we only have $\Pi(\varphi \wedge \psi) \leq \min(\Pi(\varphi),\Pi(\psi))$ in the general case, and $\Pi(\varphi \wedge \psi)$ is not a function of $\Pi(\varphi)$ and $\Pi(\psi)$ only. This completely departs from fully truth-functional calculi like multiple-valued logics. Axiom (ii) is then equivalent to $\forall\varphi, \forall\psi, N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$ and, as a consequence, letting $\psi = \neg\varphi$, we get $N(\varphi) > 0 \Rightarrow \Pi(\varphi) = 1$. Moreover we only have $N(\varphi \vee \psi) \geq \max(N(\varphi), N(\psi))$ by duality. We adopt the following conventions :

- $N(\varphi) = 1$ means that, given the available knowledge, φ is certainly true.
- $1 > N(\varphi) > 0$ that φ is somewhat certain and $\neg\varphi$ not certain at all (since the axioms imply that $\forall\varphi$, min $(N(\varphi), N(\neg\varphi)) = 0$).
- $N(\varphi) = N(\neg\varphi) = 0$ (equivalent to $\Pi(\varphi) = \Pi(\neg\varphi) = 1$) corresponds to the case of total ignorance ; it expresses that, from the available knowledge, nothing enables us to say if φ is rather true or rather false.
- $0 < \Pi(\varphi) < 1$ (equivalent to $1 > N(\neg\varphi) > 0$) means that φ is somewhat impossible, i.e. that $\neg\varphi$ is somewhat certain and φ not certain at all.
- $\Pi(\varphi) = 0$, means that φ is certainly false.

The use of min and max operators suggests that the precise values of the necessity (or possibility) degrees is not so important, the essential being the *ordering* on the formulae induced by the numbers.

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1 In the case $\mathscr{L}$ is infinite (e.g. in the first order case), (ii) has to be replaced by the more general (ii') $\Pi(\vee_{i\in I}\, \varphi_i) = \sup_{i\in I} \Pi(\varphi_i)$.

We point out that possibilistic logic is not a *fuzzy logic* strictly speaking, since fuzzy logic assigns *degrees of truth* in [0,1] to vague (non-classical) formulae, measuring their *conformity* with the reference knowledge, generally supposed to be complete. A statement like "it is 0.7-true that John is tall", which may be translated in fuzzy logic, by Truth(Tall(John)) = 0.7, expresses that the conformity of John's height with the interpretation of the vague predicate "tall" is 0.7, i.e. that, knowing John's height precisely, it is rather true that John is tall. Whereas the statement "it is 0.7-certain that the contract will be signed" may be translated in possibilistic logic, by N(Signed(contract)) ≥ 0.7, expressing that the *non-vague* event "the contract will be signed" is *rather certain*, and not *rather true*.

2.2. Possibilistic Logic : Language and Semantics

Let us define a *necessity-valued formula* as a pair $(\varphi\ \alpha)$, where φ is a classical propositional or first-order closed formula of $\mathcal{L}$, and α a valuation in [0,1]. A *necessity-valued knowledge base* is then defined as a finite set (in the conjunctive meaning) of necessity-valued formulae ; these are the basic elements of the language of *necessity-valued possibilistic logic*, which is a fragment of (general) possibilistic logic. The latter also involves possibility-valued formulae ; see [11], [27].

Let us now consider the semantic aspects of necessity-valued logic. Necessity-valued formulae will be interpreted by means of possibility distributions. Let π be a possibility distribution on Ω (not necessarily normalized), and $(\varphi\ \alpha)$ a necessity-valued formula. Then we define the notion of satisfaction by :

$$\pi \models (\varphi\ \alpha) \quad \text{iff} \quad N(\varphi) \geq \alpha$$

where N is the necessity measure induced by π. If $\mathcal{F} = \{(\varphi_1\ \alpha_1), \ldots, (\varphi_n\ \alpha_n)\}$ is a set of necessity-valued formulae then

$$\pi \models \mathcal{F} \quad \text{iff} \quad \forall\, i \in \{1, \ldots, n\}, \pi \models (\varphi_i\ \alpha_i).$$

Then, the notion of logical consequence is defined in a very natural way : $\mathcal{F}$ being a set of necessity-valued formulae and $(\varphi\ \alpha)$ a necessity-valued formula,

$$\mathcal{F} \models (\varphi\ \alpha) \quad \text{iff} \quad \forall\pi, \pi \models \mathcal{F} \text{ implies } \pi \models (\varphi\ \alpha)$$

i.e. the set of possibility distributions satisfying $\mathcal{F}$ is included in the set of possibility distributions satisfying $(\varphi\ \alpha)$.

Thus, the models of a set of necessity-valued formulas $\mathcal{F}$ are possibility distributions on the set Ω of all interpretations for $\mathcal{L}$. Measuring the consistency of $\mathcal{F}$ consists then in evaluating to what degree there is at least an interpretation completely possible for $\mathcal{F}$, i.e. to what degree the set of possibility distributions satisfying $\mathcal{F}$ contains normalized possibility distributions ; the quantity[2] $\text{Cons}(\mathcal{F}) = \sup_{\pi \models \mathcal{F}} \sup_{\omega \in \Omega} \pi(\omega)$ will be called *consistency degree* of $\mathcal{F}$, and its complement to 1,

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[2] The notations $\sup_{\pi \models \mathcal{F}}$, $\sup_{\omega \in \Omega}$ express that the supremum is taken among π such that $\pi \models \mathcal{F}$, and among ω such that $\omega \in \Omega$ respectively.

$$\text{Incons}(\mathcal{F}) = 1 - \sup_{\pi \vDash \mathcal{F}} \sup_{\omega \in \Omega} \pi(\omega) = \inf_{\pi \vDash \mathcal{F}} SN(\pi)$$

is called the *inconsistency degree* of $\mathcal{F}$. Thus, necessity-valued logic enables the gradation of inconsistency. If Incons($\mathcal{F}$) = 0 then $\mathcal{F}$ will be said *completely consistent* ; if Incons($\mathcal{F}$) = 1 then $\mathcal{F}$ will be said *completely inconsistent*, and if $0 < \text{Incons}(\mathcal{F}) < 1$ then $\mathcal{F}$ will be said *partially inconsistent*. It easily comes down (see [27], [11] for the proofs) that

Proposition 2.1. : $\text{Incons}(\mathcal{F}) = \inf\{N(\bot) \mid \pi \vDash \mathcal{F}\}$ where N is the necessity measure induced by π.

Proposition 2.2. : let $\mathcal{F} = \{(\varphi_1\ \alpha_1), ..., (\varphi_n\ \alpha_n)\}$ be a set of necessity-valued formulas and let us define the possibility distribution $\pi^*_{\mathcal{F}}$ by

$$\begin{aligned} \pi^*_{\mathcal{F}}(\omega) &= \min\{1 - \alpha_i \mid \omega \vDash \neg\varphi_i, i = 1, \ldots, n\} \\ &= 1 \text{ if } \forall i, \omega \vDash \varphi_i ; \end{aligned}$$

then for any possibility distribution π on Ω, π satisfies $\mathcal{F}$ if and only if $\pi \leq \pi^*_{\mathcal{F}}$, i.e. $\forall\ \omega \in \Omega, \pi(\omega) \leq \pi^*_{\mathcal{F}}(\omega)$. $\pi^*_{\mathcal{F}}$ is said to be the least specific (i.e. the largest) possibility distribution satisfying $\mathcal{F}$.

Corollary 2.3. : $\mathcal{F} \vDash (\varphi\ \alpha)$ iff $\pi^*_{\mathcal{F}} \vDash (\varphi\ \alpha)$.

Corollary 2.4. : $\text{Incons}(\mathcal{F}) = 1 - \sup_{\omega \in \Omega} \pi^*_{\mathcal{F}}(\omega) = SN(\pi^*_{\mathcal{F}})$.

Then, computing the inconsistency degree of $\mathcal{F}$ reduces to compute the degree of subnormalization of the possibility distribution $\pi^*_{\mathcal{F}}$. The quantity $\pi^*_{\mathcal{F}}(\omega)$ represents the compatibility degree of ω with $\mathcal{F}$.

The inconsistency degree of a possibilistic knowledge base can be seen as a threshold below which any deduction is trivial : indeed, if Incons($\mathcal{F}$) = α, then any possibility distribution satisfying $\mathcal{F}$ verifies $N(\bot) \geq \alpha$ and a fortiori for any formula φ, $N(\varphi) \geq N(\bot) \geq \alpha$; thus, any deduction whose form is $\mathcal{F} \vDash (\varphi\ \beta)$ with $\beta \leq \alpha$ is trivial. Allowing non-trivial deductions only makes the consequence operator nonmonotonic (see [16] for a connection with nonmonotonic logics).

A lot of results can be proved about deduction in necessity-valued logic. They can be found in [11] or [27]. The most important ones extend deduction and refutation to possibilistic logic :

Proposition 2.6. (deduction) : $\mathcal{F} \cup \{(\varphi\ 1)\} \vDash (\psi\ \alpha)$ iff $\mathcal{F} \vDash (\varphi \rightarrow \psi\ \ \alpha)$.

Corollary 2.7. (refutation) : $\mathcal{F} \vDash (\varphi\ \alpha)$ iff $\mathcal{F} \cup \{(\neg\varphi\ 1)\} \vDash (\bot\ \alpha)$.

Thus, if we want to know whether $(\varphi\ \alpha)$ is a logical consequence of $\mathcal{F}$ or not, it is sufficient to compute the inconsistency degree of $\mathcal{F} \cup \{(\neg\varphi\ 1)\}$, which is equal to the largest α such that $\mathcal{F} \vDash (\varphi\ \alpha)$.

2.3 . Clausal Form and Resolution

A *necessity-valued* clause is a necessity-valued formula $(c\ \alpha)$ where c is a first-order clause. A *necessity-valued clausal form* is a finite set of necessity-valued clauses. If $(\varphi\ \alpha)$ is a necessity-valued formula and if $\{c_1, \ldots, c_n\}$ is a clausal form of φ then a clausal form of $(\varphi\ \alpha)$ is $\{(c_1\ \alpha), \ldots, (c_n\ \alpha)\}$; if $\mathcal{F}$ is a set of necessity-valued formulas then the set of necessity-valued clauses $\mathcal{C}$ obtained by replacing each necessity-valued formula by one of its clausal forms, is the clausal form of $\mathcal{F}$, and is proved to have the same inconsistency degree as $\mathcal{F}$. The resolution rule for necessity-valued possibilistic logic is the following :

$$(c_1\ \alpha_1)\ ,\ (c_2\ \alpha_2) \vdash (c'\ \min(\alpha_1, \alpha_2))$$

where c' is a resolvent of clauses c_1 and c_2. Possibilistic resolution for necessity-valued clauses is proved to be sound and complete for refutation, i.e. if $\mathrm{Incons}(\mathcal{C}) = \alpha$ then there is a deduction of $(\bot\ \alpha)$, called an α-refutation, from $\mathcal{C}$, and this refutation is optimal, i.e. there is no β-refutation from $\mathcal{C}$ where $\beta > \alpha$. See [11].

Let us notice first that the semantics of possibilistic logic only requires the definition of necessity measures on a logical language $\mathcal{L}$. Furthermore, to define these necessity measures from $\mathcal{L}$ to [0,1] we only needed three operations on [0,1] : the minimum and maximum operators (which underlie the *ordering structure*) and the *order reversing operation* $(1 - (\cdot))$. Thus, from a theoretical point of view, a straightforward generalization is to map possibility distributions and necessity measures, no longer into [0,1] but more generally into any *complete distributive lattice* L. In such a case we shall use the name "*lattice-based logics*".

In the following, we shall use the *Boolean lattice* $L = 2^T$, where T is a given set, equipped with the set intersection, union, and complementation (the ordering being the set inclusion). The reference set T will be interpreted in the rest of the section as a temporal scale (discrete or continuous) : T will be assumed to be totally ordered ; for the sake of simplicity we assume that T is a real closed interval $T = [T_{min}, T_{max}]$ where T_{min} and T_{max} may be equal to $-\infty$ or $+\infty$. In Section 4, instead of 2^T, we shall work with the set $[0,1]^T$ of fuzzy subsets of T.

3. A Reified Temporal Logic as a Lattice-Based Possibilistic Logic

Now we actualize the definitions of Section 2, in the framework of Boolean-valued possibilistic logic, keeping in mind the temporal interpretation. $\mathcal{L}$ still denotes a logical propositional or first-order language, Ω the set of interpretations associated with $\mathcal{L}$. The valuation lattice L is the Boolean lattice 2^T, equipped with the inclusion ordering $(\subset)$ (note that this symbol will denote *non-strict inclusion*), the union $(\cup)$, intersection $(\cap)$, and complementation (denoted by an overbar) operations.

A *temporal possibility distribution* (or temporal distribution, for short) on $\mathcal{L}$ is a mapping $\pi : \Omega \to L$; π is said to be normalized iff $\cup\{\pi(\omega), \omega \in \Omega\} = T$; otherwise, the subnormalization level of π is defined as :

$$SN(\pi) = \overline{\cup \{\pi(\omega), \omega \in \Omega\}} = \cap \{\overline{\pi(\omega)}, \omega \in \Omega\}.$$

A *temporal (possibility) distribution* is then an allocation of a set of time instants to each interpretation. Writing down that $\pi(\omega) = \tau$ means that at any instant t of τ the interpretation ω is not excluded (it may be the *actual* state of facts at time t) and that at any instant of $\bar{\tau}$, ω is completely excluded. When π is normalized there is an interpretation that is considered as possible at any time instant. If a temporal distribution is subnormalized then there are instants t in T such that no interpretation may be the actual state of facts at time t : hence these instants are in a situation of inconsistency. More precisely, the set of *inconsistent instants* according to π is $SN(\pi)$, also called *inconsistency lapse* of π.

The *temporal possibility function* induced by π is the mapping $\Pi : \mathcal{L} \to L$ defined by :

$$\Pi(\varphi) = \cup \{\pi(\omega), \omega \vDash \varphi\}.$$

It is the set of instants when φ is possibly true (or equivalently, when we do not know explicitely that φ is false). The *temporal certainty function* induced by π is the mapping $N : \mathcal{L} \to L$ defined by :

$$N(\varphi) = \overline{\cup \{\pi(\omega), \omega \vDash \neg\varphi\}} = \cap \{\overline{\pi(\omega)}, \omega \vDash \neg\varphi\}.$$

As the relation $N(\varphi) = \overline{\Pi(\neg\varphi)}$ suggests, $N(\varphi)$ is the set of instants when it is impossible that $\neg\varphi$ be true, or equivalently, when it is explicitely certain that φ is true. Using Π and N we can distinguish between the instants when we are sure that φ is true and the ones when it is only possible. It can be checked that the following properties hold :

i) $N(\top) = T$

meaning that tautologies are certain at any instant of T ;

ii) $N(\bot) = SN(\pi)$ (= $\varnothing$ if π is normalized)

meaning that contradictions are sure during an inconsistency lapse ;

iii) $N(\varphi \wedge \psi) = N(\varphi) \cap N(\psi)$

meaning that $\varphi \wedge \psi$ is certainly true when and only when both φ and ψ are certainly true ; in particular, $N(\varphi) \cap N(\neg\varphi) = N(\bot) = SN(\pi)$;

iv) $N(\varphi \vee \psi) \supset (N(\varphi) \cup N(\psi))$

meaning that $\varphi \vee \psi$ is certainly true whenever one of φ and ψ is known to be certainly true (but generally $N(\varphi \vee \psi) \neq N(\varphi) \cup N(\psi)$) ; for instance, if nothing is known about φ then $N(\varphi) = N(\neg\varphi) = \emptyset$ and however $N(\varphi \vee \neg\varphi) = N(\top) = T$;

v) if $\varphi \models \psi$ then $N(\varphi) \subset N(\psi)$

i.e. N is monotonic with respect to logical entailment ;

vi) $(N(\varphi) \cap N(\varphi \rightarrow \psi)) \subset N(\psi)$

which is a timed version of the modus ponens.

A *timed certainty-valued formula* (timed formula, for short) is a pair $(\varphi\ \tau)$ where φ is a classical propositional formula and τ is a subset of T, i.e. $\tau \subset T$. Writing down $(\varphi\ \tau)$ expresses that $N(\varphi) \supset \tau$, i.e. that φ is certainly true at least during τ. In practice τ will be an interval or a union thereof. A *timed certainty-valued knowledge base* is a finite set of timed certainty-valued formulas.

It is clear that our timed (certainty-valued) logic is nothing but a reified temporal logic, where the time component has been separated from the purely logical component. From that point of view it is similar to what is done in Joubel and Raiman [25], where time periods are considered as assumptions in an hypothetical reasoning system.

The semantics of the timed logic is easily defined, like in possibilistic logic : if π is a temporal possibility distribution then

$$\pi \models (\varphi\ \tau) \text{ iff } N(\varphi) \supset \tau, \text{ where N is induced by } \pi \text{ ;}$$
$$\pi \models \{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\} \text{ iff } \forall i, \pi \models (\varphi_i\ \tau_i) \text{ ;}$$
$$\{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\} \models (\psi\ \tau) \text{ iff } \pi \models \{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\} \text{ entails } \pi \models (\psi\ \tau).$$

Let $\mathcal{F} = \{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\}$ be a timed certainty-valued knowledge base and ψ a formula. Then the deduction problem is to find the greatest set of instants when, according to $\mathcal{F}$, the formula ψ is certainly true. This set will be denoted by $\mathrm{Cert}(\psi ; \mathcal{F})$. It comes immediately that

$$\mathrm{Cert}(\psi ; \mathcal{F}) = \cup \{\tau , \mathcal{F} \models (\psi\ \tau)\} = \cap \{N(\psi) , \text{N induced by } \pi \text{ and } \pi \models \mathcal{F}\}.$$

Taking $\psi = \bot$, $\mathrm{Cert}(\bot ; \mathcal{F}) = \cap \{SN(\pi) , \pi \models \mathcal{F}\}$ is the set of instants which are inconsistent according to any temporal distribution satisfying $\mathcal{F}$. It will be called the *inconsistency lapse of* $\mathcal{F}$ and denoted by $\mathrm{Incons}(\mathcal{F})$. Given a temporal distribution, from the fact that if $\varphi \models \psi$ then $N(\varphi) \subset N(\psi)$ and since $\bot \models \varphi$, it comes immediately that $\forall \varphi \in \mathcal{L}$, $N(\varphi) \supset N(\bot)$; so $\mathrm{Cert}(\psi ; \mathcal{F})$ always contains $\mathrm{Incons}(\mathcal{F})$. As a consequence, $\mathcal{F} \models (\varphi\ \tau)$ with $\tau \subset \mathrm{Incons}(\mathcal{F})$ is a trivial deduction. On the contrary, $\mathcal{F} \models (\varphi\ \tau)$ is non-trivial if τ is not included in $\mathrm{Incons}(\mathcal{F})$, and completely non-trivial if $\tau \cap \mathrm{Incons}(\mathcal{F}) = \emptyset$. The non-trivial part of a timed formula $(\varphi\ \tau)$ deduced from a timed knowledge base $\mathcal{F}$ is the timed formula

$(\varphi\ \ \tau \setminus \text{Incons}(\mathcal{F}))$ where \ is the set difference. The greatest non-trivial set of instants where, according to $\mathcal{F}$, the formula φ is certainly true is

$$\text{Cert}^*(\varphi ; \mathcal{F}) = \text{Cert}(\varphi ; \mathcal{F}) \setminus \text{Incons}(\mathcal{F}).$$

Let us now give a simple characterization of the set of temporal possibility distributions satisfying $\mathcal{F}$. Let $\mathcal{F} = \{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\}$ then

$$\begin{aligned}
\pi \vDash \mathcal{F} \quad & \text{iff } \forall i,\ N(\varphi_i) \supset \tau_i \\
& \text{iff } \forall i,\ \cap\ \{\overline{\pi(\omega)},\ \omega \vDash \neg\varphi_i\} \supset \tau_i \\
& \text{iff } \forall\ \omega \in \Omega,\ \forall\ i \in \{1, \ldots, n\} \text{ such that } \omega \vDash \neg\varphi_i,\ \overline{\pi(\omega)} \supset \tau_i \\
& \text{iff } \forall\ \omega \in \Omega,\ \overline{\pi(\omega)} \supset (\cup\ \{\tau_i,\ \omega \vDash \neg\varphi_i\}) \\
& \text{iff } \forall\ \omega \in \Omega,\ \pi(\omega) \subset (\cap\ \{\overline{\tau_i},\ \omega \vDash \neg\varphi_i\}) \\
& \text{iff } \forall\ \omega \in \Omega,\ \pi(\omega) \subset \pi_{\mathcal{F}}^*(\omega) \text{ where}
\end{aligned}$$

$$\begin{aligned}
\pi_{\mathcal{F}}^*(\omega) &= \cap\ \{\overline{\tau_i},\ \omega \vDash \neg\varphi_i\} \\
&\quad (= T \text{ if } \forall i,\ \omega \vDash \varphi_i).
\end{aligned}$$

Thus, the set of temporal possibility distributions satisfying $\mathcal{F}$ has a maximal element $\pi_{\mathcal{F}}^*$. This characterization is interesting because

$$\text{Cert}(\varphi ; \mathcal{F}) = N_{\mathcal{F}}^*(\varphi)$$
$$\text{Incons}(\mathcal{F}) = N_{\mathcal{F}}^*(\bot) = SN(\pi_{\mathcal{F}}^*)$$

where $N_{\mathcal{F}}^*$ is the temporal necessity function induced by $\pi_{\mathcal{F}}^*$. The proof is similar to the one in possibilistic logic [11], [27]. Thus, the knowledge of $\pi_{\mathcal{F}}^*$ is sufficient for the deduction problem. $\pi_{\mathcal{F}}^*$ is the largest temporal distribution satisfying $\mathcal{F}$, in a sense very similar to the principle of minimum of specificity [17]. Indeed $N_{\mathcal{F}}^*(\varphi)$ contains all instants when we are certain that φ is true (according to the knowledge contained in $\mathcal{F}$) and all instants when $\mathcal{F}$ is inconsistent, and only these instants.

Instead of focusing on the set of time instants where a given formula is certainly true, we may dually consider the set of certainly true formulas at a given time $t \in T$. This set defined as $\{\varphi \mid t \in N_{\mathcal{F}}^*(\varphi)\}$ is a deductively closed, possibly inconsistent set of classical formulas representing what is known for sure at time t. It leads to a more intuituive view of the given semantics. Instead of considering $N_{\mathcal{F}}^* : \mathcal{L} \to 2^T$ we may consider, in an equivalent way, the collection of mappings $(N_t^*)_{t \in T}$ defined from $\mathcal{L}$ to $\{0,1\}$ by

$$N_t^*(\varphi) = \begin{cases} 1 \text{ iff } N_{\mathcal{F}}^*(\varphi) \text{ contains } t \text{ ;} \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that $N_t^*(\varphi) = 1$ if and only if according to $\mathcal{F}$, either it is certain that φ is true, or $\mathcal{F}$ is inconsistent at time t. It easily comes that for all t, N_t^* is a crisp (i.e. bivalued) necessity measure, i.e. a mapping from $\mathcal{L}$ to $\{0,1\}$ satisfying the axioms

$$N_t^*(\top) = 1$$
$$\forall \varphi, \psi \in \mathcal{L},\ N_t^*(\varphi \wedge \psi) = \min(N_t^*(\varphi), N_t^*(\psi)).$$

Then considering both $N_t^*(\varphi)$ and $N_t^*(\neg\varphi)$, four different situations may happen :

(i) $N_t^*(\varphi) = N_t^*(\neg\varphi) = 1$; since $N_t^*(\bot) = \min(N_t^*(\varphi), N_t^*(\neg\varphi)) = 1$, which is equivalent to $t \in \text{Incons}(\mathcal{F})$; it means that $\mathcal{F}$ is inconsistent at time t ;

(ii) $N_t^*(\varphi) = 1$ and $N_t^*(\neg\varphi) = 0$; it means that φ is certainly true at time t ;

(iii) $N_t^*(\varphi) = 0$ and $N_t^*(\neg\varphi) = 1$; it means that φ is certainly false at time t ;

(iv) $N_t^*(\varphi) = 0$ and $N_t^*(\neg\varphi) = 0$; it means that the truth value is completely unknown at time t.

Thus for a given time instant t, temporal necessity-valued logic comes down to a four-valued logic whose four "truth values" are *true*, *false*, *unknown* and *inconsistent* (Belnap [4]).

Let us now come back to the semantics of timed logic. The following results can be established in a similar way as in possibilistic logic :

Deduction theorem :

$\mathcal{F} \cup \{(\varphi\ T)\} \vDash (\psi\ \tau)$ if and only if $\mathcal{F} \vDash (\varphi \rightarrow \psi\ \ \tau)$

Refutation theorem :

$\mathcal{F} \vDash (\varphi\ \tau)$ if and only if $\mathcal{F} \cup \{(\neg\varphi\ T)\} \vDash (\bot\ \tau)$

or equivalently

$\text{Cert}(\mathcal{F},\varphi) = \text{Incons}(\mathcal{F} \cup \{(\neg\varphi\ T)\})$.

Thus, if we want to know whether $(\varphi\ \tau)$ is a logical consequence of $\mathcal{F}$ or not, it is sufficient to compute the inconsistency lapse of $\mathcal{F} \cup \{(\neg\varphi\ T)\}$, which is equal to the union of the sets of instants τ such that $\mathcal{F} \vDash (\varphi\ \tau)$.

As already pointed out, any deduction problem in timed logic comes down to computing an inconsistency lapse, which is the set of instants when the knowledge behaves inconsistently. In order to automatize the computation, we are going to define first a clausal form equivalent to a set of timed formulas $\mathcal{F}$, and then a resolution rule similar to those used in (numerical) possibilistic logic.

Clausal form

A timed clause is a timed formula $(c\ \tau)$ where c is a propositional clause. A timed clausal form is a finite set of timed clauses. It is possible to find a (propositional) timed clausal form, for any (propositional) timed necessity-valued knowledge base :

Let $\mathcal{F} = \{(\varphi_1\ \tau_1), \ldots, (\varphi_n\ \tau_n)\}$;

For every i = 1, …, n, let $\{c_{i,1}, \ldots, c_{i,n_i}\}$ be a clausal form of φ_i ;

Let $\mathcal{C} = \cup_{i=1,\ldots,n} (\cup_{j=1,\ldots,n_i} (c_{i,j}\ \tau_i))$

Then it can be proved that $\mathcal{C}$ is equivalent to $\mathcal{F}$.

Resolution rule

Let (c τ) and (c' τ') be two timed clauses, and let c'' be any classical resolvent (if any) of c and c' ; then (c'' $\tau \cap \tau'$) is a resolvent of (c τ) and (c' τ'), which is formally written :

$$(c\ \tau), (c'\ \tau') \vdash (c''\ \ \tau \cap \tau').$$

In order to get a sound and complete procedure we must add a combination rule :

Combination rule

$$(c\ \tau), (c\ \tau') \vdash (c\ \ \tau \cup \tau').$$

Resolution, together with combination, enjoys the properties of soundness and completeness for refutation :

Soundness and completeness of resolution & combination for refutation

$$\cup\ \{\tau, \mathcal{C} \vdash (\bot\ \tau)\} = \text{Incons}(\mathcal{C}).$$

Technically, this result means that in order to find the inconsistency lapse, an elementary algorithm consists in finding all the resolution paths leading to the empty clause, and then combining them all by computing their least upper bound. Of course, it is not always necessary to find all paths (for example, in the case where there exists a deduction of the empty clause with weight τ such that $\tau = \text{Incons}(\mathcal{C})$, only this refutation is necessary). An open problem is the designing of efficient algorithms based on ordered search methods in order to compute inconsistency lapses.

Example 1

Consider the following pieces of information :

John stayed in the office until 9.00 am ± 5 mn and certainly not afterwards
Mary stayed in the office from 8.50 am ± 10 mn and certainly not before
John and Mary never met in the office this morning.

In order to translate them into timed formulas, we will slightly rewrite this information : let us take the interval T = [8.00, 10.00] as temporal scale (assuming that nothing changes between 8.00 and 8.40 and between 9.05 and 10.00) ; the first two sentences will be expressed as

"John was in the office" is certainly true between 8.00 and 8.55
"John was not in the office" is certainly true between 9.05 and 10.00
"Mary was in the office" is certainly true between 9.00 and 10.00
"Mary was not in the office" is certainly true between 8.00 and 8.40

whilst the last one is expressed in the form

"John was not in the office or Mary was not in the office"
is certainly true between 8.00 and 10.00.

Note that writing that "John was not in the office" is certainly true between 9.05 and 10.00 is another way to express that "John was in the office" is possibly true between 8.00 and 9.05. Nothing is said about what happened to John between 8.55 and 9.05. He may have come in and get out of his office several times.

Let us now translate the five last sentences by means of timed formulas. For the sake of simplicity, we use closed intervals for modeling them. Assuming that the literals J (resp. M) means that John (resp. Mary) is at the office"; the formulas of the corresponding knowledge base $\mathcal{F}$ are

$$(J \quad [8.00,\ 8.55])$$
$$(\neg J \quad [9.05,\ 10.00])$$
$$(M \quad [9.00,\ 10.00])$$
$$(\neg M \quad [8.00,\ 8.40])$$
$$(\neg J \vee \neg M \ [8.00,\ 10.00])$$

We can now compute the temporal distribution $\pi_{\mathcal{F}}*$, knowing that

$$\pi_{\mathcal{F}}*(\omega) = \cap \{\overline{\tau_i}, (\varphi_i\ \tau_i) \in \mathcal{F}, \omega \models \neg\varphi_i\}).$$

Let us respectively denote $J \wedge M$, $J \wedge \neg M$, $\neg J \wedge M$, $\neg J \wedge \neg M$ the four interpretations of Ω meaning respectively that "J and M are given the value *true*", "J is given the value *true* and M the value *false*", etc. Then using the notations [a,b[,]a,b[for intervals open in b, open both in a and in b respectively, we have

$$\pi_{\mathcal{F}}*(J \wedge M) = [8.00, 9.05[\cap]8.40, 10.00] \cap \emptyset = \emptyset$$
$$\pi_{\mathcal{F}}*(J \wedge \neg M) = [8.00, 9.05[\cap [8.00, 9.00[= [8.00, 9.00[$$
$$\pi_{\mathcal{F}}*(\neg J \wedge M) =]8.55, 10.00] \cap]8.40, 10.00] =]8.55, 10.00]$$
$$\pi_{\mathcal{F}}*(\neg J \wedge \neg M) =]8.55, 10.00] \cap [8.00, 9.00[=]8.55, 9.00[$$

Since $\cup_\omega \pi_{\mathcal{F}}*(\omega) = [8,10]$, the inconsistency lapse of $\mathcal{F}$ is $\text{Incons}(\mathcal{F}) = \emptyset$, i.e. $\mathcal{F}$ never behaves inconsistently.

We may now compute the time intervals in which some formulas are certainly or possibly true :

$$N_{\mathcal{F}}*(J) = \overline{\pi_{\mathcal{F}}*(\neg J \wedge M)} \cap \overline{\pi_{\mathcal{F}}*(\neg J \wedge \neg M)}$$
$$= [8.00, 8.55] \cap ([8.00, 8.55] \cup [9.00, 10.00])$$
$$= [8.00, 8.55]$$
$$N_{\mathcal{F}}*(\neg J) = [9.00, 10.00]$$
$$N_{\mathcal{F}}*(M) = [9.00, 10.00]$$
$$N_{\mathcal{F}}*(\neg M) = [8.00, 8.55].$$

Thus, "John is in the office" is

- certainly true during [8.00, 8.55] ;
- certainly false during [9.00, 10.00] ;
- unknown (possibly true and possibly false) during]8.55, 9.00[;

this is stronger than the information explicitly written in the knowledge base (according to which "John is in the office" was certainly false during [9.05, 10.00] only).

Example 2

This example is somehow more complex than the first one since it involves contradictory information. Consider the following pieces of informations

John stayed in the office until 9.15 ± 10 mn and certainly not afterwards
Mary stayed in the office from 8.45 ± 10 mn and certainly not before
Between 9.00 and 9.10 there was exactly one person
(among John and Mary) in the office

which, in a manner similar to Example 1, gives the set $\mathcal{F}$ of timed formulas

$$\begin{array}{l}(J \quad [8.00,\ 9.05])\\(\neg J \quad [9.25,\ 10.00])\\(M \quad [8.55,\ 10.00])\\(\neg M \quad [8.00,\ 8.35])\\((\neg J \vee \neg M) \wedge (J \vee M)\ [9.00,\ 9.10])\end{array}$$

as represented on Figure 1.

The temporal distribution $\pi_{\mathcal{F}}*$ is then :

$$\begin{aligned}\pi_{\mathcal{F}}*(J \wedge M) &= [8.00, 9.25[\cap]8.35, 10.00] \cap ([8.00, 9.00[\cup]9.10, 10.00])\\ &=]8.35, 9.00[\cup]9.10, 9.25[\\ \pi_{\mathcal{F}}*(J \wedge \neg M) &= [8.00, 9.25[\cap [8.00, 8.55[= [8.00, 8.55[\end{aligned}$$

$$\pi_{\mathcal{F}}*(\neg J \wedge M) =]8.35, 10.00] \cap]9.05, 10.00] =]9.05, 10.00]$$

$$\pi_{\mathcal{F}}*(\neg J \wedge \neg M) =]9.05, 10.00] \cap [8.00, 8.55[\cap ([8.00, 9.00[\cup]9.10, 10.00]) = \emptyset$$

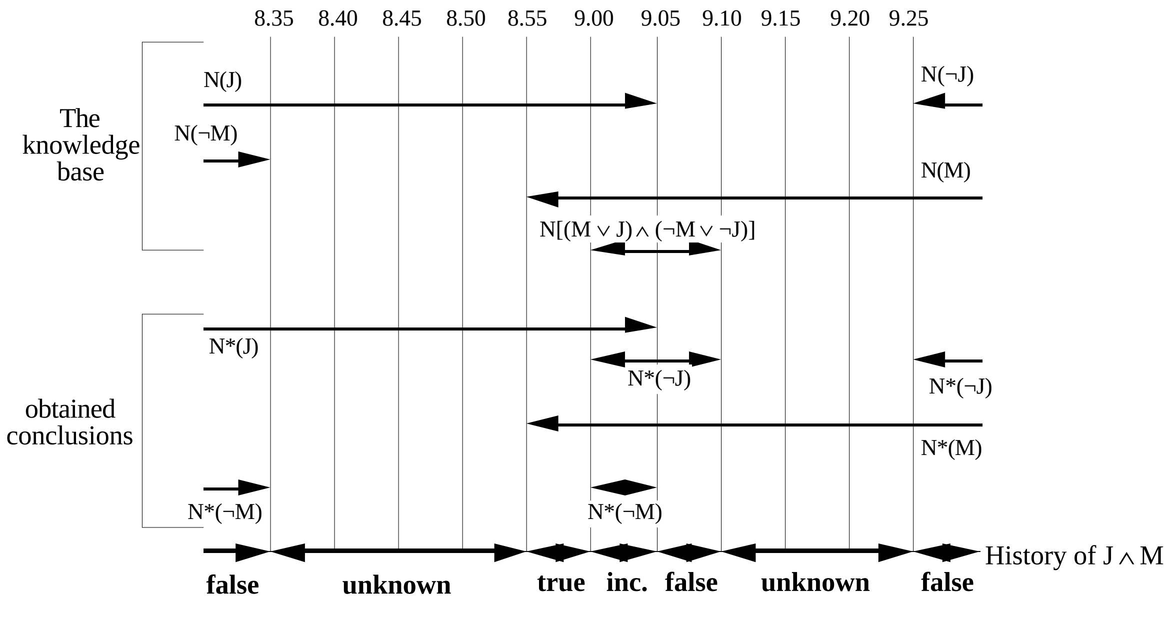


Figure 1

Let us compute the inconsistency lapse of $\pi_{\mathcal{K}}*$.

$$\cup_{\omega}\ \pi_{\mathcal{K}}*(\omega) = [8.00, 9.00[\ \cup\ ]9.05, 10.00]$$

Hence

$$\text{Incons}(\mathcal{K}) = \text{SN}(\pi_{\mathcal{K}}*) = [9.00, 9.05]$$

Thus the knowledge base behaves inconsistently between 9.00 and 9.05, due to the fact that it is explicitely written in $\mathcal{K}$ that both John and Mary are in the office and that there is exactly one person in it during this time interval. Then we have (see Figure 1 where N* is short for $N_{\mathcal{K}}*$)

$N_{\mathcal{K}}*(J) = [8, 9.05]$
$N_{\mathcal{K}}*(\neg J) = [9, 9.10] \cup [9.25, 10.00]$
$N_{\mathcal{K}}*(M) = [8.55, 10.00]$
$N_{\mathcal{K}}*(\neg M) = [8, 8.35] \cup [9.00, 9.05]$
$N_{\mathcal{K}}*(J \wedge M) = [8.55, 9.05]$
$N_{\mathcal{K}}*(\neg J \vee \neg M) = [8, 8.35] \cup [9.00, 9.10] \cup [9.25, 10.00]$

but since $\mathcal{K}$ is partially inconsistent, we have to drop out the inconsistent instants in order to keep only the non-trivial part of these deductions. For instance,

$\text{Cert}*(J \wedge M) = N_{\mathcal{K}}*(J \wedge M) \setminus \text{Incons}(\mathcal{K}) = [8.55, 9.00]$
$\text{Cert}*(\neg J \vee \neg M) = [8, 8.35] \cup\ ]9.05, 9.10] \cup [9.25, 10.00]$

Lastly, let us focus on $J \wedge M$; according to the last equalities, the state of knowledge $V*_t$ about $J \wedge M$ when considering a given instant t is

$$V*_t(J \wedge M) = \begin{cases} \text{inconsistent } \forall\, t \in [9.00, 9.05] \\ \text{true } \forall\, t \in [8.55, 9.00[\\ \text{false } \forall\, t \in [8.00, 8.35] \cup\ ]9.05, 9.10] \cup [9.25, 10.00] \\ \text{unknown } \forall\, t \in\ ]8.35, 8.55[\ \cup\ ]9.10, 9.25[\end{cases}$$

which is represented in Figure 1.

These results can be obtained using refutation by resolution :

(i) computing the maximal set of instants τ such that $\mathcal{K} \vDash (J \wedge M\ \ \tau)$ comes down to compute Incons ($\mathcal{K} \cup \{(\neg J \vee \neg M\ \ T)\}$). Let us consider the clausal form $\mathcal{C}$ of $\mathcal{K}$ where the last formula $((\neg J \vee \neg M) \wedge (J \vee M)\ [9.00, 9.10])$ has been transformed into $\{(\neg J \vee \neg M\ [9.00, 9.10]), (J \vee M\ [9.00, 9.10])\}$; let us then add $(\neg J \vee \neg M\ T)$ to $\mathcal{C}$ and let us find all deductions of the empty clause (Figure 2).

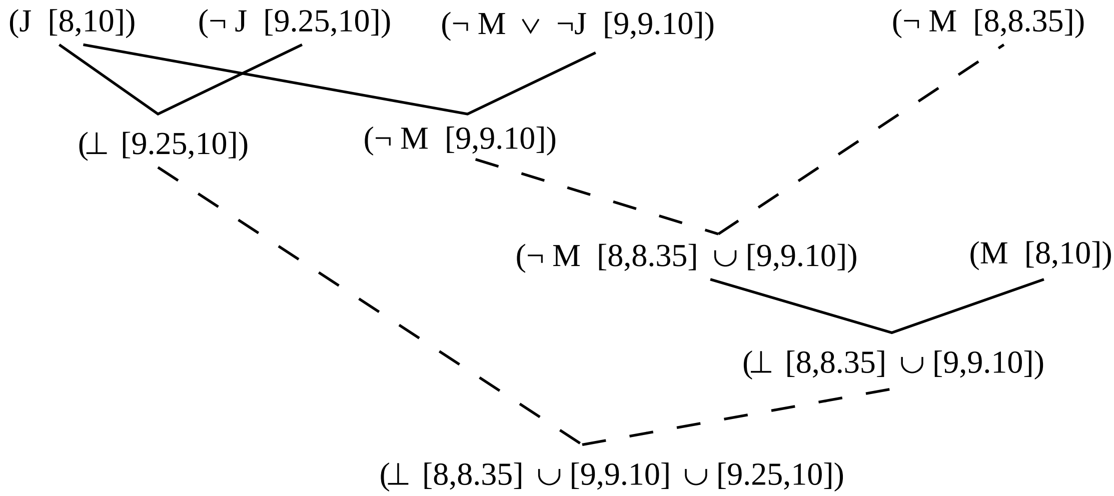
$(\neg J \vee \neg M\ [8,10])$ $(J\ \ [8,9.05])$

$(\neg M\ \ [8,9.05])$ $(M\ \ [8.55,10])$

$(\bot\ \ [8.55,9.05])$

Conclusion : $N_{\mathcal{K}}*(J \wedge M) = [8.55, 9.05]$

Figure 2



Conclusion : $N_{\mathcal{K}}*(\neg J \vee \neg M) = [8, 8.35] \cup [9.00, 9.10] \cup [9.25, 10.00]$

Figure 3

(ii) computing the maximal value τ such that $\mathcal{K} \models (\neg J \vee \neg M\ \tau)$ comes down to compute Incons($\mathcal{K} \wedge (J \wedge M\ \ T)$) where $(J \wedge M\ \ T)$ is equivalent to the two clauses {(J T), (M T)} (Figure 3).

Remark

Among worth-considering extensions, we may think of existentially quantified time instants, which would enable us for instance to deal with clipping problems like the following one : "φ and ψ cannot be simultaneously true", i.e. $(\neg\varphi \vee \neg\psi\ \ T)$, "φ is true in the interval $[t_1,x]$", where t_1 is a constant and x is unknown, i.e. $\exists x\ (\varphi\ [t_1,x])$, "ψ is true in the interval $[y,t_2]$", where t_2 is a constant and y is unknown, i.e. $\exists y\ (\psi\ [y,t_2])$, where $t_1 < t_2$. We want to establish that we should have $x < y$. Indeed resolution yields $(\bot\ [y,x])$, i.e. the knowledge base is consistent only if $y > x$, as well as $(\neg\varphi\ [y,t_2])$ and $(\neg\psi\ [t_1,x])$. This extension requires symbolic treatment of time periods.

5. A Reified Temporal Logic : the Fuzzy Case

It is not always realistic to assume that the interval where things happen are always perfectly defined. In the previous section "John left the office between 9.05 and 9.25" meant that it is completely possible that he left it at any time between 9.05 and 9.25, all instants being equally possible, and that it is completely impossible that he left his office at a time outside this interval. On the contrary we could wish to express that John left the office at "about 9.15", meaning, for example, that all times between 9.10 and 9.20 are completely possible, all times before 9.05 and after 9.25 completely impossible and, the closer to 9.05 (resp. 9.25) the less possible John left the office. This leads us to allow the interpretation of pieces of knowledge as formulas being certain during a *fuzzy time interval* (Dubois and Prade [15]), or more generally a fuzzy subset of the temporal scale.

The new structure is now defined as follows : T is still a temporal scale ; $L = [0,1]^T$ is the lattice of fuzzy subsets of T, equipped with the *fuzzy inclusion* ordering : if $\tilde{\tau}$ and $\tilde{\tau}'$

are two fuzzy subsets of T defined by their membership functions $\mu_{\tilde{\tau}}$ and $\mu_{\tilde{\tau}'}$ from T to [0,1], then

$$\tilde{\tau} \subset \tilde{\tau}' \text{ iff } \forall\, t \in T,\ \mu_{\tilde{\tau}}(t) \leq \mu_{\tilde{\tau}'}(t)$$

The lattice operators on L are the fuzzy intersection, union and complementation operators which are defined respectively by

$$\mu_{\tilde{\tau} \cap \tilde{\tau}'}(t) = \min(\mu_{\tilde{\tau}}(t), \mu_{\tilde{\tau}'}(t))$$
$$\mu_{\tilde{\tau} \cup \tilde{\tau}'}(t) = \max(\mu_{\tilde{\tau}}(t), \mu_{\tilde{\tau}'}(t))$$
$$\mu_{\bar{\tilde{\tau}}}(t) = 1 - \mu_{\tilde{\tau}}(t)$$

(L, $\cap$, $\cup$) is a complete distributive lattice but it is not Boolean since generally $\tilde{\tau} \cup \bar{\tilde{\tau}} \neq T$ and $\tilde{\tau} \cap \bar{\tilde{\tau}} \neq \emptyset$.

Now, the following definitions : temporal possibility distribution, temporal possibility function, temporal certainty function, timed (certainty-valued) formula, timed knowledge base , inconsistency lapse, logical consequence are formally the same as the ones given in Section 3. Incons($\mathcal{F}$) is now a fuzzy subset of T, meaning that among inconsistent instants, some are more inconsistent than others.

The resulting logical model is a *fuzzy reified temporal logic* that we shall name *fuzzy timed logic*. Let N be a temporal certainty function in the fuzzy case. N(φ) is a (non necessarily normalized) fuzzy subset of T. Then, given a formula φ and an instant t, the membership degree of t to N(φ) estimates to what extent φ is certainly true at time t. Note that this degree is meaningless if it is not strictly greater than the membership degree of t to the inconsistency lapse of the knowledge base (see Section 2). Considering a fixed time t, the mapping $\varphi \mapsto \mu_{N(\varphi)}(t)$ defines a necessity measure, satisfying the axioms given in Section 2. Thus, an intuitive way to consider a fuzzy temporal necessity function is to view it as a collection (indexed by time instants of T) of necessity measures N_t. Similarly, a fuzzy timed knowledge base $\mathcal{F} = \{(\varphi_i\ \tilde{\tau}_i), i = 1,n\}$ can be viewed as a collection of possibilistic knowledge bases $\mathcal{F}_t = \{(\varphi_i\ \mu_{\tilde{\tau}_i}(t)), i = 1,n\}$, indexed by $t \in T$, where $N_t(\varphi_i) \geq \mu_{\tilde{\tau}_i}(t)$. In other words, fuzzy timed logic is also a timed possibilistic logic.

The deduction and refutation theorems still hold in fuzzy timed logic, as well as the resolution procedure. Let us point out that in order to select the non-trivial part of a deduction $\mathcal{F} \vDash (\varphi\ \tilde{\tau})$ we have to know Cert(φ ; $\mathcal{F}$), Incons($\mathcal{F}$) and then to compute the fuzzy set-difference Cert*(φ ; $\mathcal{F}$) = Cert(φ ; $\mathcal{F}$) \ Incons($\mathcal{F}$). There are several existing definitions for this operation [12]. The question is how to define the fuzzy set-difference in accordance with the algebraic structure of ($[0,1]^T$, max, min, 1 – (·)) while preserving intuitive properties. Especially, the non-trivial deduction in fuzzy timed logic must be in accordance with non-trivial deduction in possibilistic logic. If $\mathcal{F}$ is a possibilistic knowledge base with Incons($\mathcal{F}$) = $\alpha > 0$, then the non-trivial deduction operation $|\approx$ is defined as follows

$\mathcal{F} \mid\approx (\varphi\ \beta)$ if and only if $\mathcal{F} \vDash (\varphi\ \beta)$ with $\beta > \alpha$
$\mathcal{F} \mid\approx (\varphi\ 0)$ otherwise

Back to fuzzy timed logic viewing α as $\mu_{Incons(\mathcal{F})}(t)$ and β as $\mu_{Cert(\varphi;\mathcal{F})}(t)$ for some $t \in T$, the fuzzy set difference induced by the treatment of non-trivial deduction in the static case is such that

$$\mu_{Cert(\varphi;\mathcal{F}) \setminus Incons(\mathcal{F})}(t) = \mu_{Cert(\varphi;\mathcal{F})}(t) \ominus \mu_{Incons(\mathcal{F})}(t)$$

with $\beta \ominus \alpha = 0$ if $\beta \leq \alpha$ and $\beta \ominus \alpha = \beta$ otherwise. This definition has nice properties, i.e. the non-trivial part $Cert^*(\varphi ; \mathcal{F})$ verifies the two properties

- if $Cert(\varphi ; \mathcal{F}) \subset Incons(\mathcal{F})$ then $Cert^*(\varphi ; \mathcal{F}) = \emptyset$
- $Cert(\varphi ; \mathcal{F}) \cup Incons(\varphi ; \mathcal{F}) = Cert^*(\varphi ; \mathcal{F}) \cup Incons(\varphi ; \mathcal{F})$

$Cert^*(\varphi ; \mathcal{F})$ is even the smallest fuzzy set that satisfies the above equality, i.e. $\beta \ominus \alpha = \inf\{x \mid \max(x,\alpha) = \max(\beta,\alpha)\}$. This fuzzy set difference operation has been introduced in Dubois and Prade [12], noticing that in the crisp case $B \setminus A = \bigcap\{E \mid B \subset A \cup E\}$, which yields $\beta \ominus \alpha = \inf\{x, \beta \leq \max(\alpha,x)\}$ as well.

Example 3

This example is a fuzzification of Example 2. Consider the following pieces of information

John stayed in the office until about 9.15 and certainly not afterwards
Mary stayed in the office from about 8.45 and certainly not before
Between 9.00 and 9.10 (strictly) there was exactly one person (among John and Mary) in the office

Let us consider that the fuzzy temporal intervals representing "about t_0" are "trapezoïdal" fuzzy intervals (see Dubois and Prade [14]) of the form shown in Figure 4 (here for t_0 = 8.45 and t_0 = 9.15), meaning that if an event occurs at "about t_0", it is completely possible

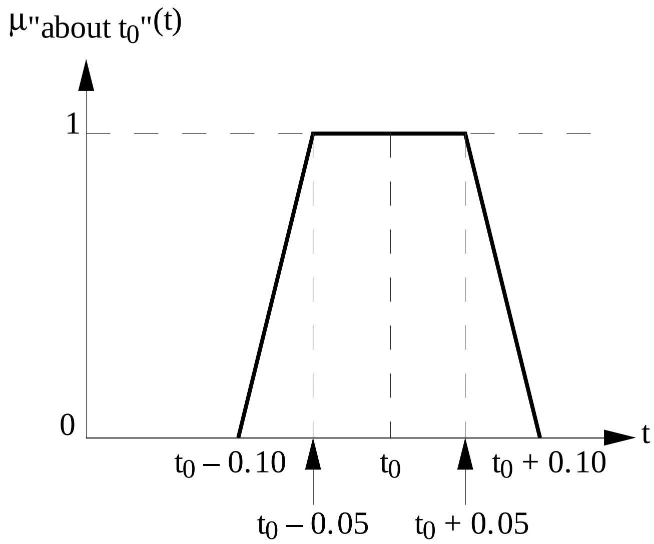
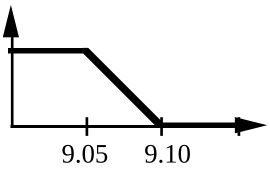


Figure 4

that it actually occurs between $t_0 - 5$ mn and $t_0 + 5$ mn, completely impossible that it actually happens before $t_0 - 10$ mn or after $t_0 + 10$ mn, and that the membership function of the possibility it actually happens at time t is linearly increasing on $[t_0 - 0.10, t_0 - 0.05]$ and linearly decreasing on $[t_0 + 0.05, t_0 + 0.10]$.

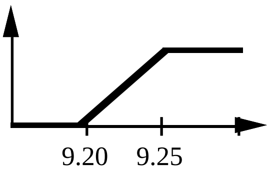
In a manner similar to Examples 1 and 2, the logical translation of the previous sentences gives the set of fuzzy temporal necessity-valued formulas

$(J\ \tilde{\tau}_1)$ where $\tilde{\tau}_1$ is

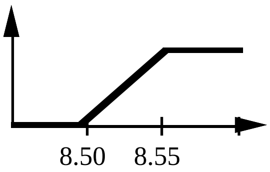


($\tilde{\tau}_1$ is the fuzzy set of instants necessarily before "about 9.15", i.e. $\mu_{\tilde{\tau}_1}(t) = \inf_{t \geq s} 1 - \mu_{\text{"about 9.15"}}(s)$ which is the necessity measure of "t before s" (t < s), where possible values of s are restricted by $\mu_{\text{"about 9.15"}}$ [15]).

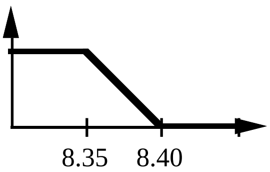
$(\neg J\ \tilde{\tau}_2)$ where $\tilde{\tau}_2$ is



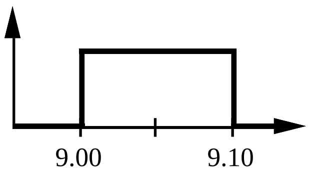
$(M\ \tilde{\tau}_3)$ where $\tilde{\tau}_3$ is



$(\neg M\ \tilde{\tau}_4)$ where $\tilde{\tau}_4$ is



$((\neg J \vee \neg M) \wedge (J \vee M)\ \ \tilde{\tau}_5)$ where $\tilde{\tau}_5$ is



Let us first compute the fuzzy temporal possibility distribution $\pi_{\mathcal{H}}*$:

$$\pi_{\mathcal{H}}*(J \wedge M) = \overline{\tilde{\tau}_2} \cap \overline{\tilde{\tau}_4} \cap \overline{\tilde{\tau}_5}$$

$$\pi_{\mathcal{H}}*(J \wedge \neg M) = \overline{\tilde{\tau}_2} \cap \overline{\tilde{\tau}_3}$$

$$\pi_{\mathcal{H}}*(\neg J \wedge M) = \overline{\tilde{\tau}_1} \cap \overline{\tilde{\tau}_4}$$

$$\pi_{\mathcal{H}}*(\neg J \wedge \neg M) = \overline{\tilde{\tau}_1} \cap \overline{\tilde{\tau}_3} \cap \overline{\tilde{\tau}_5}$$

These objects are fuzzy subsets of T (they cannot be named "fuzzy intervals" since they are generally not convex). Incons($\mathcal{H}$) is the fuzzy complementary of the union of these four fuzzy subsets, i.e.

$$
\begin{aligned}
\text{Incons}(\mathcal{H}) &= \overline{\pi_{\mathcal{H}}{}^*(J \wedge M) \cup \pi_{\mathcal{H}}{}^*(\neg J \wedge M) \cup \pi_{\mathcal{H}}{}^*(J \wedge \neg M) \cup \pi_{\mathcal{H}}{}^*(\neg J \wedge \neg M)} \\
&= (\tilde{\tau}_2 \cup \tilde{\tau}_4 \cup \tilde{\tau}_5) \cap (\tilde{\tau}_2 \cup \tilde{\tau}_3) \cap (\tilde{\tau}_1 \cup \tilde{\tau}_4) \cap (\tilde{\tau}_1 \cup \tilde{\tau}_3 \cup \tilde{\tau}_5) \\
&= (\tilde{\tau}_2 \cup \tilde{\tau}_4 \cup \tilde{\tau}_5) \cap \tilde{\tau}_3 \cap \tilde{\tau}_1 \cap (\tilde{\tau}_1 \cup \tilde{\tau}_3 \cup \tilde{\tau}_5) \\
&= \tilde{\tau}_1 \cap \tilde{\tau}_3 \cap (\tilde{\tau}_2 \cup \tilde{\tau}_4 \cup \tilde{\tau}_5) \text{ (using the peculiarities of the } \tau_i\text{'s)} \\
&= (\tilde{\tau}_1 \cap \tilde{\tau}_2 \cap \tilde{\tau}_3) \cup (\tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_4) \cup (\tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_5) \\
&= \tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_5 = \tilde{\tau}_1 \cap \tilde{\tau}_5 \text{ (using the peculiarities of the } \tau_i\text{'s)}
\end{aligned}
$$

which is represented in Figure 5.

Let us now compute $N_{\mathcal{H}}{}^*(J)$ and $N_{\mathcal{H}}{}^*(M)$.

$$
\begin{aligned}
N_{\mathcal{H}}{}^*(J) &= \overline{\pi_{\mathcal{H}}{}^*(\neg J \wedge M) \cup \pi_{\mathcal{H}}{}^*(\neg J \wedge \neg M)} = \overline{(\overline{\tilde{\tau}_1} \cap \overline{\tilde{\tau}_4}) \cup (\overline{\tilde{\tau}_1} \cap \overline{\tilde{\tau}_3} \cap \overline{\tilde{\tau}_5})} \\
&= (\tilde{\tau}_1 \cup \tilde{\tau}_4) \cap (\tilde{\tau}_1 \cup \tilde{\tau}_3 \cup \tilde{\tau}_5) = (\tilde{\tau}_1 \cup \tilde{\tau}_4) \cap T = \tilde{\tau}_1
\end{aligned}
$$

whose non-trivial part is Cert*(J) = $\tilde{\tau}_1$ \ Incons($\mathcal{H}$).

$$N_{\mathcal{H}}{}^*(M) = (\tilde{\tau}_2 \cup \tilde{\tau}_3) \cap (\tilde{\tau}_1 \cup \tilde{\tau}_3 \cup \tilde{\tau}_5) = \tilde{\tau}_3$$

$$N_{\mathcal{H}}{}^*(J \wedge M) = N_{\mathcal{H}}{}^*(J) \cap N_{\mathcal{H}}{}^*(M) = \tilde{\tau}_1 \cap \tilde{\tau}_3$$

$$N_{\mathcal{H}}{}^*(\neg J \vee \neg M) = \overline{\pi_{\mathcal{H}}{}^*(J \wedge M)} = \tilde{\tau}_2 \cup \tilde{\tau}_4 \cup \tilde{\tau}_5.$$

$N_{\mathcal{H}}{}^*(J \wedge M)$ and Cert*($J \wedge M$) = $N_{\mathcal{H}}{}^*(J \wedge M)$ \ Incons($\mathcal{H}$) are represented on Figure 6.1. $N_{\mathcal{H}}{}^*(\neg J \vee \neg M)$ and Cert*($\neg J \vee \neg M$) are represented on Figure 6.2. On Figure 6.3. we have represented Cert*($J \wedge M$) and Poss*($J \wedge M$) = $\overline{\text{Cert}^*(\neg J \vee \neg M)}$ which is the fuzzy set of time instants where $J \wedge M$ is possibly true ; it can be seen on the Figure 6.3 that Poss*($J \wedge M$) contains Cert*($J \wedge M$), which is in accordance with our intuition.

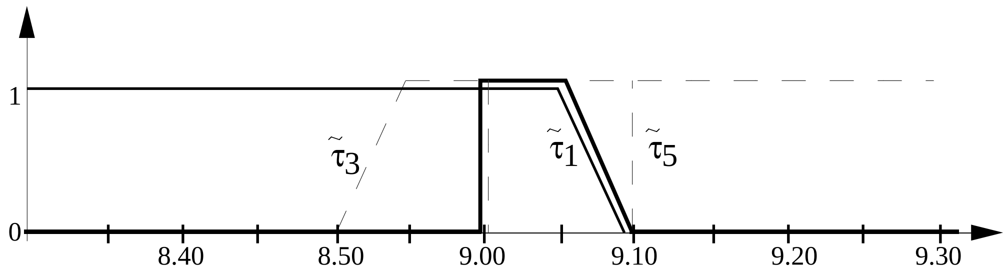


Figure 5 : Incons($\mathcal{H}$) = $\tilde{\tau}_1 \cap \tilde{\tau}_5$ (——)

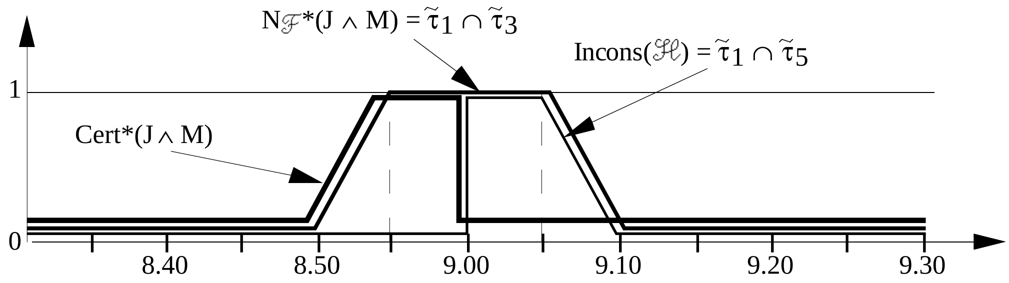


Figure 6.1 : Cert*(J ∧ M)

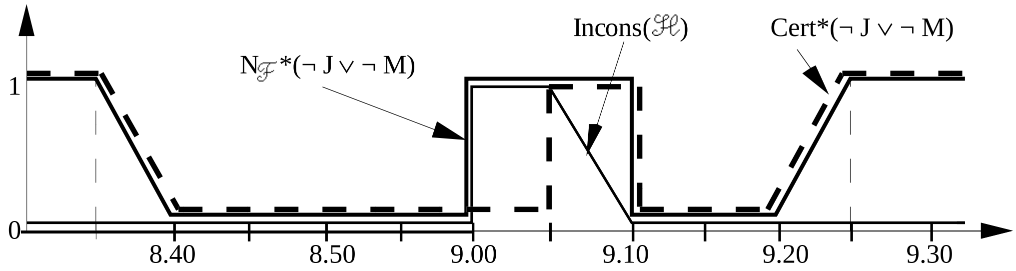


Figure 6.2 : Cert*(¬J ∨ ¬M)

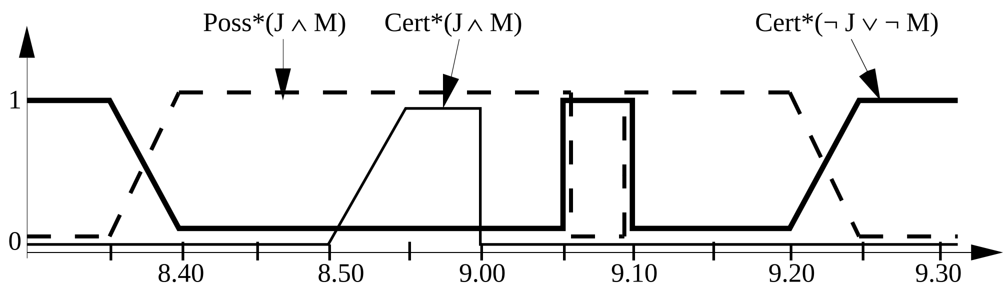
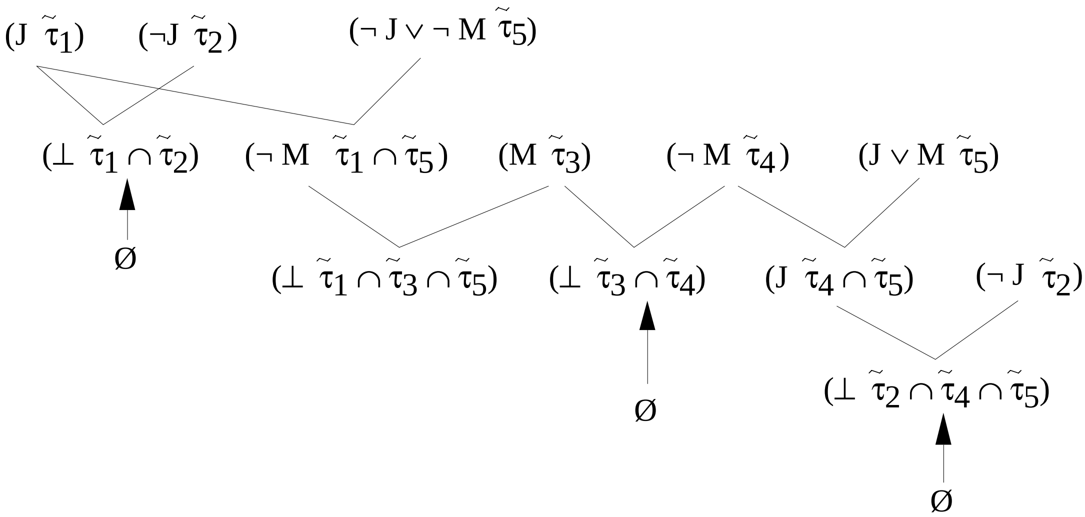


Figure 6.3 : Poss*(J ∧ M)

Lastly, let us check that Incons($\mathcal{H}$), $N_{\mathcal{H}}$*(J ∧ M) and $N_{\mathcal{H}}$*(¬J ∨ ¬M) can be computed using refutation by resolution. This is done in Figures 7, 8, 9

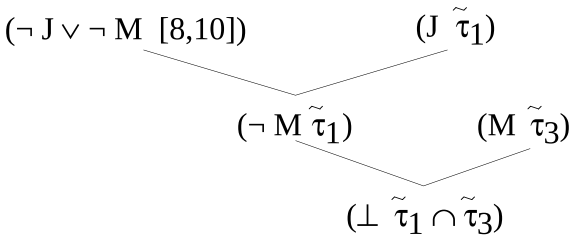
• Figure 7 : collecting the fuzzy sets attached to ⊥ leads to

$$\begin{aligned} \text{Incons}(\mathscr{H}) &= (\tilde{\tau}_1 \cap \tilde{\tau}_2) \cup (\tilde{\tau}_3 \cap \tilde{\tau}_4) \cup (\tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_5) \cup (\tilde{\tau}_2 \cap \tilde{\tau}_4 \cap \tilde{\tau}_5) \\ &= \emptyset \cup \emptyset \cup (\tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_5) \cup \emptyset \\ &= \tilde{\tau}_1 \cap \tilde{\tau}_3 \cap \tilde{\tau}_5 = \tilde{\tau}_1 \cap \tilde{\tau}_5 \end{aligned}$$



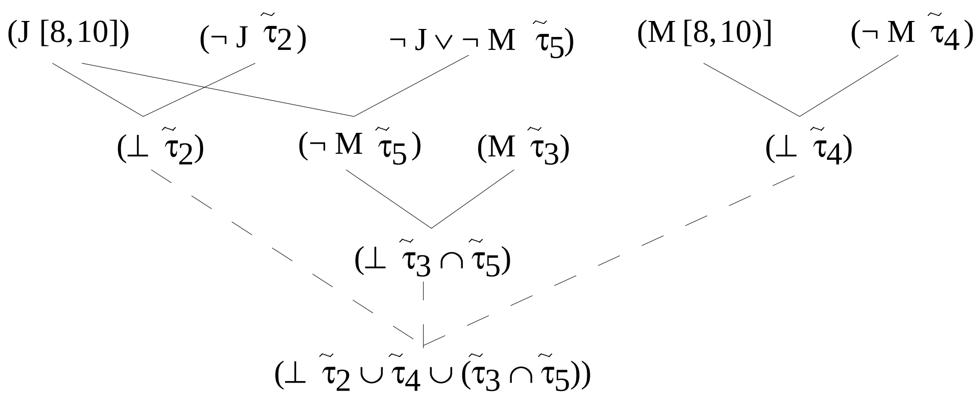
<u>Figure 7</u> : Incons($\mathscr{H}$)

• Figure 8 : $(\neg J \vee \neg M\ T)$ has been added to $\mathscr{H}$, and it leads to $N_{\mathscr{H}}{}^*(J \wedge M) = \tilde{\tau}_1 \cap \tilde{\tau}_3$



<u>Figure 8</u> : $N_{\mathscr{H}}{}^*(J \wedge M)$

• Figure 9 : $(J\ T)$ and $(M\ T)$ have been added to $\mathscr{H}$ and it leads to $N_{\mathscr{H}}{}^*\ (\neg J \vee \neg M) = \tilde{\tau}_2 \cup \tilde{\tau}_4 \cup (\tilde{\tau}_3 \cap \tilde{\tau}_5) = \tilde{\tau}_2 \cup \tilde{\tau}_4 \cup \tilde{\tau}_5$.



<u>Figure 9</u> : $N_{\mathscr{H}}{}^*(\neg J \vee \neg M)$

On the whole, timed logic makes possible to compute the fuzzy set of instants when we are more or less certain that a proposition φ is true, and the fuzzy set of instants when we are more or less certain that φ is false. By complementation, this latter fuzzy set yields the fuzzy set of instants when φ is more or less possibly true (a fuzzy set which includes the first mentioned one, since $N_t(\varphi) > 0$ entails $\prod_t(\varphi) = 1 - N_t(\neg\varphi) = 1$). Then from these two fuzzy sets, defined by $N_t(\varphi)$ and $\prod_t(\varphi)$ when t varies, we can compute the fuzzy bounds of intervals when φ is true ; see [15] for the definition of these bounds. Considering now two formulas φ and ψ, describing events, we can then compute to what extent it is certain (in the sense of a necessity measure) that, for instance, φ takes place before ψ, using the extension of Allen's [1] interval relationships to fuzzily bounded intervals ; see [15] for details. Note that a piece of information like "it is somewhat certain that φ takes place before ψ" can be easily handled in standard possibilistic logic.

5. Conclusion

In this paper we have provided the basis for a reified temporal logic with a semantics, while handling (graded) uncertainty. Timed possibilistic logic, as presented here, has some limitations. Let us mention some of them. In our logic we cannot presently represent information like "there exists at least an instant t in the time period τ where it is more or less certain that φ is true". The evaluation of such a statement would require the use of a quantity like $\sup_{t\in\tau} N_t(\varphi)$. We cannot either express that for instance a sentence cannot become true before another becomes false. Hence we cannot express symbolic precedence constraints as in Ghallab and Mounir-Alaoui [23]. So the language is not rich enough to formulate planning problems. The logic described in this paper is thus more adapted to handle timed information such as observations issued from a dynamic world and whose reliability decreases with time.

Lastly, reified temporal logics are basically very simple and have a limited expressive power with respect to modal temporal logic where past and future can be explicitly dealt with. Since representations of possibilistic logic in a (multi-) modal logic framework have been already explored (Dubois, Prade and Testemale [18], Fariñas del Cerro and Herzig [22]), we may think of incorporating some possibilistic logic ideas in modal temporal logics for grading modalities and still enhance their expressive power.

Another problem that can be addressed in a structure similar to possibilistic logic is the case of a knowledge base whose contents stems from several sources. To do that, we might change the temporal axis T into a set S of sources, and define N(φ) as the subset of sources which claim that φ is certainly true. The same results as obtained here would be arrived at. Besides, let us replace T by 2^A, where A is a set of assumptions ; then $L = 2^{2^A}$ is the set of all sets of assumptions, which may be interpreted as the set of all disjunctions of conjunctions of assumptions, or, in the Assumption-based Truth Maintenance Systems (ATMS) terminology [7], as the set of all labels, a label being a disjunction of environments. This idea of viewing ATMS as a particular Boolean-valued logic has been studied by Ginsberg [24].

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