Possibilistic logic[1]

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1. Introduction

Possibilistic logic is a logic of uncertainty tailored for reasoning under incomplete evidence and partially inconsistent knowledge. At the syntactic level it handles formulas of propositional or first-order logic to which are attached numbers between 0 and 1, or more generally elements in a totally ordered set. These weights are lower bounds on so-called degrees of necessity or degrees of possibility of the corresponding formulas. The degree of necessity (or certainty) of a formula expresses to what extent the available evidence entails the truth of this formula. The degree of possibility expresses to what extent the truth of the formula is not incompatible with the available evidence.

At the mathematical level, degrees of possibility and necessity are closely related to fuzzy sets (Zadeh, 1965, 1978a), and possibilistic logic is especially adapted to automated reasoning when the available information is pervaded with vagueness. A vague piece of evidence can be viewed as defining an implicit ordering on the possible worlds it refers to, this ordering being encoded by means of fuzzy set membership functions. Hence possibilistic logic is a tool for reasoning under uncertainty based on the idea of (complete) ordering rather than counting, contrary to probabilistic logic.

To figure out how possibilistic logic could emerge as a worth-studying formalism, it might be interesting to go back to the origins of fuzzy set theory and what is called "fuzzy logic". Fuzzy sets were introduced by Zadeh (1965) in an attempt to propose a mathematical tool describing the type of model people use when reasoning about systems. More particularly,

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Zadeh focused on the presence of classes without sharp boundaries in human-originated descriptions of systems, and fuzzy sets are meant to represent these classes ; the idea is to turn class membership into a gradual notion instead of the usual all-or-nothing view.

Then a fuzzy set F on a referential set Ω is simply described by a membership function μ_F that maps elements ω of Ω to the unit interval [0,1], 0 standing for non-membership, 1 for complete membership, and numbers in between for partial membership. The choice of the unit interval for a membership scale L contains some arbitrariness. It has been motivated by the fact that set characteristic functions take values in the pair {0,1} usually. Clearly any sub-interval of the reals can be used instead, including $\mathbb{R}$ itself, once completed by $-\infty$ and $+\infty$. But more abstract scales might fit as well such as any finite chain, any totally ordered set L. Clearly, a real interval is the most simple example of a totally ordered set L such that $\forall\, a \in L, b \in L$, $a < b$, then $\exists\, c \in L$, $a < c < b$. This property, (which is not satisfied by a finite chain) ensures a smooth membership gradedness. The particular choice of the unit interval also makes sense in the scope of comparing fuzzy sets and probability (e.g. Dubois and Prade, 1989).

Based on the notion of membership function, it is easy to extend many mathematical definitions pertaining to sets over to fuzzy sets. Set-theoretic operations for fuzzy sets were thus defined as follows (Zadeh, 1965)

union $F \cup G$:	$\mu_{F\cup G} = \max(\mu_F,\mu_G)$	(1)
intersection $F \cap G$:	$\mu_{F\cap G} = \min(\mu_F,\mu_G)$	(2)
complementation $\overline{F}$:	$\mu_{\overline{F}} = 1 - \mu_F$	(3)

Note that (1) and (2) just require a lattice structure for the membership scale (Goguen, 1967) while (3) further requires some order-reversing mapping on the lattice. The justification for these definitions in the unit interval came a long time later (e.g. Dubois and Prade (1985a) for a survey). Subsequently relations were extended into fuzzy relations, especially equivalence and ordering notions were turned into so-called similarity and gradual preference relations (Zadeh, 1971).

Considering the usual assimilation between predicates and their extensions as sets, it is not surprizing that fuzzy set-theoretic operations were quickly interpreted as logical connectives. In other words, the natural connections between set theory and logic has led to a similar connection between fuzzy set theory and multiple valued logic at least at the naïve level. This state of facts prompted a revival of multiple-valued logic inspired by the birth of fuzzy sets, and the name "fuzzy logic" was coined in the seventies by R.C.T. Lee (1972) who tried to extend the resolution rule to a multiple-valued logic that handle clausal forms by means of the three truth-functional basic fuzzy set connectives given above. This path was followed later on by

Mukaidono (1982), and gave birth to logic programming tools (Mukaidono et al., 1989 ; Orci, 1989 ; Ishizuka and Kanai, 1985). Another important trend in the multiple-valued logic view of fuzzy sets stems from a seminal paper by J. Goguen (1969). In this work, directly inspired by Lukasiewicz logic, the author points out the difficulty to produce a syntax for the logic of vague concepts, when this syntax is in fact the one of classical logic. Ten years later Pavelka (1979) found a solution to Goguen's problem by introducing truth values in the language (see also Novák, 1990). Apart from these two "schools" of fuzzy logic, still other works have been published on the relationship between multiple-valued logics and fuzzy sets (see Dubois, Lang and Prade (1991a) for a survey).

Interestingly enough, Zadeh himself did not participate to these logical developments, but started focusing on the representation of vague aspects in natural languages. In his (1975a) he introduced the concept of fuzzy restriction, as a fuzzy relation that restricts the possible values of a vector of variables in a flexible way, and developed a calculus of fuzzy relations which extends all basic notions of constraint propagation, and can be viewed as a pioneering work in hypergraph-based methods for reasoning under uncertainty (e.g. Shafer and Shenoy, 1990). Quite at the same time, fuzzy truth-values were proposed as a means to evaluate the truth of a vague statement, in the face of another vague statement that is taken as the reference. Zadeh (1975b) names "fuzzy logic" a logic that handles vague statements and fuzzy truth-values. At this point a misunderstanding apparently grew up between Zadeh and the community of logicians. "Fuzzy logic" after Zadeh was severely criticized (Morgan and Pelletier, 1977 ; Haack, 1979) for basically two reasons : first, Zadeh's fuzzy logic was claimed not to be a logic. Indeed it has no syntax, and the problem of developping a syntax for a logic of vague concepts has indeed never been addressed by Zadeh who adopted a computational view of fuzzy reasoning, based on non-linear programming. A second critique addressed the usefulness and meaningfulness of fuzzy truth-values, and the question of truth-functionality. Zadeh's works were viewed as a debatable attempt to extend truth-functional logics beyond classical logic, and fuzzy truth-values as a strange and gratuitous extension of numerical truth-values, whose meaning has ever been controversial for logicians themselves. Apparently, Zadeh's main thrust, namely that a truth-value was a local notion (Bellman and Zadeh, 1977), pertaining to a state of knowledge itself expressed in fuzzy terms, was missed, perhaps due to the term "fuzzy logic" that was used by other researchers to denote multiple-valued logic.

In his (1978a), Zadeh introduced the measure of possibility as a scalar index that evaluates the consistency of a fuzzy proposition with respect to a state of knowledge expressed by means of a fuzzy restriction. Attempts at introducing a non-probabilistic view of uncertainty, similarly to Zadeh's approach can be traced back to a proposal by Shackle (1961) which was never completely formalized. The notion of fuzzy restriction corresponds to a radical change in the semantics of the membership function. A fuzzy restriction is a fuzzy set of *possible* values, and its membership function is thus called a possibility distribution. Soon after, the dual notion of certainty (Zadeh, 1979a) or necessity (Dubois and Prade, 1980) was introduced as a scalar evaluation of the strength of entailment of a fuzzy proposition from a given fuzzy restriction. At this point, it became patent that Zadeh's "fuzzy logic" was not just another multiple-valued logic, but rather an approach to reasoning under uncertainty and incomplete knowledge described by fuzzy restrictions — what Zadeh (1979b) called "approximate" reasoning. Moreover the basic scalar evaluations, possibility and necessity, are not truth-functional. Possibility and necessity valuations could then play a role similar to probabilities in logic ; in particular, they could come on top of a classical logic language. Instead of considering a probability distribution on a set of possible worlds (or interpretations), a *possibility* distribution can be considered. The result is possibilistic logic. As it will be seen, a possibility distribution on a set of possible worlds expresses a preference ordering among the worlds. Thus the semantics of possibilistic logic departs from the semantics based on a similarity relation between worlds, recently proposed by Ruspini (1991). Ruspini's semantics rather looks as a fuzzy set extension of the logic of rough sets (Fariñas del Cerro and Orlowska, 1985) which implements a semantics induced by an equivalence relation modelling indiscernibility (Pawlak, 1982).

This paper is organized as follows : Section 2 pursues the overview by introducing background material on fuzzy set and possibility theory, including comparative possibility relations that underlie possibility and necessity measures. Section 3 forms the main body of the paper and presents formal aspects of a fragment of possibilistic logic where formulas are valued by a lower bound on their degree of necessity. It includes an axiomatization and a refutation method based on extended resolution that is liable of implementation on a computer and supports partial inconsistency. The remainder of Section 3 lays bare the existing links between possibilistic logic and non-monotonic logics on the one hand, and belief revision on the other hand. It is indicated that possibilistic logic can be cast in the frameworks of preference logics after Shoham (1988), counterfactual logics after Lewis (1973) and epistemic entrenchment theory after Gärdenfors (1988). In the presence of partial inconsistency, possibilistic logic behaves as a cumulative non-monotonic logic whose properties have been suggested by Gabbay (1985), and studied by Makinson (1989) and Kraus et al. (1990). Section 4 considers a richer possibilistic logic where formulas can be weighted by a lower bound on possibility or necessity degrees. Then it briefly sketches some other potentially interesting extensions of possibilistic logic, whereby the valuation set may become partially ordered, the weights may become variable or convey the interpretation of costs, and the predicates may become vague. Section 5 describes some applications of possibilistic logic to truth-maintenance systems, inconsistency handling in logical data bases, discrete optimization and logic programming.

2. Possibility theory

Let x be a variable taking its values in a set U. A possibility distribution π_X attached to x describes a state of knowledge about the value of x. This value, although unknown, is supposed to be unique. π_X is a mapping from U to the unit interval, such that $\pi_X(u) = 1$ for at least one value u. The following conventions are adopted

$\pi_X(u) = 0$ means that x = u is impossible ;
$\pi_X(u) = 1$ means that x = u is completely allowed ;
$\pi_X(u) > \pi_X(u')$ means that x = u is preferred to x = u'.

The normalization requirement $\pi_X(u) = 1$ means that in U there is at least one value of x that is completely allowed. Zadeh (1978a) explains how fuzzy sets give birth to possibility distributions. For instance the sentence "John is tall" can be modelled by means of a fuzzy set F on a set of sizes, which represents the set of possible values of John's size. Here U is the set of sizes, x is the variable representing John's size, F the fuzzy set representing "tall" in a given context, and the possibility distribution π_X is taken as equal to μ_F. Following Zadeh (1979b), the fuzzy set F is viewed as a fuzzy restriction "which serves as an elastic constraint" on the value that may be assigned to the variable x.

In this view the sentence "John is tall" is considered as a piece of incomplete evidence, and is supposed to be the only available information about x. The advantage of using a fuzzy set in the modelling of "tall" is to convey the information that if u > u', u is a better potential value for x than u', (if μ_{tall} is an increasing membership function). More generally, the preference expressed by means of the degree of possibility $\pi_X(u)$ reflects the closeness of u to the prototypes of the fuzzy set F. This view completely differs from another interpretation of "John is tall" where the value of x is *known* (say x = 1.78 m) and "tall" is used as a linguistic substitute to this value. In this latter situation, a "degree of truth" of "John is tall" can be computed (as $\mu_F(1.78)$) if needed, because the underlying information is complete. This is the case of the multiple-valued logic understanding of fuzzy logic, where sentences can be attached degrees of truth ; however there is a danger to obtain the paradoxical situation where given a membership function and a degree of truth, the size of John may be recomputed. This paradox has often been the source of criticisms addressed to fuzzy set theory. It is completely obviated when a proposition like "John is tall", taken as a piece of evidence, is modelled as an elastic constraint.

Possibility distributions are liable of interpretations that differ from membership functions of fuzzy sets. Probabilistic interpretations include upper probability bounds, consonant random sets (Dubois and Prade, 1988) and likelihood functions (Smets, 1982), and infinitesimal probabilities (Spohn, 1988).

Let π_x and π'_x be two possibility distributions attached to x. Then, π_x is said to be more specific than π'_x (Yager, 1983) if and only if $\pi_x < \pi'_x$. Specificity refers to the level of precision of a possibility distribution. When $\pi_x < \pi'_x$, π_x is more informative than π'_x. Especially, if $\exists\ u_0 \in U$, $\pi_x(u_0) = 1$ while $\pi_x(u) = 0$ for $u \neq u_0$, the state of knowledge is said to be complete (we know that $x = u_0$). Contrastedly, a state of total ignorance is easily expressed as $\pi_x(u) = 1$, $\forall\ u \in U$. Of importance in possibility theory is the principle of minimum specificity which says that given a set of constraints restricting the value of x, π_x should be defined so as to allocate the maximal degree of possibility to each $u \in U$, in accordance with the constraints. This principle points out that given a piece of evidence of the form "x is F" where F is a fuzzy set, the equality $\pi_x = \mu_F$ is only a consequence of the constraint $\pi_x \leq \mu_F$ acting on the values of x, together with the principle of minimal specificity. Similar informational principles can be encountered in several non-monotonic inference systems (e.g. in Pearl (1990) with the idea of compact ordering of defaults, with the rational closure of a conditional knowledge base (Lehman, 1989), or when worlds are assigned to the smallest sphere in a sphere system in conditional logic (Lamarre, 1992).

More generally, Zadeh's theory of approximate reasoning considers that the available information pertaining to a situation is stored in a data base, and involves a set $\{x_1, \ldots, x_n\}$ of variables that take their values on $U_1, U_2, \ldots, U_n$. The data base contains the representation of pieces of information that define fuzzy restrictions on $U_1 \times U_2 \times \ldots \times U_n$. Zadeh has devised a method that systematically translates several kinds of natural language sentences into possibility distributions on $U_1 \times U_2 \times \ldots \times U_n$. This translation method called PRUF (Zadeh, 1978b) is not described here ; for details see also Dubois and Prade (1980, 1991d). For the sake of simplicity, we shall assume that the available knowledge is described by means of m possibility distributions $\pi_1, \ldots, \pi_m$ that model m pieces of available knowledge in a data base D. Answering a query may consist in computing the possibility distribution π_{x_i} that represents the fuzzy domain of possible values of x_i. Approximate reasoning is understood as a procedure that computes π_{x_i} from the knowledge of $\pi_1, \ldots, \pi_m$. This procedure involves three basic steps (Zadeh, 1979b) :

– cylindrical extension : if π_j is a fuzzy restriction on the Cartesian product $\times_{k \in K} U_k$, the cylindrical extension $c\pi_j$ of π_j on $U = U_1 \times U_2 \times \ldots \times U_n$ is

$$c\pi_j(u) = \pi_j(u_K)$$

where $u = (u_K, u_{\bar{K}})$, u_K is a vector of values of variables x_k, $k \in K$, $u_{\bar{K}}$ is the complementary vector of values of variables x_k, $k \notin K$;

– combination : the least specific possibility distribution induced by $\pi_1, \pi_2, \ldots, \pi_m$ is

$$\pi_D = \min_{j=1,m} c\pi_j \qquad (4)$$

which represents the state of knowledge contained in $D = \{\pi_1, \pi_2, \ldots, \pi_m\}$;

– projection : the marginal possibility distribution π_{x_i} attached to the variable x_i is defined by

$$\pi_{x_i}(u_i) = \sup_{u_k, k\neq i} \pi_D(u_1, u_2, \ldots, u_n) = \sup_{u_k, k\neq i} \min_{j=1,m} c\pi_j(u_1, \ldots, u_n).$$

This kind of procedure for approximate reasoning involving cylindrical extension, combination and projection operations has been applied to other theories of uncertain reasoning such as belief functions (Shafer, 1976 ; Shafer and Shenoy, 1990) with a different combination step. Clearly it offers a semantics for approximate reasoning but does not presuppose the existence of a syntax.

Note that in (4), it may happen that $\pi_D(u) < 1, \forall u$. Then π_D is said to be subnormalized, and subnormalization corresponds to a lack of consistency between $\pi_1, \pi_2, \ldots, \pi_m$. The degree of consistency of D is $\text{cons}(D) = \sup_u \pi_D(u)$. In Section 3, the proposed framework will cope with this situation.

Another natural type of query to a fuzzy data base D, is to check whether a given statement S is a consequence of $\{\pi_1, \pi_2, \ldots, \pi_m\}$. Let π_S be the possibility distribution which represents the fuzzy restriction on U induced by the statement S. Zadeh (1979b) introduces the notion of semantic entailment in possibility theory as follows : S follows from $D = \{\pi_1, \pi_2, \ldots, \pi_m\}$ if and only if $\pi_D \leq \pi_S$ where π_D is given by (4). In other words, the consequences of D should not be more specific than D.

The fuzzy truth-value of S is more generally defined as a possibility distribution π_τ on [0,1] such that (Zadeh, 1978b)

$$\pi_\tau(t) = \sup_{u:t=\pi_S(u)} \pi_D(u)$$

This formula computes the fuzzy set $\tau = \pi_S(D)$ of the possible truth-values of S if the available information D were precise, and in accordance with the constraints $\{\pi_1, \ldots, \pi_m\}$. It is easy to verify that

$$\text{D entails S} \iff \pi_D \leq \pi_S \iff \pi_\tau(t) \leq t$$
$$\text{D entails } \bar{S} \iff \pi_D \leq 1 - \pi_S \iff \pi_\tau(t) \leq 1 - t$$

where $\bar{S}$ denotes the statement opposite to S, represented by $1 - \pi_S$. The right-hand side inequalities can thus be interpreted as follows : the fuzzy truth-values τ such that $\pi_\tau(t) \le t$ mean "true", while the fuzzy truth-values such that $\pi_\tau(t) \le 1 - t$ mean "false". Indeed if $\pi_S = \pi_D$ then $\pi_\tau(t) = t$, if $\exists u, \pi_S(u) = t$, and if $\pi_S = 1 - \pi_D$, then $\pi_\tau(t) = 1 - t$ if $\exists u, \pi_S(u) = t$. When π_S is the characteristic function of a subset A of U, $\pi_\tau(t) = 0, \forall\ t \ne 0,1$ since S cannot be but true or false. Moreover we can compute

$$\pi_\tau(1) = \sup_{u \in A} \pi_D(u)$$
$$\pi_\tau(0) = \sup_{u \notin A} \pi_D(u).$$

These indices correspond to the degrees of possibility that S is true and that S is false. Note that the fuzzy truth value τ is local in the sense that it is relative to the state of knowledge described by D. $\pi_\tau(1)$ corresponds to the notion of possibility measure of $A \subseteq U$, introduced by Zadeh (1978a). A possibility measure is a set function Π that attaches to each subset $A \subseteq U$ a number $\Pi(A) \in [0,1]$. The basic axioms of a possibility measure are

$$\Pi(\emptyset) = 0 \;\; ; \;\; \Pi(U) = 1$$
$$\Pi(\cup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i) \qquad (5)$$

for any index set I. A possibility measure derives from a possibility distribution π, which verifies $\forall\ u \in U, \pi_x(u) = \Pi(\{u\})$. Especially we can write

$$\Pi(A) = \sup_{u \in A} \pi_x(u) \qquad (6)$$

$\Pi(A)$ expresses to what extent there is a value $u \in A$ that may stand as a value of x. The dual set function is called a necessity measure, is denoted N, and is defined by (Dubois and Prade, 1980)

$$N(A) = 1 - \Pi(\bar{A}) = \inf_{u \notin A} 1 - \pi_x(u) \qquad (7)$$

where $\bar{A}$ is the complement of A. N(A) evaluates to what extent all possible values of x belong to A, i.e. to what extent one is certain that x belongs to A. A third quantity has been recently emphasized by the authors (Dubois and Prade, 1992a), namely

$$\Delta(A) = \inf_{u \in A} \pi_x(u) \qquad (8)$$

which evaluates to what extent *all* elements in A are possible values for x. Noticeable properties of possibility and necessity measures are as follows

$$N(A \cap B) = \min(N(A),N(B)) \;\; ; \;\; \Pi(A \cup B) = \max(\Pi(A),\Pi(B)) \qquad (9)$$

but we only have inequalities $N(A \cup B) \geq \max(N(A),N(B))$, $\Pi(A \cap B) \leq \min(\Pi(A),\Pi(B))$ generally. Moreover

$$\min(N(A),N(\bar{A})) = 0 \;\; ; \;\; \max(\Pi(A),\Pi(\bar{A})) = 1. \qquad (10)$$

These three degrees of uncertainty enable uncertainty-qualified statements of the form "x is A is possible, certain, etc…" to be modelled in terms of sets of possibility distributions.

Especially, the piece of evidence "x is A is at least α-certain" where $\alpha \in [0,1]$, is equivalent to the set of possibility distributions π such that

$$N(A) = \inf_{u \notin A} 1 - \pi(u) \geq \alpha \qquad (11)$$

It is easy to see that this set of possibility distributions has a greatest element $\pi_x = \max(\mu_A, 1 - \alpha)$ where μ_A is the characteristic function of A. By virtue of the principle of minimum specificity, this greatest element is the best representation of the piece of information.

Contrastedly, the pieces of information "some u in A is at least β-possible for x" and "all u in A are γ-possible for x", can be modelled by the sets of possibility distributions π respectively defined by the following inequalities :

$$\Pi(A) = \sup_{u \in A} \pi(u) \geq \beta \qquad (12)$$
$$\Delta(A) = \inf_{u \in A} \pi(u) \geq \gamma \qquad (13)$$

The maximal element of each set of possibility measures expresses the state of total ignorance $\pi(u) = 1, \forall u$, and is obviously not informative. Indeed knowing that some or all u in A are possible for x does not prevent elements in $\bar{A}$ from being possible too. Hence there is no unique possibility distribution which may be a good representation of any of these sets. Note that $\Pi(A) \geq \Delta(A)$ so that $\Delta(A) \geq \gamma$ is more restrictive than $\Pi(A) \geq \gamma$. In possibilistic logic we consider possibility qualified statements of the form $\Pi(A) \geq \beta$ that express that A is consistent with the available knowledge to some extent. Note that possibility distributions that satisfy (13) are such that $\pi(u) \geq \min(\gamma, \mu_A(u))$, $\forall u$. Such a lower bound do not even exist with (12). At any rate, (12) and (13) must be represented by means of a family of possibility distributions, and not a single one as for (11).

When possibility distributions do not derive from probabilistic knowledge, the use of the interval [0,1] to model degrees of possibility and necessity is not compulsory. It looks natural for fuzzy numbers, i.e. possibility distributions of the real line, where $\pi(u)$ embodies proximity notions, since membership grades then evaluates to what extent a value is close to a prototype value. But as said earlier, only a totally ordered structure is requested strictly speaking. It even makes sense to consider possibility as a comparative notion, just as De Finetti (1937) did for probability.

Namely let $\geq_\Pi$ be a relation defined on a finite Boolean algebra $\mathcal{B}$ of subsets of U. $A \geq_\Pi B$ intends to mean "A is at least as possible as B". Let $>_\Pi$ and $\sim_\Pi$ denote the strict ordering and equivalence relations associated to $\geq_\Pi$. The following set of axioms has been proposed by Dubois (1986) for characterizing possibility relations

1) $U >_\Pi \emptyset$ (non triviality)
2) $A \geq_\Pi \emptyset$
3) $A \geq_\Pi B$ and $B \geq_\Pi C$ imply $A \geq_\Pi C$ (transitivity)
4) $A \geq_\Pi B$ or $B \geq_\Pi A$ (completeness)
Π) $A \geq_\Pi B$ implies $A \cup C \geq_\Pi B \cup C$, $\forall C$

The last axiom (Π) is the crucial one. It means that if A is considered as at least as consistent as B with the available knowledge, then $A \cup C$ cannot be less consistent than $B \cup C$. Note that letting $C = \bar{B}$ leads to $A \cup \bar{B} \geq_\Pi U$ so that $A \geq_\Pi B$ implies $A \cup \bar{B} \sim_\Pi U$; this means that if A is at least as possible as B, then it is possible that B implies A (since $A \sim_\Pi U$ only means that A is totally possible, and $A \sim_\Pi \emptyset$ that A is totally impossible). It can be proved (Dubois, 1986) that any set-function $\Pi : \mathcal{B} \to [0,1]$ such that

$$\Pi(U) = 1 \;;\; \Pi(\emptyset) = 0 \;;$$
$$\Pi(A) \geq \Pi(B) \Leftrightarrow A \geq_\Pi B$$

is a possibility measure and conversely. A dual relation $\geq_C$ such that $A \geq_C B$ means A is at least as certain as B, is defined as

$$A \geq_C B \text{ if and only if } \bar{B} \geq_\Pi \bar{A}$$

i.e. A is more certain than B as long as $\bar{A}$ is less possible than $\bar{B}$. The comparative certainty relation satisfies axioms 1)-4) and the following characteristic one :

N) $A \geq_C B$ implies $A \cap C \geq_C B \cap C, \forall C$

The only numerical counterparts of comparative necessity relations are necessity measures (Dubois, 1986). Note that if $C = \bar{A}$ we obtain the claim that $A \geq_C B$ implies $B \cap \bar{A} \sim_C \emptyset$, i.e. if A is at least as certain as B then $\bar{A}$ cannot be certain whatsoever in the context B. Again, $A \sim_C \emptyset$ means that A is uncertain, not that it is false. Comparative possibility relations seem to appear for the first time in the works of D. Lewis (1973) in the framework of a conditional logic of counterfactuals (see Section 3.7). Comparative certainty relations turn out to be at the core of belief revision theory (Gärdenfors, 1988) and so-called cumulative non-monotonic reasoning (Gärdenfors and Makinson, 1992) (see Section 3.10). It explains and comforts the idea that possibilistic logic, as developed in the following sections is something natural and potentially useful.

3. Possibilistic logic : the case of necessity-valued formulas

As said in Section 2, uncertain knowledge can be expressed in terms of certainty– and possibility–qualified statements ; possibilistic logic handles syntactic objects expressing inequalities resulting from these statements, i.e. constraints whose form is either $N(\varphi) \geq \alpha$ (for certainty-qualified statements) or $\Pi(\varphi) \geq \alpha$ (for possibility-qualified statements) where φ is a closed first order logic formula. These objects, called possibilistic formulas, are the basic objects of possibilistic logic ; an uncertain knowledge base is then a set of certainty– and possibility–qualified statements, and will be logically represented as a set (i.e. a conjunction) of possibilistic formulas.

This section is devoted to the study of a fragment of general possibilistic logic, where knowledge bases consist only in *necessity-qualified* statements. This fragment will be called *necessity-valued (possibilistic) logic*. After the language is presented in Section 3.1, the semantics will be studied in sections 3.2 to 3.5. Although *general* possibilistic logic is a richer framework since it enables the modelling of both necessity- and possibility-qualified statements, the necessity-valued fragment is significant, since it is sufficient for modeling a preference order upon formulae and as such, it entertains close links to the non-monotonic reasoning approach based on preferential models (Section 3.9) and belief revision theory (Section 3.10) ; moreover, it is simpler to deal with formal and algorithmic aspects by restricting to this fragment rather than by considering the general formalism. Lastly, many of the results shown in this section are easily extended to the case when possibility-qualified statements of the form $\Pi(\varphi) \geq \alpha$ are allowed. This more general possibilistic logic is considered in Section 4.

3.1. Language

A ***necessity-valued formula*** is a pair $(\varphi\ \alpha)$ where φ is a classical first-order closed formula and $\alpha \in (0,1]$ is a positive number. $(\varphi\ \alpha)$ expresses that φ is certain at least to the degree α, i.e. $N(\varphi) \geq \alpha$, where N is a necessity measure modelling our possibly incomplete state of knowledge. The right part of a possibilistic formula, i.e. α, is called the ***valuation*** of the formula (or the *weight*), and is denoted val(φ). Note that we do not consider weighted formulas of the form $(\varphi\ 0)$ since $\forall\varphi$, $N(\varphi) \geq 0$.

A ***necessity-valued knowledge base*** is then defined as a finite set (i.e. a conjunction) of necessity-valued formulas. ***PL1*** denotes the language consisting in necessity-valued formulas. From now on $\mathcal{F}^*$ denotes the set of classical formulas obtained from a set of possibilistic formulas $\mathcal{F}$, by ignoring the weights ; thus, if $\mathcal{F} = \{(\varphi_i\ \alpha_i), i = 1, \ldots, n\}$ then $\mathcal{F}^* = \{\varphi_i, i = 1, \ldots, n\}$. It will be called the *classical projection* of $\mathcal{F}$.

The language we define as such can be considered as somewhat restrictive at two different levels : firstly, only conjunctions of necessity-valued formulas are considered, not disjunctions, negations, universal and existential quantifications of possibilistic formulas, such as $(\varphi\ \alpha) \vee (\psi\ \beta)$, or $\exists x\ (\varphi\ \alpha)$, etc. ; secondly, our language does not enable valuations to be embedded, such as $((\varphi\ \alpha)\ \beta)$.

A necessity-valued knowledge base may also be seen as a collection of nested sets of (classical) formulas : α being a valuation of (0,1], let us define the *α-cut* and the *strict α-cut* of $\mathcal{F}$, denoted respectively by $\mathcal{F}_\alpha$ and $\mathcal{F}_{\overline{\alpha}}$, by

$$\mathcal{F}_\alpha = \{(\varphi\ \beta) \in \mathcal{F} \mid \beta \geq \alpha\}$$
$$\mathcal{F}_{\overline{\alpha}} = \{(\varphi\ \beta) \in \mathcal{F} \mid \beta > \alpha\}\ ;$$

their classical projections $\mathcal{F}_\alpha^*$ and $\mathcal{F}_{\overline{\alpha}}^*$ are thus

$$\mathcal{F}_\alpha^* = \{\varphi \mid (\varphi\ \beta) \in \mathcal{F} \text{ and } \beta \geq \alpha\}$$
$$\mathcal{F}_{\overline{\alpha}}^* = \{\varphi \mid (\varphi\ \beta) \in \mathcal{F} \text{ and } \beta > \alpha\}\ ;$$

thus, knowing $\mathcal{F}$ comes down to knowing the sets of classical formulas $\mathcal{F}_\alpha^*$ for α varying in (0,1]. A necessity-valued knowledge base $\mathcal{F}$ can thus be viewed as a layered knowledge base, where the higher layers (α close to 1) correspond to the most certain pieces of knowledge. Reasoning from such a knowledge base will aim at deriving conclusions by means of the most certain parts of $\mathcal{F}$. Rescher (1964) seems to be the first to think of such a layered knowledge base where the different layers reflect various levels of reliability.

Let us point out that the basic idea of possibilistic logic is to handle certainty valuations explicitly in the language. This departs for instance from Nilsson (1986)'s probabilistic logic where probability bounds are expressed by semantical constraints. Furthermore, these valuations do not appear as constants in formulae, but they are treated in a separate way, which gives the language more homogeneity: indeed they appear as *labels* associated to formulae; thus, possibilistic logic can be cast in Gabbay's Labeled Deductive Systems (Gabbay, 1991) framework, where the set of labels is the totally ordered set [0,1] and the operations defined on it follow directly from the axioms of possibility theory.

Furthermore, possibilistic logic can also be cast in Ginsberg's (1989) bilattice-based multivalued logics framework (up to a few technical differences) ; let us recall that a Ginsberg's bilattice consists in a valuation set equipped with two orderings $\leq_t$ and $\leq_k$ (based respectively on certainty and specificity). In possibilistic logic a valuation (in the sense of Ginsberg) will be a pair (α,β) where α (resp. β) is the best known lower bound of N (φ) (resp. N ($\neg\varphi$)) ; note that in the consistent case, $\alpha = 0$ or $\beta = 0$. Then the two orderings are defined by $(\alpha\ \beta) \leq_t (\alpha'\ \beta')$ iff $\alpha' \geq \alpha$ and $\beta' \leq \beta$, and $(\alpha\ \beta) \leq_k (\alpha'\ \beta')$ iff $\alpha' \geq \alpha$ and $\beta' \geq \beta$. See Lang (1991a) for more technical details.

3.2. A semantics coping with partial inconsistency

Let $\mathcal{L}$ be a classical language associated with the set $\mathcal{F}^*$ of classical formulas obtained from a set $\mathcal{F}$ of possibilistic (necessity-valued) formulas, and let Ω be the set of (classical) interpretations for $\mathcal{L}$. Let $\mathcal{L}'$ be the set of closed formulas of $\mathcal{L}$.

The semantics of a set of classical formulas $\mathcal{F}^*$ is defined by means of the subset of interpretations of $\mathcal{F}^*$ that satisfy all formulas in $\mathcal{F}^*$. Each such interpretation is called a model. In the case of a set of necessity-valued formulas, we shall consider a possibility distribution over Ω, that will represent the fuzzy set of models of $\mathcal{F}$. In other words, $\mathcal{F}$ will induce a preference relation over Ω, encoded by means of a possibility distribution. Let us consider first the valuations induced by a possibility distribution π on Ω. It is not supposed that π is necessarily normalized.

The ***possibility measure*** Π, induced (in the sense of Zadeh (1978a)) by the possibility distribution π is a function from $\mathcal{L}'$ to [0,1] defined by

$$\forall\ \varphi \in \mathcal{L}',\ \prod(\varphi) = \sup\{\pi(\omega),\ \omega \vDash \varphi\}$$ [1]

where $\omega \vDash \varphi$ means "ω is a model of φ", or "ω satisfies φ".

The dual ***necessity measure*** **N** induced by π is defined by

$$\forall\ \varphi \in \mathcal{L}',\ N(\varphi) = 1 - \prod(\neg\varphi) = \inf\{1 - \pi(\omega),\ \omega \vDash \neg\varphi\}$$

Then, necessity-valued formulas $(\varphi\ \alpha)$ express constraints of the form $N(\varphi) \geq \alpha$ on the set of possibility distributions over Ω which are compatible with the corresponding possibilistic formulas (see (11) in Section 2).

Giving up the normalisation condition $\sup\{\pi(\omega),\ \omega \in \Omega\} = 1$ slightly modifies the behavior of necessity measures with respect to the usual possibility theory : if $1 - \alpha_\pi = \sup\{\pi(\omega),\ \omega \in \Omega\} < 1$, then we have

$$\forall\varphi,\ \min(N(\varphi),N(\neg\varphi)) = \alpha_\pi > 0$$

which leads to $N(\bot) = N(\varphi \wedge \neg\varphi) = \min(N(\varphi),N(\neg\varphi)) = \alpha_\pi > 0$ instead of $N(\bot) = 0$. However the following properties still hold :

$$N(T) = 1\ ;$$
$$N(\varphi \wedge \psi) = \min(N(\varphi),N(\psi))\ ;$$
$$N(\varphi \vee \psi) \geq \max(N(\varphi),N(\psi))\ ;$$
$$\text{if } \varphi \vDash \psi \text{ then } N(\psi) \geq N(\varphi).$$

A possibility distribution π on Ω is said to ***satisfy*** the necessity-valued formula $(\varphi\ \alpha)$, iff $N(\varphi) \geq \alpha$, where N is the necessity measure induced by π. Note that a normalized possibility distribution satisfying $(\varphi\ \alpha)$ always exists (see the comment on equation (11)). We shall then use the notation $\pi \vDash (\varphi\ \alpha)$. Let $\mathcal{F} = \{\Phi_i,\ i = 1, \ldots, n\}$ be a set of possibilistic formulas $\Phi_i = (\varphi_i\ \alpha_i)$ where $\varphi_i \in \mathcal{L}'$ and $\alpha_i \in (0,1]$; a possibility distribution π is said to satisfy $\mathcal{F}$, i.e. $\pi \vDash \mathcal{F}$, iff $\forall\ i = 1, \ldots, n$, π satisfies Φ_i. Note that if π is required to be normalized, a normalized π such that $\pi \vDash \mathcal{F}$ may not exist. When this condition is not required, the "absurd possibility distribution" $\pi_\bot$ such that $\forall\ \omega \in \Omega$, $\pi_\bot(\omega) = 0$, always verifies $\pi_\bot \vDash \mathcal{F}$. If $\forall\pi$, $\pi \vDash (\varphi\ \alpha)$ is true, this is denoted $\vDash(\varphi\ \alpha)$ and $(\varphi\ \alpha)$ is said to be valid.

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1 sup{} and inf{} denote the least upper bound and greatest lower bound respectively of the subset of real numbers defined between '{}'.

Then, a possibilistic formula Φ is said to be a ***logical consequence*** of the set of possibilistic formulas $\mathcal{F}$ iff any possibility distribution satisfying $\mathcal{F}$ also satisfies Φ, i.e. $\forall\pi$, $(\pi \models \mathcal{F}) \Rightarrow (\pi \models \Phi)$. It will be denoted by $\mathcal{F} \models \Phi$.

Our semantics is somewhat similar to the semantics of Nilsson's (1986) probabilistic logic. In this logic probabilities or probability bounds are attached to formulas in propositional logic. The semantics of these weighted formulas consists of a set of probability distributions on the set of interpretations Ω, inducing probability measures on the set of closed formulas $\mathcal{L}'$, which are compatible with bounds constraining the probability of formulas in the knowledge base. The notions of logical consequences are similar in both approaches. This view of probabilistic logic goes back to De Finetti (1937) and was studied by Adams and Levine (1975) as well.

The deduction problem will then be stated in the following manner : let $\mathcal{F}$ be a set of possibilistic formulas and φ a classical formula we would like to deduce from $\mathcal{F}$; we have to compute the best valuation α (i.e. the best lower bound of a necessity degree) such that $(\varphi\ \alpha)$ is a logical consequence of $\mathcal{F}$, i.e. to compute

$$\mathrm{Val}(\varphi,\mathcal{F}) = \sup\{\alpha \in (0,1],\ \mathcal{F} \models (\varphi\ \alpha)\}$$

Example

Let $\mathcal{F} = \{(p\ 0.7), (\neg p \vee q\ \ 0.4)\}$.

$\pi \models \mathcal{F}$ $\Leftrightarrow$ $N(p) \geq 0.7$ and $N(\neg p \vee q) \geq 0.4$

$\Leftrightarrow$ $\inf\{1 - \pi(\omega), \omega \models \neg p\} \geq 0.7$ and $\inf\{1 - \pi(\omega), \omega \models p \wedge \neg q\} \geq 0.4$.

Let [p,q], [¬p,q], [p,¬q] and [¬p,¬q] be the 4 different interpretations for the propositional language generated by {p,q} (where [p,q] gives the value True to p and q, etc.). Then, it comes down to

$\pi \models \mathcal{F}$

$\Leftrightarrow$ $\pi([\neg p,q]) \leq 0.3,\ \pi([\neg p,\neg q]) \leq 0.3,\ \pi([p,\neg q]) \leq 0.6$

$\Leftrightarrow$ $\pi([\neg p,q]) \leq 0.3,\ \pi([\neg p,\neg q]) \leq 0.3,\ \pi([p,\neg q]) \leq 0.6,\ \pi([p,q]) \leq 1$.

If one looks for a normalized possibility distribution satisfying $\mathcal{F}$, this forces the equality $\pi([p,q]) = 1$. It is then obvious that $\mathcal{F} \models (q\ 0.4)$. Indeed, any possibility distribution π satisfying $\mathcal{F}$ is such that $\pi([p,\neg q]) \leq 0.6$, and thus verifies $N(q) = \min(1 - \pi([p,\neg q]),\ 1 - \pi([\neg p,\neg q])) \geq 0.4$; hence π satisfies $(q\ 0.4)$. Moreover, there is no $\alpha > 0.4$, such that $\mathcal{F} \models (q\ \alpha)$; thus $\mathrm{Val}(q,\mathcal{F}) = 0.4$.

The following properties are straightforward :

(i) $(\varphi\ \alpha) \vDash (\varphi\ \beta)\ \forall\ \alpha \geq \beta$

(ii) $\forall\ \alpha > 0, \vDash (\varphi\ \alpha)$ if and only if φ is a tautology.

Now a fundamental result can be established :

Proposition 1 : [2]

Let $\mathcal{F} = \{(\varphi_1\ \alpha_1), \ldots, (\varphi_n\ \alpha_n)\}$ be a set of necessity-valued formulas and let us define the possibility distribution $\pi_{\mathcal{F}}$ by

$$\pi_{\mathcal{F}}(\omega) = \min\{1 - \alpha_i \mid \omega \vDash \neg\varphi_i, i = 1, \ldots, n\}$$
$$= 1 \text{ if } \omega \vDash \varphi_1 \wedge \ldots \wedge \varphi$$

then for any possibility distribution π on Ω, π satisfies $\mathcal{F}$ if and only if $\pi \leq \pi_{\mathcal{F}}$, i.e. $\forall\ \omega \in\ \Omega, \pi(\omega) \leq \pi_{\mathcal{F}}(\omega)$.

Corollary 2 :

$$\mathcal{F} \vDash (\varphi\ \alpha) \text{ iff } \pi_{\mathcal{F}} \vDash (\varphi\ \alpha)$$

or in other terms $\mathrm{Val}(\varphi,\mathcal{F}) = N_{\mathcal{F}}(\varphi)$ where $N_{\mathcal{F}}$ is the necessity measure induced by $\pi_{\mathcal{F}}$.

It is worth noticing that there is an equivalence between the consistency of the classical projection $\mathcal{F}^*$ and the existence of a normalized possibility distributionπ satisfying $\mathcal{F}$. Indeed if $\pi_{\mathcal{F}}$ is normalized then $\exists\ \omega \vDash \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$. Hence $\mathcal{F}^*$ is consistent. Conversely, if $\mathcal{F}^*$ is consistent and $\omega \vDash \mathcal{F}^*$, then the possibility distribution π_ω such that $\pi_\omega(\omega) = 1$, and $\pi_\omega(\omega') = 0$ if $\omega' \neq \omega$ does satisfy $\mathcal{F}$.

Note that in the degenerate case where only two levels of possibility and certainty are used (0 and 1), possibilistic logic comes down to a "non-gradual logic of uncertainty" where a formula φ is always in one of the 3 following states[3] : TRUE (when $N(\varphi) = 1$), FALSE (when $N(\neg\varphi) = 1$) and UNKNOWN (when $N(\varphi) = N(\neg\varphi) = 0$). Thus, possibilistic logic restricted to degrees in {0,1} is a 3-valued logic, non truth-functional (indeed, as in the gradual case, it can still be the case that $N(\varphi) = N(\neg\varphi) = 0$ while $N(\varphi \vee \neg\varphi) = 1$, which happens every time φ is UNKNOWN) ; it is also a partial logic (Blamey, 1986 ; φ being UNKNOWN is then interpreted as "φ is given no truth-value"), whose semantics consists in crisp possibility

2 Proofs of the propositions can be found in the Annex.
3 If we exclude the case of complete inconsistency where $\forall\varphi$, $N(\varphi) = N(\neg\varphi) = 1$.

distributions π, which are nothing but subsets of the set of all interpretations Ω : ω is in π iff ω is a possible interpretation. This is actually a particular case of the possible worlds semantics of S5 (restricted to the case of only 1 equivalence class). Note that non-gradual possibilistic logic completely differs from Kleene's 3-valued logic, which is truth-functional and where the third truth-value must be interpreted as "half-true" and not as "unknown".

3.3. Partial inconsistency

A possibilistic knowledge base $\mathcal{F}$ whose associated possibility distribution $\pi_{\mathcal{F}}$ is such that $0 < \sup \pi_{\mathcal{F}} < 1$ is said to be partially inconsistent. Measuring the consistency of $\mathcal{F}$ consists then in evaluating to what degree there is at least one completely possible interpretation for $\mathcal{F}$, i.e. to what degree the set of possibility distributions satisfying $\mathcal{F}$ contains normalized possibility distributions ; the quantity

$$\text{Cons}(\mathcal{F}) = \sup_{\pi \models \mathcal{F}} \sup_{\omega \in \Omega} \pi(\omega) = \sup_{\omega \in \Omega} \pi_{\mathcal{F}}(\omega)$$

will be called *consistency degree* of $\mathcal{F}$, and its complement to 1, $\text{Incons}(\mathcal{F}) = 1 - \text{Cons}(\mathcal{F}) = 1 - \sup_{\omega \in \Omega} \pi_{\mathcal{F}}(\omega)$ is called the ***inconsistency degree*** of $\mathcal{F}$.

Let us now take an example when a possibilistic knowledge base is partially inconsistent.

Example

Let $\mathcal{G} = \{(\neg p \vee r\ \ 0.6), (\neg q \vee \neg r\ \ 0.9), (p\ 0.8)), (q\ 0.3)\}$. It can be checked that

$\pi \models \mathcal{G}$ and π is normalized iff

$\pi([p, q, r]) \leq 0.1$;	$\pi([p, q, \neg r]) \leq 0.4$;	$\pi([p, \neg q, r]) \leq 0.7$;
$\pi([p, \neg q, \neg r]) \leq 0.4$;	$\pi([\neg p, q, r]) \leq 0.1$;	$\pi([\neg p, q, \neg r]) \leq 0.2$;
$\pi([\neg p, \neg q, r]) \leq 0.2$;	$\pi([\neg p, \neg q, \neg r]) \leq 0.2$;	$\sup\{\pi(\omega), \omega \in \Omega\} = 1.$

This set of constraints being unsatisfiable (because of the normalization constraint), there is no normalized possibility distribution over Ω satisfying $\mathcal{G}$, which comes down to say that $\mathcal{G}$ is partially inconsistent. More specifically, $\pi_{\mathcal{G}}$, obtained by turning inequalities into equalities is such that $\sup \pi_{\mathcal{G}} = 0.7$.

It would not be fully satisfactory to define a logic which handles degrees of uncertainty without allowing for degrees of (partial) inconsistency. Indeed, let us consider the above example where we suppose that p, q and r respectively express "the hostages will be freed" (p); "Peter is going to be the victim of an affair" (q) ; "Peter will be elected" (r) respectively. Then the formulas contained in $\mathcal{F}$ express that it is moderately certain that if the hostages are freed then Peter will be elected, that it is almost certain that if Peter is victim of an affair then he will not be elected, that it is rather certain that the hostages are going to be freed and that it is weakly certain that Peter will be the victim of an affair. The inconsistency comes from the beliefs of the experts who gave the information stored in the knowledge base. However, the expert who gave the last formula was only weakly certain of what he said, so that the inconsistency should be relativized. Since the first three formulas of $\mathcal{K}$ are strictly more certain than the last one, we would like our logic to behave as if the set of formulas were only partially inconsistent, its inconsistency degree being the valuation of the weakest formula involved in the contradiction ; then, the deduction of a formula with a valuation strictly greater than this inconsistency degree should still be permitted ; since this deduction involves only a consistent part of the knowledge base made here of the most certain pieces of information in the example, we might still deduce non-trivially (r 0.6) but not, for instance, (r 1), as we shall see later. However a conclusion deduced from a partially inconsistent knowledge base should be regarded as more brittle than what is derived from a consistent one.

Partial inconsistency extends inconsistency in classical logic in the following sense : let $F = \{\varphi_i \mid i = 1, \ldots, n\}$ be a set of classical formulas and let us associate to F the set of completely certain necessity-valued formulas $\mathcal{F} = \{(\varphi_i\ 1), i = 1, \ldots, n\}$; then, it can be proved immediately that if F is consistent then Incons($\mathcal{F}$) = 0 and if F is inconsistent then Incons($\mathcal{F}$) = 1. Thus, necessity-valued logic enables the gradation of inconsistency : if Incons($\mathcal{F}$) = 0 then $\mathcal{F}$ is *completely consistent*, if Incons($\mathcal{F}$) = 1 then $\mathcal{F}$ is *completely inconsistent*, and if 0 < Incons($\mathcal{F}$) < 1 then $\mathcal{F}$ is *partially inconsistent*. The strong link, already pointed out above, between partial inconsistency and inconsistency in classical logic can be restated as follows : Incons($\mathcal{F}$) = 0 if and only if the classical projection $\mathcal{F}^*$ is consistent in the classical sense.

A partially inconsistent knowledge base entails the contradiction with a positive necessity degree, i.e. $\mathcal{F} \models (\bot\ \alpha)$ for some $\alpha > 0$. Indeed the following result is easy to establish :

Proposition 3 :

$$\text{Incons}(\mathcal{F}) = \inf\{N(\bot) \mid \pi \models \mathcal{F}\} = N_{\mathcal{F}}(\bot) = \sup\{\alpha, \mathcal{F} \models (\bot\ \alpha)\}$$

where N is the necessity distribution induced by π.

This equality achieves to justify the terminology "inconsistency degree" since Incons($\mathcal{F}$) is the smallest necessity degree of the contradiction $\bot$ for all possibility distributions satisfying $\mathcal{F}$.

3.4. Fuzzy sets of interpretations and best models

As seen above, knowing the possibility distribution $\pi_{\mathcal{F}}$ is sufficient for any deduction problem in necessity-valued logic (including the computation of the inconsistency degree). It is important to notice that

- $\pi_{\mathcal{F}}$ minimizes the inconsistency among the possibility distributions satisfying $\mathcal{F}$, i.e. $N_{\mathcal{F}}(\bot) = \text{Incons}(\mathcal{F}) = \inf\{N(\bot) \mid \pi \models \mathcal{F}\}$;
- among the possibility distributions π satisfying $\mathcal{F}$ and minimizing the inconsistency (i.e., the least subnormalized ones), $\pi_{\mathcal{F}}$ is *the least specific* one (as shown by Proposition 1). Indeed $\pi_{\mathcal{F}}$ is the possibility distribution on Ω obtained by applying the principle of minimum of specificity on the set of constraints expressed by the necessity-valued formulas of $\mathcal{F}$.

The link between our semantics and the semantics of classical logic can be precisely described as follows. In classical logic, a set of formulas $F = \{\varphi_1, \ldots, \varphi_n\}$, induces a partition of the set of interpretations Ω into two subsets : the subset M(F) of models of F and the subset M(¬F) of interpretations which do not satisfy F (also equal to the set of models of ¬F). Then F is said to be consistent if and only if M(F) is not empty, inconsistent otherwise, and valid iff $M(F) = \Omega$. Noticing that F corresponds to the set of necessity-valued formulas $\mathcal{F} = \{(\varphi_1\ 1), \ldots, (\varphi_n\ 1)\}$, we may compute the (crisp) possibility distribution $\pi_{\mathcal{F}}$:

$$\begin{aligned}\pi_{\mathcal{F}}(\omega) &= \inf\{1 - \alpha_i \mid \omega \models \neg\varphi_i,\ i = 1, ..., n\} \text{ where all } \alpha_i \text{ are } 1,\\ &= 1 \text{ if } \forall i,\ \omega \models \varphi_i\\ &= 0 \text{ otherwise.}\end{aligned}$$

Thus the least specific possibility distribution satisfying $\mathcal{F}$ is the (crisp) membership function of the set of models of F.

In necessity-valued logic, when the valuations are allowed to be intermediary, $\pi_{\mathcal{F}}$ defines a *fuzzy subset* of the interpretation set Ω, denoted $\tilde{M}(\mathcal{F})$ which can be seen as the *fuzzy set of models* of $\mathcal{F}$, its membership function being $\mu_{\tilde{M}(\mathcal{F})}(\omega) = \pi_{\mathcal{F}}(\omega)$. The quantity $\pi_{\mathcal{F}}(\omega)$ represents the *compatibility* degree of ω with $\mathcal{F}$, measuring to what degree ω is a model of $\mathcal{F}$. Lastly it is easy to check that $\mathcal{F} \models \mathcal{G}$ iff $\tilde{M}(\mathcal{F})$ is included in $\tilde{M}(\mathcal{G})$ (in the *fuzzy inclusion* sense : $\tilde{M}(\mathcal{F}) \subseteq \tilde{M}(\mathcal{G}) \Leftrightarrow \mu_{\tilde{M}(\mathcal{F})} \leq \mu_{\tilde{M}(\mathcal{G})}$).

As it turns out, to establish that φ is deducible from $\mathcal{F}$ with certainty degree α, is to say that the models of $\neg\varphi$ are compatible with $\tilde{M}(\mathcal{F})$ at most to the degree $1 - \alpha$, which reads : $\sup\{\pi_{\mathcal{F}}(\omega) \mid \omega \in M(\neg\varphi)\} \leq 1 - \alpha$. It can be checked that it corresponds to the result which would be obtained by representing each necessity-valued formula by means of the least specific possibility distribution that satisfies it (on the set of interpretations), combining conjunctively these representations and projecting the result of this combination on $M(\varphi)$ and on $M(\neg\varphi)$. This indicates that the deduction process in possibilistic logic can be viewed as a particular case of Zadeh's theory of approximate reasoning, when the universe of discourse is a set of logical interpretations.

The following proposition leads to an important definition :

Proposition 4 :

The least upper bound in the computation of Incons($\mathcal{F}$) is attained, i.e. there exists (at least) an interpretation ω* such that $\pi_{\mathcal{F}}(\omega^*) = \sup_{\omega \in \Omega} \pi_{\mathcal{F}}(\omega)$.

Then, the interpretations ω* maximizing $\pi_{\mathcal{F}}$ will be called the ***best models*** of $\mathcal{F}$. They are the most compatible with $\mathcal{F}$ among the set of all interpretations **Ω**, or equivalently the interpretations maximizing the membership degree to the fuzzy set of models of $\mathcal{F}$. The previous result shows that the set of best models is never empty.

Interpretations may be ordered according to their compatibility degrees. Thus, *ordering the formulas in the knowledge base leads to ordering the interpretations*. Then, selecting the best (one of the best) model(s) is similar to making a choice among several extensions in the sense of default logic (Reiter, 1980).

Example

Let $\mathcal{F} = \{(u\ \ \alpha), (\neg u \vee v\ \ \beta), (\neg v\ \ \gamma)\}$; then

$\pi_{\mathcal{F}}([u,v]) = 1 - \gamma$;
$\pi_{\mathcal{F}}([u,\neg v]) = 1 - \beta$;
$\pi_{\mathcal{F}}([\neg u,v]) = 1 - \max(\alpha, \gamma)$;
$\pi_{\mathcal{F}}([\neg u,\neg v]) = 1 - \alpha$;

the subnormalization degree of $\pi_{\mathcal{F}}$ being max $(1 - \alpha, 1 - \beta, 1 - \gamma)$, we get

$\text{Incons}(\mathcal{F}) = 1 - \sup \pi_{\mathcal{F}} = \min(\alpha, \beta, \gamma)$

and the set of best models of $\mathcal{F}$ is

- $\{[\neg u,\neg v]\}$ if $\alpha < \min(\beta, \gamma)$
- $\{[u,\neg v]\}$ if $\beta < \min(\alpha, \gamma)$
- $\{[u,v]\}$ if $\gamma < \min(\alpha, \beta)$
- $\{[u,\neg v], [\neg u,\neg v]\}$ if $\alpha = \beta < \gamma$
- $\{[u,v], [\neg u,v], [\neg u,\neg v]\}$ if $\alpha = \gamma < \beta$
- $\{[u,v], [u,\neg v]\}$ if $\beta = \gamma < \alpha$

- {[u,v], [¬u,v], [u,¬v], [¬u,¬v]} if $\alpha = \beta = \gamma$

This example indicates that the inconsistency degree of an inconsistent possibilistic knowledge base $\mathcal{F}$ is the valuation of the least certain formula involved in the strongest contradiction in $\mathcal{F}$. It is easy to see that $\forall\ \Phi \in \mathcal{F}$, Incons($\mathcal{F} - \{\Phi\}$) $\leq$ Incons($\mathcal{F}$). Let $\mathcal{F}' \subseteq \mathcal{F}$ such that Incons($\mathcal{F}'$) = Incons($\mathcal{F}$) > 0 and $\forall\ \Phi \in \mathcal{F}'$, Incons($\mathcal{F}' - \{\Phi\}$) = 0, i.e. $\mathcal{F}' - \{\Phi\}$ is consistent. $\mathcal{F}'$ is called a *strongly minimal inconsistent subset* of $\mathcal{F}$. Then the following result holds :

Proposition 5 :

The inconsistency degree of an inconsistent[3] possibilistic knowledge base $\mathcal{F}$ is the smallest weight of possibilistic formulas in any inconsistent subset $\mathcal{F}'$ of $\mathcal{F}$. More precisely, if Incons($\mathcal{F}$) = α > 0 then there exists at least one formula ($\varphi\ \alpha$) $\in \mathcal{F}'$ and $\forall$($\varphi'\ \beta$) $\in \mathcal{F}'$, $\beta \geq \alpha$.

3.5. Deduction under partial inconsistency

Let $\mathcal{F}$ be a partially inconsistent necessity-valued knowledge base, that is, $\mathcal{F} \models (\bot$ Incons($\mathcal{F}$)) with Incons($\mathcal{F}$) > 0 ; thus, since for any formula φ we have N(φ) $\geq$ N($\bot$), any formula φ is deducible from $\mathcal{F}$ with a valuation greater or equal to Incons($\mathcal{F}$). It means that any deduction such that $\mathcal{F} \models (\varphi\ \alpha)$ with α = Incons($\mathcal{F}$) may be only due to the partial inconsistency of $\mathcal{F}$ and has perhaps nothing to do with φ. These deductions are called ***trivial deductions*** ; on the contrary, deductions of necessity-valued formulae $\mathcal{F} \models (\varphi\ \alpha)$ with α > Incons($\mathcal{F}$) are not caused by the partial inconsistency ; they are called ***non-trivial deductions***.

Thus, Incons($\mathcal{F}$) acts as a *threshold* inhibiting all formulas of $\mathcal{F}$ with a valuation equal to or under this threshold. The following result shows its role as a threshold for the deduction problem more deeply :

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3 If $\mathcal{F}$ is consistent, i.e. Incons($\mathcal{F}$) = 0, there is of course no minimal inconsistent subsets of $\mathcal{F}$ and Proposition 5 is therefore unapplicable.

Proposition 6 :
Let $\mathcal{F}$ be a set of possibilistic formulas and let Incons($\mathcal{F}$) = inc ; then

(i) $\mathcal{F}$ is semantically equivalent to $\mathcal{F}_{inc}$ and to $\mathcal{F}_{\overline{inc}} \cup \{(\bot\ inc)\}$
(ii) $\mathcal{F}_{\overline{inc}}$ is consistent
(iii) if $\mathcal{F} \vDash (\psi\ \alpha)$ non trivially (i.e. with α > inc) then $\mathcal{F}_{\overline{inc}} \vDash (\psi\ \alpha)$.

This result shows that only the consistent part of $\mathcal{F}$ consisting of the formulas with a weight strictly greater than the inconsistency degree is significant for the deduction process. The next result establishes a link between inconsistency degrees and inconsistency in classical logic.

Proposition 7 (partial inconsistencies and α-cuts) :

(1) (Dubois and Prade, 1987) Let $\mathcal{F}$ be a set of necessity-valued formulae ; then Incons($\mathcal{F}$) = 0 if and only if $\mathcal{F}$* is consistent in the classical sense.

(2) Incons($\mathcal{F}$) $= \sup \{\alpha \mid \mathcal{F}_{\alpha}^* \text{ inconsistent}\}$
$= \inf \{\alpha \mid \mathcal{F}_{\overline{\alpha}}^* \text{ consistent}\}$

and these two bounds are reached.

So, necessity-valued logic is close to classical logic in the sense that a necessity-valued knowledge base is equivalent to a finite family of classical knowledge bases. The impact of this result on automated deduction is the possibility of computing an inconsistency degree using only classical first-order logic procedures, which leads to this result about complexity in propositional necessity-valued logic :

Proposition 8 :
Determining the inconsistency degree of a propositional necessity-valued knowledge base is a NP-complete problem.

The following results generalize the semantic versions of the classical deduction and refutation theorems to necessity-valued logic :

Proposition 9 *(deduction theorem)* ***:***

$$\mathcal{F} \cup \{(\varphi\ 1)\} \vDash (\psi\ \alpha) \quad \text{iff} \quad \mathcal{F} \vDash (\varphi \rightarrow \psi\ \ \alpha)$$

Proposition 10 *(refutation theorem)* ***:***

$$\mathcal{F} \vDash (\varphi\ \alpha) \quad \text{iff} \quad \mathcal{F} \cup \{(\neg\varphi\ 1)\} \vDash (\bot\ \alpha)$$

or equivalently :

$$\mathrm{Val}(\varphi,\mathcal{F}) = \mathrm{Incons}(\mathcal{F} \cup \{(\neg\varphi\ 1)\})$$

Thus, *any deduction problem in possibilistic logic comes down to computing an inconsistency degree* : if we want to know whether $(\varphi\ \alpha)$ is a logical consequence of $\mathcal{F}$ or not, it is sufficient to compute $\mathrm{Incons}(\mathcal{F} \cup (\neg\varphi\ 1))$, which is equal to the largest valuation α^* such that $\mathcal{F} \vDash (\varphi\ \alpha^*)$.

Lastly we give the following result, stating that in order to deduce a possibilistic formula $(\varphi\ \alpha)$, only the formulas with a weight greater or equal to α are useful for that purpose :

Proposition 11 :
Let $\mathcal{F}$ be a possibilistic knowledge base and $(\varphi\ \alpha)$ a necessity-valued formula. Then

$$\mathcal{F} \vDash (\varphi\ \alpha) \text{ if and only if } \mathcal{F}_\alpha \vDash (\varphi\ \alpha).$$

3.6. A formal system for necessity-valued logic

In this section we are interested in giving a formal system for possibilistic logic, equipped with the inconsistency-tolerant semantics. First of all, it is worth noticing that all tautologies of PL1 are the possibilistic formulas of the form $(\tau\ \alpha)$ where τ is a classical tautological formula and α a valuation (see property (ii) stated just before Proposition 1). Hence finding a formal system producing all possibilistic tautologies is straightforward. From now on we shall focus on the following problem : is there a formal system, i.e. a set of axioms and inference rules, such that from any set of possibilistic formulae $\mathcal{F}$ and for any possibilistic formula Φ, Φ is a logical consequence of $\mathcal{F}$ if and only if Φ is derivable from $\mathcal{F}$ in this formal system ?

We are proposing the following formal system for PL1 (see also Lang (1991a)).

Axioms schemata :

(A1) $(\varphi \rightarrow (\psi \rightarrow \varphi)\ \ 1)$
(A2) $((\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi))\ \ 1)$
(A3) $((\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)\ \ 1)$
(A4) $((\forall x\ (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow (\forall x\ \psi))\ \ 1)$ if x does not appear in φ and is not bound in ψ
(A5) $((\forall x\ \varphi) \rightarrow \varphi_{x|t}\ \ 1)$ if x is free for t in φ

Inference rules :

(GMP)	$(\varphi\ \alpha), (\varphi \rightarrow \psi\ \ \beta) \vdash (\psi\ \min(\alpha,\beta))$	
(G)	$(\varphi\ \alpha) \vdash ((\forall x\ \varphi)\ \alpha)$	if x is not bound in φ
(S)	$(\varphi\ \alpha) \vdash (\varphi\ \beta)$	if $\beta \leq \alpha$

The axioms are those of a well-known Hilbert formal system for classical logic weighted by 1. A rule of inference similar to GMP has been first proposed in Rescher (1976). It has been rediscovered in the fuzzy set setting in (Prade, 1982) see also (Dubois and Prade, 1985b). GMP is called *graded modus ponens* ; it has been also used in (Froidevaux and Grossetête, 1990) in the framework of graded default theories.

Proposition 12 :
The proposed formal system is sound and complete with respect to the inconsistency-tolerant semantics of possibilistic logic, i.e. for any set of possibilistic formulas $\mathcal{F}$ we have

$$\mathcal{F} \vDash (\varphi\ \alpha) \text{ if and only if } \mathcal{F} \vdash (\varphi\ \alpha)$$

where $\mathcal{F} \vdash (\varphi\ \alpha)$ means : "$(\varphi\ \alpha)$ can be derived from $\mathcal{F}$ in the above system".

Thus necessity-valued logic is *axiomatizable*.

3.7. Qualitative possibilistic logic and conditional logic

A qualitative possibilistic knowledge base is a finite set of (strict or non-strict) inequalities of the form $\varphi > \psi$ or $\varphi \geq \psi$ where "$\geq$" is a qualitative necessity relation (see end of Section 2). Thus, $\varphi > \psi$ and $\varphi \geq \psi$ respectively means $N(\varphi) > N(\psi)$ and $N(\varphi) \geq N(\psi)$. Satisfaction and entailment are defined by means of qualitative necessity measures (see end of Section 2). In (Fariñas del Cerro and Herzig, 1991), an equivalence is shown between qualitative possibilistic logic and a conditional logic studied by Lewis (1973). Briefly, a model in this conditional logic consists in a set of classical interpretations Ω and an *absolute sphere system* $\mathcal{S}$, which is a set of nested subsets of Ω, closed for union and intersection ($\mathcal{S}$ corresponds to the set of α-cuts induced by a qualitative necessity). In the finite propositional case, it has been shown that a qualitative necessity relation is equivalent to a such a model. Then, satisfiability and validity in qualitative possibilistic logic are equivalent to satisfiability and validity in the conditional logic. See Fariñas del Cerro, Herzig and Lang (1992). See Boutilier (1992) for another embedding of possibilistic logic in a modal framework. An extension of possibilistic logic where qualitative possibilistic ordering relations between propositions are handled in the language is outlined in (Benferhat, Dubois and Prade, 1992).

3.8. Automated deduction

In this section we focus on automating the computation of the inconsistency degree of a necessity-valued knowledge base. Two well-known automated deduction methods have been generalized to possibilistic logic : resolution (Dubois and Prade, 1987) and (in the propositional case) the Davis and Putnam semantic evaluation procedure (Lang, 1990). Here we focus mainly on resolution for which we give soundness and completeness results.

3.8.1. Clausal form

In order to extend resolution to possibilistic logic, a clausal form is first defined. A ***possibilistic clause*** is a possibilistic formula $(c\ \alpha)$ where c is a first-order or propositional clause and α is a valuation of (0,1]. A ***possibilistic clausal form*** is a universally quantified conjunction of possibilistic clauses.

The problem of finding a clausal form of $\mathcal{F}$ whose inconsistency degree is the same as $\mathcal{F}$ always has a solution in PL1. Indeed there exists a clausal form $\mathcal{C}$ of $\mathcal{F}$ such that Incons($\mathcal{C}$) = Incons($\mathcal{F}$), which gereralizes the result holding in classical logic about the equivalence between the inconsistency of a set of formulas and the inconsistency of its clausal form. Indeed the possibilistic clausal form $\mathcal{C}$ of $\mathcal{F}$ can be obtained by the following method :

Let $\mathcal{F} = \{(\varphi_i\ \alpha_i), i = 1, \ldots, n\}$
Put each φ_i into clausal form, i.e. $\varphi_i = (\forall) \wedge_j (c_{ij})$ where c_{ij} is a universally-quantified classical first-order clause ;
$\mathcal{C} \leftarrow (\forall) \wedge_{i,j} \{(c_{ij}\ \alpha_i)\}$

Proposition 13 : Incons($\mathcal{C}$) = Incons($\mathcal{F}$).

3.8.2. Necessity-valued resolution

Once a clausal form is defined for a given necessity-valued knowledge base, the resolution principle may be easily extended from classical first-order logic to necessity-valued logic, in order to compute its inconsistency degree.

The following possibilistic resolution rule between two possibilistic clauses $(c_1\ \alpha_1)$ and $(c_2\ \alpha_2)$ has been established by Dubois and Prade (1987) :

(R) $\quad (c_1\ \alpha_1)\ (c_2\ \alpha_2) \vdash (R(c_1, c_2)\ \min(\alpha_1, \alpha_2))$

where $R(c_1,c_2)$ is any classical resolvent of c_1 and c_2. The following result establishes the soundness of this resolution rule :

Proposition 14 (soundness of rule (R)) ***:***
Let $\mathcal{C}$ be a set of possibilistic clauses, and C = (c α) a possibilistic clause obtained by a finite number of successive applications of (R) to $\mathcal{C}$; then $\mathcal{C} \vDash C$.

The resolution rule for necessity-valued clauses locally performs at the syntactic level what the combination/projection principle (Section 2) does in approximate reasoning.

Moreover resolution for necessity-valued clauses is complete for refutation and we have the following results :

Proposition 15 (soundness and completeness of refutation by resolution in PL1) ***:***
Let $\mathcal{F}$ be a set of *necessity-valued* first-order formulas and $\mathcal{C}$ the set of necessity-valued clauses obtained from $\mathcal{F}$; then the valuation of the optimal refutation by resolution from $\mathcal{C}$ is the inconsistency degree of $\mathcal{F}$.

This result was first established in (Dubois, Lang and Prade, 1989).

***Corollary* :**
Let φ be a classical formula and $\mathcal{C}'$ the set of possibilistic clauses obtained from $\mathcal{F} \cup \{(\neg\varphi\ 1)\}$; then the valuation of the optimal refutation by resolution from $\mathcal{C}'$ is $Val(\varphi,\mathcal{F})$. This corollary immediately stems from propositions 14 and 15.

Thus refutation by resolution can be used for computing the inconsistency degree of a necessity-valued knowledge base. We consider a set $\mathcal{F}$ of possibilistic formulas (the knowledge base) and a formula φ ; we want to know the maximal valuation with which $\mathcal{F}$ entails φ, i.e. $Val(\varphi,\mathcal{F}) = \sup\{\alpha \in (0,1], \mathcal{F} \vDash (\varphi\ \alpha)\}$. This request can be answered by using refutation by resolution, which is extended to possibilistic logic as follows :

Refutation by resolution :

1. Put $\mathcal{F}$ into clausal form $\mathcal{C}$;
2. Put φ into clausal form ; let $c_1, \ldots, c_m$ the obtained clauses ;
3. $\mathcal{C}' \leftarrow \mathcal{C} \cup \{(c_1\ 1)), \ldots, (c_n\ \ 1))\}$
4. Search for a deduction of $(\bot\ \bar{\alpha})$ by applying repeatedly the resolution rule (R) from $\mathcal{C}'$, with $\bar{\alpha}$ maximal ;
5. $Val(\varphi,\mathcal{F}) \leftarrow \bar{\alpha}$.

An implementation based on an A*-like ordered search method has been proposed for finding out the refutation with $\bar{\alpha}$ maximal first. See (Dubois, Lang and Prade, 1987).

Illustrative example

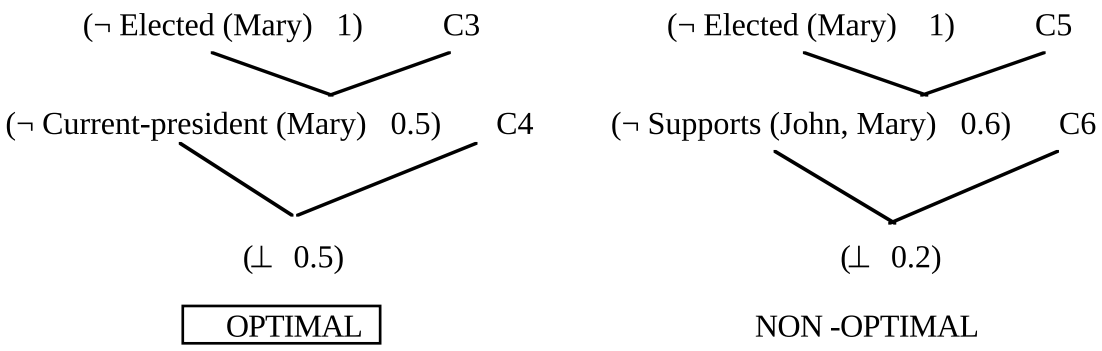
Let $\mathcal{F}$ = {Φ1, …, Φ6} be the following possibilistic knowledge base, concerning an election whose only candidates are Mary and Peter :

Φ_1 ((Elected(Peter) ∨ Elected(Mary)) ∧ (¬Elected(Peter) ∨ ¬Elected(Mary)) 1)
Φ_2 (∀x ¬Current-president(x) ∨ Elected(x) 0.5)
Φ_3 (Current-president(Mary) 1)
Φ_4 (∀x ¬Supports(John,x) ∨ Elected(x) 0.6)
Φ_5 (Supports(John,Mary) 0.2)
Φ_6 (∀x ¬Victim-of-an-affair(x) ∨ ¬Elected(x) 0.7)

$\mathcal{F}$ is equivalent to the set of possibilistic clauses $\mathcal{C}$ = {C_1, …, C_7} :

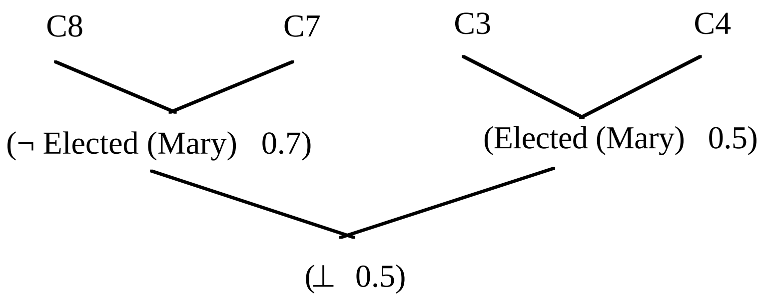
C_1 (Elected(Peter) ∨ Elected(Mary) 1)
C_2 (¬Elected(Peter) ∨ ¬Elected(Mary) 1)
C_3 (¬Current-president(x) ∨ Elected(x) 0.5)
C_4 (Current-president(Mary) 1)
C_5 (¬Supports(John,x) ∨ Elected(x) 0.6)
C_6 (Supports(John,Mary) 0.2)
C_7 (¬Victim-of-an-affair(x) ∨ ¬Elected(x) 0.7)

We cannot find any refutation from $\mathcal{C}$; hence, $\mathcal{C}$ is completely consistent, i.e. Incons($\mathcal{C}$) = 0. Let us now find the best lower bound of the necessity degree of the formula "Elected(Mary)". Let $\mathcal{C}$' = $\mathcal{C}$ ∪ {(¬Elected(Mary) 1)} ; then there exist two distinct refutations by resolution from $\mathcal{C}$', which are :



Hence we conclude that $\mathcal{C} \models$ (Elected(Mary) 0.5), i.e. it is moderately certain that Mary will be elected ; this degree 0.5 is maximal, i.e. Val(Elected(Mary), $\mathcal{C}$) = 0.5.

Then we learn that Mary is being the victim of an affair, which leads us to update the knowledge base by adding to $\mathcal{C}$ the possibilistic clause C_8 : (Victim-of-an-affair(Mary) 1)). Let $\mathcal{C}_1$ be the new knowledge base, $\mathcal{C}_1 = \mathcal{C} \cup \{C_8\}$. Then, we can find a 0.5-refutation from $\mathcal{C}_1$ (which is optimal) :



Hence $\mathcal{C}_1$ is partially inconsistent, with Incons ($\mathcal{C}_1$) = 0.5.

The refutation which had given N(Elected(Mary)) $\geq$ 0.5 can still be obtained from $\mathcal{C}_1$ but since its valuation is not greater than Incons($\mathcal{C}_1$), it becomes a trivial deduction. Contrarily, adding to $\mathcal{C}_1$ the possibilistic clause (Elected(Mary) 1), we find this time a 0.7-refutation ; and since 0.7 > Incons($\mathcal{C}_1$), the deduction $\mathcal{C}_1 \models$ (¬Elected(Mary) 0.7) is non-trivial ; it could be shown that we also have $\mathcal{C}_1 \models$ (Elected(Peter) 0.7).

3.8.3. <u>Semantic evaluation (propositional case)</u>

Resolution is a syntactic proof procedure ; as it is the case in classical logic, semantic procedures for necessity-valued logic are interesting because they are more constructive than syntactic ones. Indeed, necessity-valued resolution only gives the inconsistency degree of a necessity-valued knowledge base $\mathcal{F}$, but it does not give the best model(s) of $\mathcal{F}$ (as defined in Section 3.4). As said in Section 3.3, computing Incons($\mathcal{F}$) comes down to computing the degree of consistency sup $\pi_{\mathcal{F}}$ of the possibility distribution $\pi_{\mathcal{F}}$.

A naive idea would consist in computing $\pi_{\mathcal{F}}(\omega)$ for all ω in Ω, for example by building a semantic tree as in classical propositional logic. The main problem is that complexity is prohibitive, since it requires to compute 2^p values of $\pi_{\mathcal{F}}(\omega)$ where p is the number of atomic propositions in $\mathcal{F}$. A semantic evaluation algorithm for necessity-valued logic, based on an possibilistic extension of the Davis and Putnam (1960) procedure, computes also the best models (or one of the best models, if preferred) of a clausal necessity-valued propositional knowledge base $\mathcal{C}$ by building a (small) part of a semantic tree for $\mathcal{C}$, by evaluating literals successively. Some techniques improve the efficiency of semantic evaluation by transforming it into the search in a min-max tree, and then pruning branches by two techniques, one being the well-known alpha-beta pruning method, the other one being a generalisation of the so-called "model partition theorem" obtained for (classical) propositional logic by Jeannicot, Oxusoff and Rauzy (1988). See (Lang, 1990) for further details.

3.9. Non-monotonic aspects

Whether possibilistic logic can be considered as monotonic or not, depends on what kind of deduction is allowed. The operator $\vDash$ is monotonic : indeed, due to the definitions of satisfiability, any possibility distribution π satisfying $\mathcal{F} \cup \mathcal{F}'$ also satisfies $\mathcal{F}$, hence the result :

$$\text{if } \mathcal{F} \vDash (\varphi\ \alpha) \text{ then } \mathcal{F} \cup \mathcal{F}' \vDash (\varphi\ \alpha)$$

where $\mathcal{F}$ and $\mathcal{F}'$ are any sets of possibilistic formulas and $(\varphi\ \alpha)$ is any necessity-valued formula.

However we may wish to take into account only non-trivial deductions (we recall that the deduction $\mathcal{F} \vDash (\varphi\ \alpha)$ is non-trivial iff $\alpha > \text{Incons}(\mathcal{F})$; see Section 3.5). Let us then define the *non-trivial deduction operator* $|\!\approx$ by

$$\mathcal{F} \mid\!\approx (\varphi\ \alpha) \text{ iff } \mathcal{F} \vDash (\varphi\ \alpha) \text{ and } \alpha > \text{Incons}(\mathcal{F}).$$

Then it can be the case that $\mathcal{F} \mid\!\approx (\varphi\ \alpha)$ and $\mathcal{F} \cup \mathcal{F}' \mid\!\not\approx (\varphi\ \alpha)$; indeed this situation occurs when $\text{Incons}(\mathcal{F}) < \alpha \leq \text{Incons}(\mathcal{F} \cup \mathcal{F}')$. Hence $|\!\approx$ is non-monotonic.

We now give a detailed example illustrating this non-monotonic behaviour.

Example

We now consider again the knowledge base of Section 3.8.2. It can be established that $\text{Incons}(\mathcal{F}) = 0$, i.e. $\mathcal{F}$ is completely consistent. We are interested in knowing who will be elected and with the maximal certainty degree. It can be proved that

$$\mathcal{F} \vDash (\text{Elected(Mary)}\ 0.5)\ ;\ \mathcal{F} \vDash (\neg\text{Elected(Mary)}\ 0)$$
$$\mathcal{F} \vDash (\text{Elected(Peter)}\ 0)\ ;\ \mathcal{F} \vDash (\neg\text{Elected(Peter)}\ 0.5)$$

i.e. it is moderately certain that Mary will be elected (or equivalently that Peter will not) ; this degree 0.5 is maximal, i.e. $\text{Val}(\mathcal{F}, \text{Elected(Mary)}) = 0.5$. Since $\text{Incons}(\mathcal{F}) = 0$, we may also write that

$$\mathcal{F} \mid\!\approx (\text{Elected(Mary)}\ 0.5)\ ;\ \mathcal{F} \mid\!\approx (\neg\text{Elected(Peter)}\ 0.5)$$

Then, we learn that Mary is being the victim of an affair (which is a completely certain information). This leads us to update the knowledge base by adding to $\mathcal{F}$ the possibilistic formula

$$\Phi_7 : (\text{Victim-of-an-affair(Mary)}\ 1)$$

Let $\mathcal{F}_1$ be the new knowledge base, $\mathcal{F}_1 = \mathcal{F} \cup \{\Phi_7\}$. It can be proved that $\mathcal{F}_1$ is partially inconsistent, with Incons($\mathcal{F}_1$) = 0.5. Indeed the new information leads to prove that Mary will not be elected, whereas the previous knowledge base $\mathcal{F}$ leads to prove that Mary will be elected (each time with a non-total certainty). It can always be proved that

$$\mathcal{F}_1 \vDash (\text{Elected(Mary)}\ 0.5)$$
$$\mathcal{F}_1 \vDash (\neg\text{Elected(Peter)}\ 0.5)$$

but these deductions are now invalidated by the inconsistency threshold, hence trivial. Using the non-trivial deduction operator, it comes down to writing that the previously non-trivial deductions $\mathcal{F}$ |≈ (Elected(Mary) 0.5) and $\mathcal{F}$ |≈ (¬Elected(Peter) 0.5) can no longer be made with $\mathcal{F}_1$. In this case we capture a non-monotonic behaviour.

Besides, we have now

$$\mathcal{F}_1 \vDash (\neg\text{Elected(Mary)}\ 0.7)$$
$$\mathcal{F}_1 \vDash (\text{Elected(Peter)}\ 0.7)$$

and these deductions are non-trivial since 0.7 > Incons($\mathcal{F}_1$), i.e.

$$\mathcal{F}_1 \mid\approx (\neg\text{Elected(Mary)}\ 0.7)$$
$$\mathcal{F}_1 \mid\approx (\text{Elected(Peter)}\ 0.7)$$

meaning that it is now almost certain that Mary will not be elected and that Peter will be. Hence, updating the knowledge base leads us to an opposite conclusion.

The links between nonmonotonic reasoning and possibilistic logic (first pointed out in Dubois, Lang and Prade, 1989 and Léa Sombé, 1990) are now more deeply investigated. Let $\mathcal{F}$ be a possibilistic knowledge base containing necessity-valued formulas and $\pi_{\mathcal{F}}$ be the corresponding minimally specific possibility distribution on interpretations, namely if $\mathcal{F}$ = $\{(\varphi_i\ \alpha_i), i=1,n\}$, then $\pi_{\mathcal{F}}(\omega) = \min\{1 - \alpha_i \mid \omega \vDash \neg\varphi_i, i=1,n\}$.

Consider the preference relation, denoted $\sqsubset$, on the set of interpretations Ω, defined by

$$\omega \sqsubset \omega' \Leftrightarrow \pi_{\mathcal{F}}(\omega) < \pi_{\mathcal{F}}(\omega') \ .$$

This relation equips Ω with a strict partial order as requested in Shoham (1988). He defines a preferred model of a formula φ as an interpretation ω such that $\omega \vDash \varphi$ and $\nexists\ \omega' \neq \omega$, $\omega' \vDash \varphi$ and $\omega \sqsubset \omega'$. It is a maximal element in $\{\omega' \mid \omega' \vDash \varphi\}$ in the sense of $\sqsubset$ and this is denoted $\omega \vDash_{\sqsubset} \varphi$. Moreover φ is said to preferentially entail ψ, denoted $\varphi \vDash_{\sqsubset} \psi$ if and only if all preferred models of φ satisfy ψ ; more precisely

$$\varphi \vDash_{\sqsubset} \psi \Leftrightarrow \forall \omega,\ \omega \vDash_{\sqsubset} \varphi \text{ implies } \omega \vDash \psi$$

If $\sqsubset$ is induced by a possibility distribution $\pi_{\mathcal{F}}$, it is easy to verify that $\omega \vDash_{\sqsubset} \varphi$ if and only if $\pi_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi)$ that is, ω is a best model of $\mathcal{F} \cup \{(\varphi\ 1)\}$. When $\pi_{\mathcal{F}}(\omega) = 0$, however, the concept of preferred model is debatable since $\pi_{\mathcal{F}}(\omega) = 0$ means that ω is impossible. In the following, we shall restrict Shoham's definition to the case when $\pi_{\mathcal{F}}(\omega) > 0$, and let

$$\omega \vDash_{\sqsubset} \varphi \text{ if and only if } \pi_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi) > 0$$

Preferential entailment is then defined as above. Note that contrary to Shoham's conventions, we cannot have $\bot \vDash_{\sqsubset} \varphi$ since $\vDash_{\sqsubset}$ does not apply to an inconsistent set of premises. It may sound natural that while the contradiction entails anything, it preferentially entails nothing. This convention is different from the one adopted by Gärdenfors and Makinson (1992) for whom preferential entailment should subsume the classical notion, a point of view which is not chosen here.

The following result can then be established :

$$\exists\ \alpha > 0,\ \mathcal{F} \cup (\varphi\ 1) \mid\!\approx (\psi\ \alpha) \text{ if and only if } \varphi \vDash_{\sqsubset} \psi$$

where $\mid\!\approx$ denotes the non-trivial deduction operator. In other words, ψ is a non trivial deduction of $\mathcal{F}$ augmented with φ, if φ preferentially entails ψ in the sense of the ordering of models induced by $\mathcal{F}$. This result is formally proved in Dubois and Prade (1991a). It can be explained as follows : the possibility distribution $\pi^*_{\mathcal{F}}$ associated to $\mathcal{F} \cup \{(\varphi\ 1)\}$ is

$$\pi^*_{\mathcal{F}}(\omega) = \pi_{\mathcal{F}}(\omega) \text{ if } \omega \vDash \varphi$$
$$= 0 \text{ otherwise .}$$

It is clear that

$$\max_{\omega \in \Omega} \pi^*_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi), \text{ and } \mathrm{Incons}(\mathcal{F} \cup \{(\varphi\ 1)\}) = 1 - \Pi_{\mathcal{F}}(\varphi).$$

Now $\mathcal{F} \cup \{(\varphi\ 1)\}$ |≈ $(\psi\ \alpha)$ means that $N_{\mathcal{F}\cup\{(\varphi\ 1)\}}(\psi) = \alpha > \text{Incons}\ (\mathcal{F} \cup \{(\varphi\ 1)\})$. Now if ω is a preferred model of φ, we have $\pi_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi)$. Then the interpretation ω should satisfy ψ, for otherwise

$$N_{\mathcal{F}\cup\{(\varphi\ 1)\}}(\psi) = \min_{\omega'\models\neg\psi} 1 - \pi^*_{\mathcal{F}}(\omega') \leq 1 - \pi_{\mathcal{F}}(\omega) = \text{Incons}(\mathcal{F} \cup \{(\varphi\ 1)\}) < \alpha\ .$$

Hence φ preferentially entails ψ. The converse is as easy to establish. Again, the non-trivial deduction operation |≈ is not a generalization of the classical semantic entailment ⊨ since if F is an inconsistent classical knowledge base, then F ⊨ φ for any φ while $\mathcal{F}$ |≈ φ never holds, all deductions being trivial.

A link between preferential entailment and conditional possibility measures has been established. Namely let $\Pi(\psi \mid \varphi)$ be the possibility of ψ conditioned on φ. It is defined by the implicit equation (Hisdal, 1978) :

$$\Pi(\varphi \wedge \psi) = \min(\Pi(\psi \mid \varphi), \Pi(\varphi))$$

of which we must select the greatest solution, when $\psi \neq \bot$. The conditional possibility is then defined as

$$\forall\varphi, \forall\psi, \Pi(\psi \mid \varphi) = 1 \text{ if } \Pi(\varphi \wedge \psi) = \Pi(\varphi)$$
$$= \Pi(\varphi \wedge \psi) \text{ otherwise .}$$

The corresponding possibility distribution is $\pi(\cdot \mid \varphi)$, such that

$$\pi(\omega \mid \varphi) = 1 \text{ if } \pi(\omega) = \Pi(\varphi) > 0 \text{ and } \omega \models \varphi$$
$$= \pi(\omega) \text{ if } \pi(\omega) < \Pi(\varphi) \text{ and } \omega \models \varphi$$
$$= 0 \text{ if } \Pi(\varphi) > 0 \text{ and } \omega \models \neg\varphi$$
$$= 1 \text{ if } \Pi(\varphi) = 0$$

Hence when $\Pi(\varphi) > 0$, $\pi(\omega \mid \varphi) = 1$ if and only if ω is a preferred model of φ. The conditional necessity is defined by $N(\psi \mid \varphi) = 1 - \Pi(\neg\psi \mid \varphi)$. Note that $\Pi(\varphi) = \max(\Pi(\psi \wedge \varphi), \Pi(\neg\psi \wedge \varphi))$ so that the definition of the conditional possibility measure is also expressed by

$$\Pi(\psi \mid \varphi) = 1 \text{ if } \Pi(\varphi \wedge \psi) \geq \Pi(\varphi \wedge \neg\psi)$$
$$= \Pi(\varphi \wedge \psi) \text{ otherwise}$$

and the conditional necessity measure by

$$N(\psi \mid \varphi) = N(\neg\varphi \vee \psi) \text{ if } N(\neg\varphi \vee \psi) > N(\neg\varphi \vee \neg\psi)$$
$$= 0 \text{ otherwise.}$$

$\Pi(\psi \mid \varphi)$ and $N(\psi \mid \varphi)$ are numerical counterparts of Lewis (1973) "might" conditional (if it were that φ it might be that ψ) and "would" conditional (if it were that φ it would be that ψ) respectively.

The modified preferential entailment $\varphi \vDash_{\sqsubset} \psi$ can then be expressed in terms of the conditional possibility or necessity $\Pi_{\mathcal{F}}(\psi \mid \varphi)$ and $N_{\mathcal{F}}(\psi \mid \varphi)$ as follows (Dubois and Prade, 1991a) :

$\varphi \vDash_{\sqsubset} \psi$ if and only if $\Pi_{\mathcal{F}}(\psi \mid \varphi) > \Pi_{\mathcal{F}}(\neg\psi \mid \varphi)$
if and only if $N_{\mathcal{F}}(\psi \mid \varphi) > 0$
if and only if $\Pi_{\mathcal{F}}(\psi \wedge \varphi) > \Pi_{\mathcal{F}}(\neg\psi \wedge \varphi)$
if and only if $N_{\mathcal{F}}(\neg\varphi \vee \psi) > N_{\mathcal{F}}(\neg\varphi \vee \neg\psi)$

Indeed if $\{\omega \mid \pi_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi) > 0\} \subseteq \{\omega \mid \omega \vDash \psi\}$, it means that

$$\Pi_{\mathcal{F}}(\varphi \wedge \psi) = \max_{\omega \vDash \varphi \wedge \psi} \pi_{\mathcal{F}}(\omega) = \Pi_{\mathcal{F}}(\varphi)$$

while

$$\Pi_{\mathcal{F}}(\varphi \wedge \neg\psi) = \max_{\omega \vDash \varphi \wedge \neg\psi} \pi_{\mathcal{F}}(\omega) < \Pi_{\mathcal{F}}(\varphi)$$

since no preferred model of φ satisfies $\neg\psi$. Again we do not need as Gärdenfors and Makinson (1992) to add the supplementary condition $\varphi \vDash \psi$ to $N_{\mathcal{F}}(\psi \mid \varphi) > 0$, because we do not allow for $\bot \vDash_{\sqsubset} \varphi$. Hence ψ is a non-trivial consequence of $\mathcal{F} \cup \{(\varphi\ 1)\}$ as soon as $N(\psi \mid \varphi) > 0$. It also means that the non-trivial consequence relationship $|\approx$ can be characterized at the semantic level by means of the conditional possibility distribution $\pi_{\mathcal{F}}(\cdot \mid \varphi)$, since the only useful part of $\pi_{\mathcal{F}}$ when computing the non-trivial consequences of $\mathcal{F} \cup \{(\varphi\ 1)\}$ is its restriction to the set $\{\omega \mid \omega \vDash \varphi$ and $\pi(\omega) < \Pi(\varphi)\}$, as pointed out in (Dubois and Prade, 1991a). Indeed $N_{\mathcal{F}}(\psi \mid \varphi) > 0 \Leftrightarrow N_{\mathcal{F}}(\psi \mid \varphi) = N_{\mathcal{F}}(\neg\varphi \vee \psi) = 1 - \sup_{\omega \vDash \varphi \wedge \neg\psi} \pi_{\mathcal{F}}(\omega) > 1 - \Pi_{\mathcal{F}}(\varphi) = \text{Incons}(\mathcal{F} \cup \{(\varphi\ 1)\})$, i.e. we may just normalize the restriction of $\pi_{\mathcal{F}}$ to the models of φ by assigning a possibility degree equal to 1 to its maxima, and work with this normalized possibility distribution. Note that $\pi_{\mathcal{F}}(\omega \mid \bot) = 1, \forall\ \omega \in \Omega$, i.e. conditioning with the contradiction leads to total ignorance.

The non-trivial consequence relationship satisfies the rules of a well-behaved consequence relationship as first introduced by Gabbay (1985) :

$\varphi \vDash_{\sqsubset} \varphi$ when $\varphi \neq \bot$	(reflexivity, up to the contradiction)
$\varphi \vDash_{\sqsubset} \psi, \varphi \wedge \psi \vDash_{\sqsubset} \xi \Rightarrow \varphi \vDash_{\sqsubset} \xi$	(cut)
$\varphi \vDash_{\sqsubset} \psi, \varphi \vDash_{\sqsubset} \xi \Rightarrow \varphi \wedge \psi \vDash_{\sqsubset} \xi$	(restricted monotonicity)

$$\varphi \models_{\sqsubset} \xi, \psi \models_{\sqsubset} \xi \Rightarrow \varphi \vee \psi \models_{\sqsubset} \xi \quad \text{(OR)}$$

In terms of conditional necessity measures, these properties read (Dubois and Prade, 1991b)

$$N(\varphi \mid \varphi) = 1 \ \text{ for } \varphi \neq \bot$$
$$N(\psi \mid \varphi) > 0, N(\xi \mid \varphi \wedge \psi) > 0 \Rightarrow N(\xi \mid \varphi) \geq \min(N(\psi \mid \varphi), N(\xi \mid \varphi \wedge \psi))$$
$$N(\psi \mid \varphi) > 0, N(\xi \mid \varphi) > 0 \Rightarrow N(\xi \mid \varphi \wedge \psi) \geq \min(N(\psi \mid \varphi), N(\xi \mid \varphi))$$
$$N(\xi \mid \varphi) > 0, N(\xi \mid \psi) > 0 \Rightarrow N(\xi \mid \varphi \vee \psi) \geq \min(N(\xi \mid \varphi), N(\xi \mid \psi)).$$

Rational monotony

$$\varphi \not\models_{\sqsubset} \neg\psi \text{ and } \varphi \models_{\sqsubset} \xi \Rightarrow \varphi \wedge \psi \models_{\sqsubset} \xi$$

is also satisfied in possibilistic logic (Benferhat, Dubois and Prade, 1992) under the form

$$N(\neg\psi \mid \varphi) = 0 \text{ and } N(\xi \mid \varphi) > 0 \Rightarrow N(\xi \mid \varphi \wedge \psi) > 0.$$

Consequently possibilistic logic belongs to the family of non-monotonic logics based on preferential models.

3.10. Belief revision and possibilistic logic

Gärdenfors (1990) has suggested that non-monotonic reasoning and belief revision were two sides of the same coin ; see (Makinson and Gärdenfors, 1991) for a complete exposition. This is true for non-monotonic logic based on preferential models since Gärdenfors has shown how to translate the postulates of belief revision (Gärdenfors, 1988) into the axioms of preferential model-based non-monotonic logics, and that the latter have been given a semantics in accordance with Shoham's preference logic (Kraus et al., 1990). Thus, no wonder if there is a connection between possibilistic logic and belief revision.

More specifically Gärdenfors (1988) considers a belief set as a set K of propositions closed by the consequence relation. The expansion of K by a formula φ is simply K^+_φ = closure(K ∪ {φ}) which may contain all formulas (absurd belief set) if φ is inconsistent with K. The revision of K by a formula φ results in a consistent belief set K^*_φ even if K and φ are inconsistent together. The axioms which a rational revision procedure should satisfy are as follows

$K^*1)$ K^*_φ is a belief set

$K^*2)$ $\varphi \in K^*_\varphi$

$K^*3)$ $K^*_\varphi \subseteq K^+_\varphi$

K*4) if $\neg\varphi \notin K$ then $K^{+}_{\varphi} \subseteq K^{*}_{\varphi}$

K*5) K^{*}_{φ} is the absurd belief state if and only if $\varphi = \bot$

K*6) $\vDash \varphi \leftrightarrow \psi$ implies $K^{*}_{\varphi} = K^{*}_{\psi}$

K*7) $K^{*}_{\varphi\wedge\psi} \subseteq (K^{*}_{\varphi})^{+}_{\psi}$

K*8) if $\neg\psi \notin K^{*}_{\varphi}$ then $(K^{*}_{\varphi})^{+}_{\psi} \subseteq K^{*}_{\varphi\wedge\psi}$

While these postulates leave the choice of the revision procedure quite open, Gärdenfors (1988) proves that any such revision procedure underlies an ordering $\geq_E$ on the formulas of a belief set, that guides the revision procedure. Gärdenfors names this ordering "epistemic entrenchment". More specifically denoting $>_E$ the strict part of $\geq_E$

$$\psi \in K^{*}_{\varphi} \text{ if and only if } \neg\varphi \vee \psi >_E \neg\varphi \vee \neg\psi$$

In Dubois and Prade (1991c) we have pointed out that the relation $\geq_E$ has exactly the same properties as a comparative necessity relation, with the additional constraint $\top >_E \varphi$ if φ is not a tautology. Hence the only numerical counterpart to epistemic entrenchment relations are necessity measures.

Gärdenfors' methodology goes from postulates of belief revision to the characterization of epistemic entrenchment relations. Since a possibilistic knowledge base, containing necessity-valued formulas, obeys the laws of epistemic entrenchment, one may expect that by deleting the least certain formulas in order to cope with inconsistency, one gets a rational revision procedure.

First, a necessity-valued knowledge base $\mathcal{F}$ is not *explicitly* closed under the consequence relation, however it is *implicitely* : as shown in Section 3.2 it is equivalent to a possibility distribution $\pi_{\mathcal{F}}$ over its interpretations, through the principle of minimum specificity, and this possibility distribution enables the degree of necessity of any formula to be evaluated. This is achieved at the syntactic level by means of the extended resolution principle. Hence a partially defined epistemic entrenchment relation on a belief set K, expressed by assigning weights to some of the formulas, can be canonically extended to the whole belief set. We call it an ordered belief set.

The expansion of an ordered belief set K consists in adding a formula $(\varphi\ 1)$ and to compute the closure of $K \cup \{(\varphi\ 1)\}$ by means of the extended resolution principle.

The revision of an ordered belief set K consists in computing the non-trivial consequences of $K \cup \{(\varphi\ 1)\}$, that is, K^{*}_{φ} is made by all $(\psi\ \alpha)$ such that $K \cup \{(\varphi\ 1)\} \vDash (\psi\ \alpha)$ with

$\alpha > \text{Incons}(K \cup \{(\varphi\ 1)\})$. The obtained ordered belief set K^*_φ contains none of the formulas $(\xi\ \beta) \in K$ such that $\beta \le \text{Incons}\ (\mathcal{K} \cup \{(\varphi\ \ 1)\})$. In fact K^*_φ violates axiom K^*_5 in the sense that $K^*_\varphi = \emptyset$ in particular if $\varphi = \bot$. This is again due to the fact that $\bot \mid\!\approx \psi$ is not accepted in our approach. But it is really a matter of detail. Note that $K^*_\bot = \emptyset$ is consistent with $\pi_K(\omega \mid \bot) = 1, \forall\omega$, as defined above, while if $K^*_\bot$ contain all formulas, it would require the convention $\pi_K(\omega \mid \bot) = 0, \forall\omega \in \Omega$. See Dubois and Prade (1992b) for a detailed discussion.

At the semantic level, the expansion K^+_φ of K consists in turning the corresponding possibility distribution π into π^+_φ such that

$$\pi^+_\varphi(\omega) \begin{cases} = \pi(\omega) \text{ if } \omega \vDash \varphi \\ = 0 \text{ otherwise} \end{cases}$$

Revision consists in turning π into $\pi^*_\varphi = \pi(\cdot \mid \varphi)$. It is proved in Dubois and Prade (1992b) that π^+_φ and π^*_φ satisfy all rationality postulates of well-behaved expansions and revisions respectively (up to the question of defining $K^*_\bot$). And both are at work in the inconsistency-tolerant deduction machinery of possibilistic logic.

4. Generalizations of possibilistic logic

The "basic" version of possibilistic logic that we have discussed so far may be not sufficient to model some kinds of incomplete information we may wish to handle, such as :

– possibility-qualified sentences, for instance

"it is possible that John comes"

– conditional sentences, whose condition depends on a fuzzy predicate

"the *later* John arrives, the more certain the meeting will not be quiet"

– sentences involving vague predicates, for instance

"if the temperature is high then there will be only a few participants"

In order to enable the handling of such knowledge, we present in this section some formalisms which are either extensions of possibilistic logic (where the basic language and semantics of possibilistic logic are enriched) or generalizations (where possibilistic logic is considered as a particular case of more general logical models).

4.1. Possibilistic logic with possibility- and necessity-qualified formulas

In Section 3 we have studied in detail the semantics, axiomatics and some automated deduction procedures of necessity-valued logic where necessity-qualified statements are represented by necessity-valued formulas. However this fragment of possibilistic logic cannot handle *possibility-qualified* statements. As seen in Section 2, if the statement "(at least one value in A) is (at least) α-possible for x" is given, then finding an underlying possibility distribution restricting the values of x comes down to solve the equation $\Pi(A) \geq \alpha$. In order to handle both possibility- and necessity- qualified statements, the (extended) language should be able to syntactically model constraints in terms of lower bounds of a necessity or of a possibility measure. It comes down to allow for two kinds of weighted formulas : necessity-valued formulas expressing that $N(\varphi) \geq \alpha$ as already seen and possibility-valued formulas expressing that $\Pi(\varphi) \geq \alpha$. The valuations will be denoted w and stand for $(\Pi\ \alpha)$ or $(N\ \alpha)$ according to whether α is a lower bound of a possibility or a necessity measure.

Thus, a ***possibilistic formula*** is either a pair $(\varphi\ (N\ \alpha))$ where φ is a classical first-order closed formula and $\alpha \in (0,1]$, (α should be strictly positive) or a pair $(\varphi\ (\Pi\ \beta))$ where $\beta \in [0,1]$. $(\varphi\ (N\ \alpha))$ expresses that φ is certain at least to the degree α, i.e. $N(\varphi) \geq \alpha$, and $(\varphi\ (\Pi\ \beta))$ expresses that φ is possible in some world at least to the degree β, i.e. $\Pi(\varphi) \geq \beta$, where Π and N are dual measures ($\Pi(\varphi) = 1 - N(\neg\varphi)$) of possibility and necessity modelling our incomplete state of knowledge. $\Pi(\varphi) \geq \beta$ expresses to what extent we consider that φ cannot be refuted (or equivalently $\neg\varphi$ cannot be proved). More specifically, $\Pi(\varphi) \geq \beta$ expresses that φ is consistent with the remainder of the knowledge base to which φ belongs, at least at level β. Particularly, if both $(\varphi\ (\Pi\ 1))$ and $(\neg\varphi\ (\Pi\ 1))$ are stated, it means that neither φ nor $\neg\varphi$ is allowed to be a consequence of the remainder of the knowledge base. Hence the use of possibility-qualified statements allows us for claiming that some propositions cannot be established nor refuted. We can express knowledge about ignorance.

The right part of a possibilistic formula, i.e. $(N\ \alpha)$ or $(\Pi\ \beta)$, is called the ***valuation*** of the formula, and is denoted val(φ). $\mathcal{V}$ will denote the set of all possible valuations w, i.e.

$$\mathcal{V} = \{(N\ \alpha) \mid 0 < \alpha \leq 1\} \cup \{(\Pi\ \alpha) \mid 0 \leq \alpha \leq 1\}$$

Since $N(\varphi) > 0$ entails $\Pi(\varphi) = 1$, $(\varphi\ (N\ \alpha))$ is stronger than $(\varphi\ (\Pi\ \beta))$ for any $\alpha > 0$, $\beta \geq 0$; this leads us to define the following ordering among valuations :

$$(N\ \alpha) \leq (N\ \beta) \text{ iff } \alpha \leq \beta\ ;\ (\Pi\ \alpha) \leq (\Pi\ \beta) \text{ iff } \alpha \leq \beta\ ;\ (\Pi\ \alpha) \leq (N\ \beta)\ \forall\alpha, \forall\ \beta > 0.$$

Hence the maximal and minimal elements of $\mathcal{V}$ are respectively (N 1) (expressing that a formula is completely certain) and (Π 0) (expressing that we do not know anything about the truth, the falsity nor the consistency of a formula). Again, we never explicitly handle formulas of the form (φ (Π 0)) since $\forall\varphi$, $\Pi(\varphi) \geq 0$. The difference between (φ (Π 1)) and (φ (Π 0)) is that by stating (φ (Π 1)) it is claimed that for sure $\neg\varphi$ cannot be proved, while (φ (Π 0)) expresses our ignorance about whether $\neg\varphi$ can be proved or not.

A ***possibilistic knowledge base*** is then defined as a finite set (i.e. a conjunction) of possibilistic formulas. A possibilistic formula whose valuation is of the form (N α) (respectively (Π α)) will be called a ***necessity-valued*** (resp. ***possibility-valued***) ***formula***. Let ***PL2*** denote the language consisting in possibilistic formulas (both necessity-valued as well as possibility-valued ones). We recall that the language consisting only in necessity-valued formulas was denoted PL1. The classical projection $\mathcal{F}^*$ of $\mathcal{F}$ still denotes the set of classical formulas obtained from a set of possibilistic formulae $\mathcal{F}$, by ignoring the weights ; thus, if $\mathcal{F} = \{(\varphi_i\ w_i)\ , i = 1, \ldots, n\}$ then $\mathcal{F}^* = \{\varphi_i\ , i = 1, \ldots, n\}$. As for necessity-valued knowledge bases, a possibilistic knowledge base may also be seen as a collection of nested sets of (classical) formulas (since $\mathcal{V}$ is ordered) : w being a valuation of $\mathcal{V}$, the *w-cut* and the *strict w-cut* of $\mathcal{F}$, denoted respectively by $\mathcal{F}_w$ and $\mathcal{F}_{\overline{w}}$, are defined by

$$\mathcal{F}_w = \{(\varphi\ v) \in \mathcal{F} \mid v \geq w\}\ ;$$
$$\mathcal{F}_{\overline{w}} = \{(\varphi\ v) \in \mathcal{F} \mid v > w\}.$$

4.1.1. Semantics

In this subsection we extend the necessity-valued semantics to PL2 in a natural way . A lot of results being very similar to those of necessity-valued logic, we are often just stating them with very few comments or examples ; we are mainly focusing on the differences induced by the extension of the language, i.e. properties which do not hold anymore in full possibilistic logic. A more detailed treatment of PL2 can be found in (Lang et al., 1991).

We first associate to a set of possibilistic formulas the set of *normalized* possibility distributions on Ω satisfying it. For the possibility distribution π inducing the possibility measure Π and the necessity measure N, satisfaction is defined as :

$$\pi \models (\varphi\ (N\ \alpha)) \text{ if and only if } N(\varphi) \geq \alpha\ ;$$
$$\pi \models (\varphi\ (\Pi\ \alpha)) \text{ if and only if } \Pi(\varphi) \geq \alpha\ ;$$
$$\pi \models \mathcal{F} = \{(\varphi_i\ w_i), i = 1, \ldots, n\} \text{ if and only if } \forall\ i = 1, \ldots, n,\ \pi \models (\varphi_i\ w_i).$$

and logical consequence as :

$\mathcal{F} \models \Phi$ if and only if for all π $(\pi \models \mathcal{F}) \Rightarrow (\pi \models \Phi)$.

The function Val is naturally extended by

$\text{Val}(\varphi,\mathcal{F}) = \text{Sup}\ \{w \mid \mathcal{F} \models (\varphi\ w)\}$.

Example

Let $\mathcal{F} = \{(p\ (N\ 0.7)), (\neg p \vee q\ (\Pi\ 0.8))\}$.
$\pi \models \mathcal{F} \Leftrightarrow N(p) \geq 0.7$ and $\Pi(\neg p \vee q) \geq 0.8$
$\Leftrightarrow \text{Inf}\{1 - \pi(\omega), \omega \models \neg p\} \geq 0.7$ and $\sup\{\pi(\omega), \omega \models \neg p \vee q\} \geq 0.8$.

Let $\Omega = \{[p,q], [\neg p,q], [p,\neg q], [\neg p,\neg q]\}$ be the 4 different interpretations for the propositional language generated by $\{p,q\}$ (where $[p,q]$ gives the value True to p and q, etc.). Then, $\pi \models \mathcal{F}$ is equivalent to $\Pi(\neg p) \leq 0.3$ and $\Pi(\neg p \vee q) \geq 0.8$, which leads to

$$\begin{cases} \max(\pi([\neg p,q]),\ \pi([\neg p,\neg q])) \leq 0.3 \\ \max(\pi([p,q]),\ \pi([\neg p,q]),\ \pi([\neg p,\neg q])) \geq 0.8 \\ \max(\pi([p,q]),\ \pi([\neg p,q]),\ \pi([p,\neg q]),\ \pi([\neg p,\neg q])) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi([\neg p,q]) \leq 0.3 \\ \pi([\neg p,\neg q]) \leq 0.3 \\ \pi([p,q]) \geq 0.8 \\ \max(\pi([p,q]),\ \pi([p,\neg q])) = 1 \end{cases}$$

It is then obvious that $\mathcal{F} \models (q\ (\Pi\ 0.8))$. Indeed, any possibility distribution π satisfying $\mathcal{F}$ is such that $\pi([p,q]) \geq 0.8$, and thus verifies $\Pi(q) = \max(\pi([p,q]), \pi([\neg p,q])) \geq 0.8$; hence π satisfies $(q\ (\Pi\ 0.8))$.
Moreover, $\forall\ w > (\Pi\ 0.8)$, we do not have $\mathcal{F} \models (q\ w)$; thus $\text{Val}(q,\mathcal{F}) = (\Pi\ 0.8)$.

The following properties are straightforward :

(i) $(\varphi\ w) \models (\varphi\ w')\ \forall w' \leq w$
(ii) $\forall w > (\Pi\ 0)$, $\models (\varphi\ w)$ if and only if φ is a tautology.

There is a strong analogy between the definitions of satisfiability in possibilistic logic and in multi-modal logics. Satisfiability of $\Diamond_\alpha\varphi$, i.e. $M, w \models \Diamond_\alpha\varphi$ iff $\exists w'$, $R_\alpha(w,w')$ and $w' \models \varphi$ (where M is a Kripke model, w a world and $\{R_\alpha, 0 < \alpha \leq 1\}$ a family of accessibility relations) corresponds to $\pi \models (\varphi\ (\Pi\ \alpha))$ iff $\exists\omega$, $\pi(\omega) \geq \alpha$ and $\omega \models \varphi$,which leads us to interpret possibilistic logic in terms of a multi-modal system. See Dubois, Prade and Testemale

(1988) for a first study and Fariñas del Cerro and Herzig (1991) for a recent multi-modal axiomatics of qualitative possibilistic logic.

There are two kinds of inconsistencies in a possibilistic knowledge base $\mathcal{F}$:

– inconsistencies generated by (only) contradictory necessity-valued formulae ; they can be solved (as already seen) by allowing non-zero value for $N(\bot)$.
– inconsistencies involving both possibility- and necessity-valued formulas.

In order to equip inconsistent possibilistic knowledge bases with semantics, one approach is to add to the set of interpretations Ω an extra-element, noted $\omega_\bot$ in which any formula is "true", i.e. $\forall \varphi \in \mathcal{L}'$, $\omega_\bot \models \varphi$ which corresponds to the idea of an "absurd interpretation" discussed by Stalnaker (1968). Let $\Omega_\bot = \Omega \cup \{\omega_\bot\}$. A possibility distribution on $\Omega_\bot$ is a mapping $\hat{\pi}$ from $\Omega_\bot$ to [0,1] such that $\exists\ \omega \in \Omega_\bot$, $\hat{\pi}(\omega) = 1$ (normalization over $\Omega_\bot$). Then we define two functions from $\mathcal{L}'$ to [0,1] induced by $\hat{\pi}$: $\hat{\Pi}(\varphi) = \sup\{\hat{\pi}(\omega), \omega \in \Omega_\bot, \omega \models \varphi\}$; $\hat{N}(\varphi) = \inf\{1 - \hat{\pi}(\omega), \omega \in \Omega_\bot, \omega \not\models \varphi\}$. Note that $\hat{N}(\varphi)$ does not take $\hat{\pi}(\omega_\bot)$ into account, while $\hat{\Pi}(\varphi)$ does ; note also that $\omega \not\models \varphi$ is no longer equivalent to $\omega \models \neg\varphi$, since $\omega_\bot \models \varphi$ and $\omega_\bot \models \neg\varphi$. The idea of adding an extra-element to the referential of a possibility distribution has been already used for dealing with the case of an attribute which does not apply to an item of a data base. However the extensions of the possibility and necessity measures which are used for the evaluations of queries in incomplete information data bases differ from $\hat{\Pi}$ and $\hat{N}$ defined here ; see Chapter 6 of Dubois and Prade (1985b).

The classical possibility and necessity measures $\Pi(\varphi) = \sup\{\hat{\pi}(\omega), \omega \in \Omega, \omega \models \varphi\}$ and $N(\varphi) = \inf\{1 - \hat{\pi}(\omega), \omega \in \Omega, \omega \models \neg\varphi\}$ deriving from the (possibly un-normalized) restriction of $\hat{\pi}$ to Ω are related to $\hat{\Pi}$ and $\hat{N}$ as follows

$$\hat{\Pi}(\varphi) = \max(\Pi(\varphi), \hat{\pi}(\omega_\bot))$$
$$\hat{N}(\varphi) = N(\varphi) = 1 - \Pi(\neg\varphi)$$

$\hat{\Pi} = \Pi$ if and only if $\hat{\pi}(\omega_\bot) = 0$; in this case $\hat{\pi}$ is normalized on Ω. Note that $\hat{\Pi}$ is not a possibility measure with respect to Ω, but only with respect to $\Omega_\bot$. We shall call $\hat{\Pi}$ and $\hat{N}$ *inconsistency-tolerant possibility* (resp. *necessity*) *measures.*

Each possibilistic formula $(\varphi\ (\Pi\ \alpha))$ or $(\varphi\ (N\ \alpha))$, is now interpreted as $\hat{\Pi}(\varphi) \geq \alpha$ (respectively $\hat{N}(\varphi) \geq \alpha$), i.e. we take into account the absurd interpretation in our understanding of uncertainty-qualified statements. For instance, $(\varphi\ (\Pi\ \alpha))$ expresses that "it is possible at least to the degree α that either φ is true or that we are in an absurd situation". This leads us to the following definitions paralleling the definitions of Section 3.2 replacing Ω (respectively π, $\prod$, N) by $\Omega_{\perp}$ (respectively $\hat{\pi}$, $\hat{\prod}$, $\hat{N}$).

– ***satisfaction*** : $\hat{\pi} \hat{\models} (\varphi\ (\prod \alpha))$ iff $\hat{\prod}(\varphi) \geq \alpha$; $\hat{\pi} \hat{\models} (\varphi\ (N\ \alpha))$ iff $\hat{N}(\varphi) \geq \alpha$, where $\hat{\prod}$ and $\hat{N}$ are the inconsistency-tolerant possibility and necessity measures induced by $\hat{\pi}$; $\hat{\pi} \hat{\models} \mathcal{F}$ iff $\hat{\pi}$ satisfies all formulae of $\mathcal{F}$;
– ***logical consequence*** : $\mathcal{F} \hat{\models} \Phi$ iff $\forall \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$ implies $\hat{\pi} \hat{\models} \Phi$.

The results about the characterization of the set of possibility distributions satisfying a necessity-valued knowledge base via a single possibility distribution $\pi_{\mathcal{F}}$ cannot be generalized to possibilistic logic with possibility-qualified formulas, since this set has generally no longer an upper bound on $\Omega_{\perp}$.

The inconsistency-tolerant semantics for full possibilistic logic subsumes the (inconsistency-tolerant) semantics for necessity-valued logic (as it is intended to be). Indeed, the previously subnormalized possibility distributions on Ω are now artificially renormalized on $\Omega_{\perp}$ by adding the constraint $\hat{\pi}(\omega_{\perp}) = 1$.

As pointed out above we can distinguish between two different types of inconsistencies. Let $\mathcal{F}$ be a set of possibilistic formulas ; considering the possibility distributions on $\Omega_{\perp}$ satisfying $\mathcal{F}$, three situations may occur :

(i) $\exists \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$ such that $\hat{\pi}(\omega_{\perp}) = 0$: in this case, $\mathcal{F}$ is consistent in both semantics ; $\mathcal{F}$ is then said to be *completely consistent*.

(ii) $\forall \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$, $\hat{\pi}(\omega_{\perp}) > 0$ but $\exists \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$ such that $\sup\{\hat{\pi}(\omega), \omega \neq \omega_{\perp}\} = 1$: then, for any $\hat{\pi}$ satisfying $\mathcal{F}$, we have $\hat{\prod}(\perp) = \hat{\pi}(\omega_{\perp}) > 0$ and $\hat{N}(\perp) = 1 - \sup\{\hat{\pi}(\omega), \omega \neq \omega_{\perp}\} = 0$. Thus $\mathcal{F}$ induces only a "possible inconsistency" (contradiction being possible to a strictly positive degree). The minimal value of $\hat{\pi}(\omega_{\perp})$ among the possibility distributions $\hat{\pi}$ on $\Omega_{\perp}$ satisfying $\mathcal{F}$ gives the *inconsistency degree* of $\mathcal{F}$. Let $\alpha = \inf\{\hat{\pi}(\omega_{\perp}), \hat{\pi} \hat{\models} \mathcal{F}\}$; then Incons($\mathcal{F}$) = $(\prod \alpha)$.

(iii) $\forall \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$, $\sup\{\hat{\pi}(\omega), \omega \neq \omega_{\perp}\} < 1$ (which entails that $\forall \hat{\pi}$, $\hat{\pi} \hat{\models} \mathcal{F}$, $\hat{\pi}(\omega_{\perp}) = 1$). In this case, for any $\hat{\pi}$ satisfying $\mathcal{F}$, we have $\hat{\pi}(\omega_{\perp}) = 1$ and $\hat{N}(\perp) = 1 - \sup\{\hat{\pi}(\omega), \omega \neq \omega_{\perp}\} > 0$. We thus recover the notion of partial inconsistency introduced in LP1.

The ***inconsistency degree*** of $\mathcal{F}$ is now a valuation of the form $(\prod \alpha)$ or $(N\ \alpha)$, defined as

$$\text{Incons}(\mathcal{F}) = \sup\{w \in \mathcal{V} \mid \mathcal{F} \hat{\models} (\perp w)\}$$

$\mathcal{F}$ is *completely consistent* iff Incons($\mathcal{F}$) = (Π 0). If $\forall \hat{\pi}$, $\hat{\pi} \,\hat{\models}\, \mathcal{F}$, $\sup\{\hat{\pi}(\omega), \omega \neq \omega_\perp\} = 0$, then Incons($\mathcal{F}$) = (N 1) and $\mathcal{F}$ is *completely inconsistent*. If Incons($\mathcal{F}$) = (Π α) with $\alpha > 0$ then $\mathcal{F}$ is said to be *weakly inconsistent*. If Incons($\mathcal{F}$) = (N β) with $\beta < 1$ then $\mathcal{F}$ is *partially inconsistent*. The following scale (Figure 1) shows the hierarchy of inconsistencies :

Incons($\mathcal{F}$)	
(N 1)	complete inconsistency
(N α)	partial inconsistency
(Π 1) / (Π α)	weak inconsistency
(Π 0)	complete consistency

Figure 1

It should be clear that when $\mathcal{F}$ is consistent, or partially inconsistent, then the two semantic entailments $\models$ and $\hat{\models}$ are equivalent.

Examples

The knowledge base $\mathcal{G}$ of Section 3.3 gives an example of a degree of inconsistency equal to (N 0.3).

An example of a knowledge base with a degree of inconsistency of the form (Π α) is given by $\mathcal{H}$ = {(p (Π 0.7)), ($\neg$p (N 0.6))}. Clearly

π satisfies $\mathcal{H}$
$\Leftrightarrow \hat{\Pi}(p) \geq 0.7$ and $\hat{N}(\neg p) \geq 0.6$
$\Leftrightarrow \hat{\Pi}(p) \geq 0.7$ and $\Pi(p) \leq 0.4$
$\Leftrightarrow \exists\, \omega \models p$ such that $\hat{\pi}(\omega) \geq 0.7$ and $\forall\, \omega \models p, \omega \neq \omega_\perp, \hat{\pi}(\omega) \leq 0.4$.
$\Leftrightarrow \hat{\pi}(\omega_\perp) = 0.7$ and $\forall \omega \models p, \omega \neq \omega_\perp, \hat{\pi}(\omega) \leq 0.4$.

Hence Incons ($\mathcal{H}$) $\geq$ (Π 0.7); let $\hat{\pi}_0$ be such that $\hat{\pi}_0(\omega_\perp) = 0.7$; $\forall\, \omega \neq \omega_\perp$ such that $\omega \models p$, then $\hat{\pi}_0(\omega) = 0.4$ and $\forall\, \omega \neq \omega_\perp$ such that $\omega \models \neg p$, then $\hat{\pi}_0(\omega) = 1$; $\hat{\pi}_0$ satisfies $\mathcal{H}$. Hence Incons($\mathcal{H}$) = (Π 0.7).

In the second example (p (Π 0.7)) states that $\neg$p cannot be proved while ($\neg$p (N 0.6)) states that $\neg$p is true with some certainty. Hence weak consistency comes from a clash between the claim that some propositions are true and the claim that these propositions cannot be proved. Partial inconsistency corresponds to the simultaneous statement that a proposition *should* be true and false.

Both examples indicate that the inconsistency degree of a possibilistic knowledge base $\mathcal{F}$ is the valuation of the weakest formula (in the sense of the ordering in $\mathcal{V}$) involved in the strongest contradiction in $\mathcal{F}$. Let w ∈ $\mathcal{V}$ such that Incons($\mathcal{F}$) = w. It is easy to see that $\forall\ \Phi \in \mathcal{F}$, Incons($\mathcal{F}$ – {Φ}) ≤ w. Let $\mathcal{F}' \subseteq \mathcal{F}$ such that Incons($\mathcal{F}$') = Incons($\mathcal{F}$) and $\forall\ \Phi \in \mathcal{F}'$, Incons($\mathcal{F}$' – {Φ}) < Incons($\mathcal{F}$'). $\mathcal{F}$' is called a *strongly minimal inconsistent subset* of $\mathcal{F}$. Proposition 5 is then completed by

Proposition 5' *(Lang et al., 1991)* ***:***

The inconsistency degree of an inconsistent possibilistic knowledge base $\mathcal{F}$ is the smallest weight of possibilistic formulas in any strongly minimal inconsistent subset $\mathcal{F}$' of $\mathcal{F}$. Especially, if Incons($\mathcal{F}$) = (∏ α) then there is a unique possibility-valued formula in $\mathcal{F}$' of the form (φ (∏ α)).

The unicity of (φ (∏ α)) is due to the fact that any two ∏-valued formulas (φ (∏ α)) and (¬φ (∏ β)) never contradict each other. Hence the equivalence of the consistency of $\mathcal{F}$ and the consistency of its classical projection $\mathcal{F}^*$ does not hold anymore when handling possibility-valued formulas : indeed $\mathcal{F}$ = {(φ (∏ 1)), (¬φ (∏ 1))} is completely consistent whereas $\mathcal{F}^*$ = {φ, ¬φ} is inconsistent. However we have the weaker result (Lang, 1991a): if Incons($\mathcal{F}$) > (∏ 0) then $\mathcal{F}^*$ is inconsistent.

Proposition 6 for necessity-valued knowledge bases remains true for the possibility-valued case, i.e. the degree of inconsistency inc ∈ $\mathcal{V}$ inhibits all formulas (φ w) with w ≤ inc and it is equivalent to work with $\mathcal{F}_{inc}$. A consequence of the above Proposition 5' is that in order to calculate the inconsistency degree of $\mathcal{F}$, it is enough to consider possibility-valued formulas separately. Namely if $\mathcal{F} = \mathcal{F}_N \cup \mathcal{F}_\Pi$ where $\mathcal{F}_N$ contains only the necessity-valued formulas of $\mathcal{F}$ and $\mathcal{F}_\Pi$ the possibility-valued formulas, then we have the following counterpart of Proposition 7 for weak inconsistency :

Proposition 7' :

Incons($\mathcal{F}$) = (∏ α) if and only if $\mathcal{F}_N^*$ is consistent and α = max{β, ∃ (φ (∏ β)) ∈ $\mathcal{F}_\Pi$, $\mathcal{F}_N^* \cup$ {φ} inconsistent}.

As a consequence, computing Incons($\mathcal{F}$) remains NP-complete as in the necessity-valued case since it comes down to check either the partial inconsistency of $\mathcal{F}_N$ (then Proposition 8 applies) or to check the inconsistency of $\mathcal{F}_N^* \cup$ {φ}, ∀ φ ∈ $\mathcal{F}_\Pi^*$. The counterpart of Propositions 9 and 10 hold as well. Lastly, the non-monotonic behaviour we pointed out in the necessity-valued case remains the same in full possibilistic logic.

4.1.2. Axiomatization of possibilistic logic involving possibility- and necessity-qualified formulas

The formal system for necessity-valued logic can be extended in order to handle possibility-valued formulas. Basically, it consists in extending the graded modus ponens over to $w \in U$ so as to enable the derivation of a possibility-valued formula from a necessity-valued formula and a possibility-valued formula. The extended formal system for full possibilistic logic PL2 uses the same axioms schemata as for PL1, where each axiom is necessity-valued by (N 1), (see Section 3.6), and the following inference rules :

(GMP) $(\varphi\ w_1)\ (\varphi \rightarrow \psi\ \ w_2) \vdash (\psi\ w_1 * w_2)$
(G) $(\varphi\ w) \vdash ((\forall x\ \varphi)\ w)$ (if x is not bound in φ)
(S) $(\varphi\ w) \vdash (\varphi\ w')\quad \forall\ w' \leq w$

where the operation $*$ is defined by

$$(N\ \alpha) * (N\ \beta) = (N\ \min(\alpha,\beta))\ ;$$
$$(N\ \alpha) * (\Pi\ \beta) = \begin{cases} (\Pi\ \beta) \text{ if } \alpha + \beta > 1\ ; \\ (\Pi\ 0) \text{ if } \alpha + \beta \leq 1. \end{cases}$$
$$(\Pi\ \alpha) * (\Pi\ \beta) = (\Pi\ 0).$$

Proposition 12' :
The proposed formal system is sound and complete with respect to the inconsistency-tolerant semantics of possibilistic logic, i.e. for any set of possibilistic formulas $\mathcal{F}$ we have

$$\mathcal{F} \mathrel{\hat{\models}} (\psi\ w) \text{ if and only if } \mathcal{F} \vdash (\psi\ w)$$

where $\mathcal{F} \vdash (\psi\ w)$ means : "(ψ w) can be derived from $\mathcal{F}$ in the above formal system" (the proof is in the annex). Thus possibilistic logic PL2 is *axiomatisable*.

4.1.3. Automated deduction

In this section we briefly extend, to possibilistic logic PL2, the results established in Section 3.8. about automated deduction procedures devoted to the computation of the inconsistency degree. We may define clausal forms as in the necessity-valued case : a ***possibilistic clause*** is a possibilistic formula (c w) where c is a first-order or propositional clause and w is a valuation of $\mathcal{V}$. A ***possibilistic clausal form*** is a universally quantified conjunction of possibilistic clauses. We denote by CPL2 the language consisting in possibilistic clauses (necessity- or possibility-valued).

We have seen (Proposition 13) that the problem of finding a clausal form of $\mathcal{F}$ whose inconsistency degree is the same as $\mathcal{F}$ always has a solution in PL1, i.e. if $\mathcal{F}$ contains only necessity-valued classical formulas. If $\mathcal{F}$ contains also possibility-valued formulas, then generally we cannot compute from $\mathcal{F}$ a clausal form having the same inconsistency degree as $\mathcal{F}$, even in propositional possibilistic logic. For instance, the intuitive clausal form we can compute from $\mathcal{F} = \{(p \wedge q\ (\Pi\ \alpha)), (\neg p \vee \neg q\ (N\ 1))\}$ $(\alpha > 0)$ is $\mathcal{C} = \{(p\ (\Pi\ \alpha)), (q\ (\Pi\ \alpha)), (\neg p \vee \neg q\ (N\ 1))\}$, but it can be checked that Incons($\mathcal{F}$) = $(\Pi\ \alpha)$ whereas Incons($\mathcal{C}$) = $(\Pi\ 0)$. This negative result comes from the non-compositionnality of possibility measures for conjunction. Indeed $(p \wedge q\ (\Pi\ \alpha))$ is much stronger than $(p\ (\Pi\ \alpha)) \wedge (q\ (\Pi\ \alpha))$, since $(p \wedge q\ (\Pi\ \alpha))$ means $\hat{\Pi}(p \wedge q) \geq \alpha$, i.e. $\exists\ \omega \in \Omega_\perp$ such that $\omega \vDash p \wedge q$ and $\hat{\pi}(\omega) \geq \alpha$, whereas $(p\ (\Pi\ \alpha)) \wedge (q\ (\Pi\ \alpha))$, means $\exists \omega, \omega' \in \Omega_\perp$ such that $\omega \vDash p$, $\omega' \vDash q$ and $\hat{\pi}(\omega) \geq \alpha$, $\hat{\pi}(\omega') \geq \alpha$. This problem also appears in modal logics (Fariñas del Cerro and Herzig, 1988) and could be solved in our framework by similarly "colouring" the "Π" valuations.

The following possibilistic resolution rule between two possibilistic clauses $(c_1\ w_1)$ and $(c_2\ w_2)$ established by Dubois and Prade (1990) extends the rule (R) of Section 3.8.2 :

$$(c_1\ w_1), (c_2\ w_2) \vdash (R(c_1, c_2)\ w_1 * w_2) \qquad\qquad (R')$$

where $R(c_1,c_2)$ is a classical resolvent of c_1 and c_2, and $*$ is the operation defined at the end of the preceding Section 4.1.2. The similarity between (R) and resolution patterns existing in modal logics has been pointed out ; see (Dubois and Prade, 1990). The soundness result is easily extended (see Lang et al., 1991) :

Proposition 14' (soundness of rule (R')) ***:***
Let $\mathcal{C}$ be a set of possibilistic clauses, and C a possibilistic clause obtained by a finite number of successive applications of (R') to $\mathcal{C}$; then $\mathcal{C} \hat{\vDash}$ C.

Refutation by resolution is very similar to the necessity-valued case, changing valuations $(N\ \alpha)$ into $w \in \mathcal{V}$, and we look for an optimal refutation, i.e. one leading to $(\perp\ \bar{w})$ with $\bar{w}$ maximal. However when the knowledge base consists in both necessity-valued and possibility-valued formulas, then, because the transformation into clausal form is not complete (it does not preserve the inconsistency degree), *we must suppose that* $\mathcal{F}$ *is a set of possibilistic clauses right away* ; in this case, $\mathcal{C} = \mathcal{F}$ and step 1 of the refutation procedure given in Section 3.8.2 is omitted. Soundness and completeness results then hold for possibilistic resolution when the knowledge is propositional (Lang et al., 1991) :

Proposition 15' (soundness and completeness of refutation by resolution in propositional clausal possibilistic logic CPL2) ***:***

If $\mathcal{C}$ is a set of *propositional* necessity- or possibility-valued *clauses*, then the valuation of the optimal refutation by resolution from $\mathcal{C}$ is equal to the inconsistency degree of $\mathcal{C}$.

***Corollary* :**
Let φ be a classical formula and $\mathcal{C}'$ the set of possibilistic clauses obtained from $\mathcal{C} \cup \{(\neg\varphi\ (N\ 1))\}$; then the valuation of the optimal refutation by resolution from $\mathcal{C}'$ is $Val(\varphi,\mathcal{C})$.

Soundness and completeness of refutation by resolution in first-order PL1 were a consequence of Propositions 10 and 13 together with the expression of the resolution rule ; it does not hold for *first-order possibility-valued* clauses : for instance, if $\mathcal{C} = \{(p(x)\ (\Pi\ \alpha))\}$, x being a (universally quantified) variable and $\alpha > 0$, and $\varphi = p(a) \wedge p(b)$, then there is no $(\Pi\ \alpha)$-refutation by resolution from $\mathcal{C} \cup \{(\neg p(a) \vee \neg p(b)\ (N\ 1))\}$, whereas $\mathcal{C} \mathrel{\hat{\models}} (p(a) \wedge p(b)\ (\Pi\ \alpha))$. It does not hold either for possibility-valued formulas, since the tranlation into clausal form does not preserve the inconsistency degree if a knowledge base contains possibility-valued formulas. As already mentioned, completeness can be recovered by indexing the "Π" symbols in the $(\Pi\ \alpha)$-valuations, in the same spirit as in modal logics (Fariñas del Cerro and Herzig, 1988). Lastly, as the existence of a possibility distribution $\hat{\pi}_{\mathcal{F}}$ such as $\hat{\pi} \mathrel{\hat{\models}} \mathcal{F}$ iff $\hat{\pi} \leq \hat{\pi}_{\mathcal{F}}$ is generally not satisfied in PL2, semantic evaluation cannot be easily extended to full possibilistic logic.

4.2. Variable valuations

In "standard" possibilistic logic we considered only weighted formulas of the form $(\varphi\ w)$ where φ is a *closed* formula of first-order logic, i.e. it is only allowed to quantify "inside" the scope of a valuation.

In ***quantified possibilistic logic***, we allow formulas of the form $(Qx_1 Qx_2 \ldots Qx_n)$ $(\varphi\ w)$ where Q is a quantifier (either $\forall$ or $\exists$) and φ a formula of first-order logic where variables $x_1, \ldots, x_n$ are free. The further step is then to allow the valuations w to depend on the free variables x_i of φ : this is ***quantified possibilistic logic with variable valuations***. Thus, the possibilistic formula $(\forall x)\ (\varphi\ w(x))$ expresses that for any x, φ is true for x with a possibility or necessity degree (at least) w(x).

The following example illustrates some potentials of universally quantified possibilistic formulas with variable valuations[4] . Let $\mu_{\tilde{P}}$ be the membership function of a fuzzy predicate $\tilde{P}$;

[4] The meaning of existentially-quantified valuations is less intuitive.

then the possibilistic formula $(\varphi(x)\ (N\ \mu_{\tilde{P}}(x)))$ enables us to express that "the more x satisfies $\tilde{P}$, the more certain $\varphi(x)$ is satisfied". For instance, we wish to translate the statement "the later John will arrive to the meeting, the more certain the meeting will be quiet ; and if John does not come at all then it is certain that the meeting will be quiet". First define the vague predicate "late" on the universe of time U = [0, 24] by its membership function "late" (see Figure 2) :

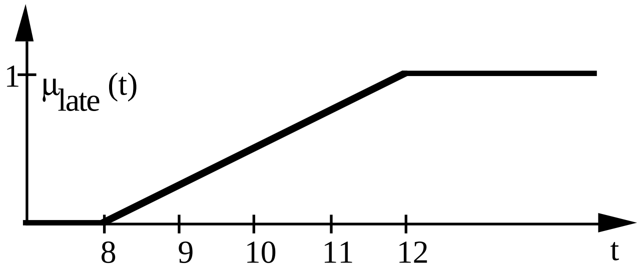


Figure 2 : Membership function of the vague predicate "late"

and then we translate the statement by the possibilistic formulas

$$\forall t\ (\text{Arrives}(\text{John, meeting}, t) \rightarrow \text{Quiet}(\text{meeting})\ (N\ \mu_{late}(t)))$$
$$((\forall t\ \neg\text{Arrives}(\text{John, meeting}, t)) \rightarrow \text{Quiet}(\text{meeting})\ (N\ 1))$$

In the first possibilistic formula above, the quantifier $\forall$ is outside the scope of the valuation (which entails that the later depends on t), while in the second formula (which is a "standard" possibilistic formula) the quantifier is inside the scope of the valuation. Generalizing the semantics of possibilistic logic in order to take into account variable valuations is not straightforward (see Lang (1991a) for further details).

Another kind of variable valuations is encountered in hypothetical reasoning. As pointed out in (Dubois et al., 1989), the weighted clause $(\neg\varphi \vee \psi\ \ \alpha)$ is semantically equivalent to the weighted clause $(\psi\ \ \min(\alpha, v(\varphi)))$ where $v(\varphi)$ is the truth value of φ, i.e. $v(\varphi) = 1$ if φ is True and $v(\varphi) = 0$ if φ is False. Indeed, for any necessity-valued proposition $(\varphi\ \alpha)$ we can write the membership function of the fuzzy set of models of $(\varphi\ \alpha)$, $\mu_{M(\varphi\ \alpha)}(\omega)$ under the form $\max(v_\omega(\varphi), 1 - \alpha)$, where $v_\omega(\varphi)$ is the truth-value assigned to φ by interpretation ω. Then we have :

$$\forall\omega,\ \mu_{M(\neg\varphi\vee\psi\ \alpha)}(\omega) = \max(v_\omega(\neg\varphi \vee \psi), 1-\alpha) = \max(1-v_\omega(\varphi), v_\omega(\psi), 1-\alpha)$$
$$= \max(v_\omega(\psi), 1-\min(v_\omega(\varphi), \alpha)) = \mu_{M(\psi\ \min(v_\omega(\varphi),\alpha))}(\omega).$$

This remark is very useful for hypothetical reasoning, since by "transferring" an atom φ from a clause to the weight part of the formula we are introducing an explicit assumption. Indeed changing $(\neg\varphi \vee \psi\ \ \alpha)$ into $(\psi\ \ \min(v(\varphi), \alpha))$ leads to state the piece of knowledge under the form "ψ is certain at the degree α, *provided that* φ is true". Then the weight is no more just a degree but in fact a label which expresses the *context* in which the piece of knowledge is more or less certain.

4.3. L-possibilistic logics

The choice of the unit interval for the necessity and possibility degrees is not compulsory. Basically what is needed is a partially ordered set such that any pair of elements possesses a least upper bound (sup) and a greatest lower bound (inf), and that possesses as well a top and a bottom element (denoted $\mathbb{1}$ and $\mathbb{0}$ respectively) so as to valuate ⊤ and ⊥. In other words L must be a complete lattice (being furthermore distributive)[5] . Then a lattice-valued necessity measure (L-necessity for short) is such that :

$$N(\varphi \wedge \psi) = \inf(N(\varphi),N(\psi))$$

The sup-operation is needed in case more than one proof path concludes on φ ; indeed, from $N(\varphi) \geq \alpha$, $N(\varphi) \geq \beta$, we would like to conclude on $N(\varphi) \geq \sup(\alpha,\beta)$. The use of a (non-necessarily totally ordered) lattice as the set of certainty degrees attached to logical formulas has already be studied especially in the scope of non-monotonic logics and logic programming ; see Sandewall (1985), Ginsberg (1988), Fitting (1991), Subrahmanian (1989), Froidevaux and Grossetête (1990). In this section we extend possibilistic logic in this direction.

Clearly the resolution-based refutation machinery works with such a structure, in the case of necessity-valued possibilistic logic. Introducing possibility-qualified statements is not easy because no inversion may exist on L. One simple way out of this problem is to introduce this kind of propositions using upper bound on necessity-measures, i.e. the following syntax could be used

$$(\varphi\ \alpha^+) \text{ means } N(\varphi) \geq \alpha$$
$$(\varphi\ \alpha^-) \text{ means } N(\neg\varphi) \leq \alpha$$

In that case the resolution rule R' works as follows in the hybrid case

5 See Goguen (1969) for argument in favour of completeness and distributiveness for lattice-valued fuzzy sets.

$$(\varphi\ \alpha^+)\ ;\ (\psi\ \beta^-) \vdash (\mathrm{Res}(\varphi,\psi)\ \alpha^+ * \beta^-)$$

such that $\alpha^+ * \beta^- = \beta^-$ if $\alpha > \beta$

$= \mathbb{1}^-$ otherwise

It is easy to check that when L = [0,1], the above resolution rule is equivalent to the one of possibilistic logic letting $\Pi(\varphi) = 1 - N(\neg\varphi)$; especially, it highlights the fact that the operation $(N\ \alpha) * (\Pi\ \beta) = \beta$ if $\alpha + \beta > 1$ and 0 otherwise owes nothing to additivity in [0,1]. Interesting examples of such lattices L are :

- a finite chain of symbolic certainty levels ;
- a Boolean lattice ; for instance generated by a partition of a time scale, $N(\varphi)$ being a time period when φ is certainly true : we get a reified temporal logic called *timed possibilistic logic*. This can be generalized to a lattice of fuzzy sets, if $N(\varphi)$ is a fuzzy time period when φ is *more or less* certainly true (Dubois, Lang, Prade 1991d) ;
- a lattice of fuzzy sets of sources, where $N(\varphi)$ is the fuzzy set of sources according to which φ is more or less certainly true (Dubois, Lang and Prade, 1992).
- the set of convex fuzzy sets on [0,1] (that may model linguistic values pertaining to certainty qualification) ;

4.4. Weighted logics based on decomposable set-functions

Keeping the [0,1] interval, one may wish to relax the axiom

$$\Pi(\varphi \vee \psi) = \max(\Pi(\varphi),\Pi(\psi)).$$

Then we can work with a very general class of [0,1]-valued set-functions (including possibility measures) introduced by Dubois and Prade (1982) and also studied by Weber (1984). Let g denote such a set-function. Possible candidates should obey the axioms : (i) $g(\bot) = 0$; (ii) $g(\top) = 1$; (iii) if $\varphi \wedge \psi = \bot$ then $g(\varphi \vee \psi) = g(\varphi) * g(\psi)$, where $*$ is a mapping from $[0,1]^2$ to [0,1] which is a semi-group with unit 0 and absorbing element 1 on the unit interval, also called a *triangular co-norm* (Schweizer and Sklar, 1983). The property (iii) is called *decomposability*. A dual mapping $\bar{g}$ is defined from g by

$$\bar{g}(\varphi) = 1 - g(\neg\varphi)$$

and verifies (i), (ii) and (iii'): if $\varphi \vee \psi = \top$ then $\bar{g}(\varphi \wedge \psi) = \bar{g}(\varphi)\ \bar{*}\ \bar{g}(\psi)$, where $\bar{*}$ is defined from $*$ by : $\forall\alpha, \beta \in [0,1]$, $\alpha\ \bar{*}\ \beta = 1 - (1 - \alpha) * (1 - \beta)$. We obtain thus a second class of decomposable measures, which includes necessity measures. This setting is more general than both probability and possibility theories, and can be characterized by suitable comparative relations on a set of propositions (Dubois, 1986).

In the particular case where $\alpha * \beta = \min(1, \alpha + \beta)$ and $\alpha \mathbin{\bar{*}} \beta = \max(0, \alpha + \beta - 1)$, then g can be chosen as a probability measure (provided that $\bar{g} = g$) and we recover thus *probabilistic logic*, in the sense of (Nilsson, 1986). Apart from $* = \max$, the choice of $\alpha * \beta = \alpha + \beta - \alpha\beta$ corresponds to still another family of set-functions. The setting of decomposable measures thus encompasses both probabilistic and possibilistic logics.

Instead of [0,1], the lattice $L = [0, +\infty]$, equipped with the opposite ordering (such that $\mathbb{0} = +\infty$ and $\mathbb{1} = 0$) and $a \mathbin{\bar{*}} b = a + b$ (which corresponds to $\bar{*}$ = product by the one-to-one mapping $a = -\ln(\alpha)$ from [0,1] to $[0,+\infty]$) leads to *toll logic* (see Dubois and Prade, 1991e) where $\bar{g}(\varphi)$ is the cost for the realization of φ, the simultaneous realization of two formulas φ and ψ such that $\varphi \vee \psi = \top$ being the sum of the costs for the realizations of φ and ψ, i.e. $\bar{g}(\varphi \wedge \psi) = \bar{g}(\varphi) \mathbin{\bar{*}} \bar{g}(\psi) = \bar{g}(\varphi) + \bar{g}(\psi)$ when $\varphi \vee \psi = \top$.

4.5. Possibilistic logic with vague predicates

Preliminary work aiming at extending the resolution rule over to the case when possibility and necessity-valued formulas involve *fuzzy* predicates, that is predicates whose extensions are fuzzy sets is proposed in Dubois and Prade (1990). When fuzzy predicates are involved the basic problems are the lack of a Boolean structure for the language quotiented by the logical equivalence relation, and the question of a proper definition of the certainty and possibility of fuzzy statements. The approach proposed in the above-mentioned reference consists in keeping the same syntax as possibilistic logic, but modifying the resolution rule in order to account for the possible overlap of models of φ and $\neg\varphi$ in the fuzzy case. It seems difficult to define right away what an interpretation is for a fuzzy proposition. We assume that we can start with a set Ω of possible worlds, and that each world ω is compatible with a vague proposition φ to a degree, say α ; let us denote it $\omega \vDash_\alpha \varphi$, where $\alpha \in [0,1]$; fuzzy set complementation leads to consider $\omega \vDash_\alpha \neg\varphi$ as equivalent to $\omega \vDash_{1-\alpha} \varphi$. Fuzzy set union and intersection suggest that

$$\omega \vDash_\alpha \varphi \text{ and } \omega \vDash_\beta \psi \Rightarrow \omega \vDash_{\min(\alpha,\beta)} \varphi \wedge \psi \text{ ; } \omega \vDash_{\max(\alpha,\beta)} \varphi \vee \psi$$

The degree of consistency of φ and ψ is evaluated as

$$\mathrm{Cons}(\varphi,\psi) = \sup_\omega \{\min(\alpha,\beta) \mid \omega \vDash_\alpha \varphi \text{ ; } \omega \vDash_\beta \psi\}$$

As a consequence, $\mathrm{Cons}(\varphi,\neg\varphi) \le \sup_\alpha \min(\alpha, 1 - \alpha) = 0.5$ only, i.e. φ and $\neg\varphi$ are no longer always totally contradictory.

Let π be a possibility distribution on Ω and φ be a fuzzy formula. The degree of certainty $C(\varphi)$ of φ in the face of π should verify the following properties :

i) $C(\varphi) = 1$ if and only if $\forall\omega, \omega \vDash_\alpha \varphi \Rightarrow \pi(\omega) \le \alpha$, i.e. the fuzzy set of worlds satisfying φ contains the fuzzy set of possible worlds described by π,
ii) $C(\varphi) \ge \beta$ if and only if $\forall\omega$ such that $\pi(\omega) > 1 - \beta$, $\omega \vDash_\alpha \varphi$ implies $\pi(\omega) \le \alpha$. In other words the inclusion relationship between π and the fuzzy extension of φ may fail to hold for $\pi(\omega)$ small enough ; and the higher β the lower the level below which this inclusion may fail to hold.

Letting $\mu_\varphi(\omega) = \alpha$ be equivalent to $\omega \vDash_\alpha \varphi$, a certainty index that satisfies these two requirements is (Dubois and Prade, 1991d) :

$$C(\varphi) = \inf_{\omega \in \Omega} \pi(\omega) \rightarrow \mu_\varphi(\omega)$$

where $\rightarrow$ is the reciprocal of Gödel's implication, i.e. $\alpha \rightarrow \beta = 1$ if $\alpha \le \beta$, and $1 - \alpha$ otherwise. Then we have the following equivalence

$$C(\varphi) \ge \beta \Leftrightarrow \forall\omega, \omega \vDash_\alpha \varphi \Rightarrow \pi(\omega) \le \max(\alpha, 1 - \beta)$$

which by the principle of minimal specificity forces $\pi(\omega) = \max(\mu_\varphi(\omega), 1 - \beta)$. It is an extension of the necessity qualification, as introduced in the crisp case in Section 2, and used for interpreting necessity-valued clauses.

This leads to the property, $\forall\ \beta \in [0,1]$, $\omega \vDash_\alpha \varphi$ implies $\omega \vDash_{\max(\alpha,1-\beta)} (\varphi\ (N\ \beta))$, viewing $(\varphi\ \ (N\ \beta))$ as equivalent to φ' with $\mu_{\varphi'} = \max(\mu_\varphi, 1 - \beta)$. Moreover the satisfaction relation for a possibility distribution π writes $\pi \vDash (\varphi\ (N\ \beta))$ if and only if $C(\varphi) \ge \beta$, when φ is a fuzzy proposition. Note that when φ is non-fuzzy, $C(\varphi)$ and $N(\varphi)$ coincide which justifies the notation $(\varphi\ (N\ \beta))$.

The resolution principle is then extended in the proposition case to a cut operation between two fuzzy clauses $\varphi \vee \psi$ and $\varphi' \vee \xi$ as follows :

$$(\varphi \vee \psi\ \ (N\ \alpha)) \wedge (\varphi' \vee \xi\ \ (N\ \beta)) \vdash (\psi \vee \xi\ \ (N \min(\alpha, \beta, 1 - \mathrm{Cons}(\varphi,\varphi')))).$$

Clearly, the less contradictory φ and φ', the less informative is the result provided by the cut rule. Especially, if $\varphi' = \neg\varphi$, the certainty degree associated with $\psi \vee \xi$ will generally be upper bounded by 0.5. When Cons(φ,φ') = 1 which happens for instance under φ' = φ, then the resolution rule leads to a completely uninformative result, which is satisfactory since φ' and φ are not contradictory at all in that case. The notion of possibility-valued fuzzy propositions and the corresponding resolution rule can be extended likewise (Dubois and Prade, 1990). However further research is needed to fully justify resolution rules for fuzzy formulas at a semantic level.

5. Some applications

5.1. Possibilistic management of assumptions

The principle of assumption-based truth-maintenance systems (ATMS) is to distinguish between two types of literals in a knowledge base, one being called *assumptions*. A knowledge base is viewed as a set of propositional formulas (usually clauses) called justifications. The problem solved by an ATMS, is to calculate, given a literal p the configuration of assumptions which enable p to be derived.

Classical ATMS (De Kleer, 1986a,b) require that the clauses contained inside the knowledge base (justifications and possibly disjunctions of assumptions) be certain ; but we may wish to handle more or less uncertain information without losing the capacities of the ATMS. The basic principle of the possibilistic ATMS is to associate to each clause a weight α which is a lower bound of its necessity degree. Assumptions may also be weighted, i.e. the user or the inference engine may decide at any time to believe an assumption with a certainty degree that he/she will give. The capabilities of possibilistic logic for dealing with assumptions are to be related to the way contexts can be handled in the weight part of a possibilistic formula, as mentioned in Section 4.2. A possibilistic ATMS (Dubois, Lang and Prade, 1990a, b) is capable of answering the following questions :

(i) Under what configuration of assumptions is the proposition p certain to the degree α ? (i.e., what assumptions shall we consider as true, and with what certainty degrees, in order to have p certain to the degree α ?)

(ii) What is the inconsistency degree of a given configuration of assumptions ?

(iii) In a given configuration of assumptions, to what degree is each proposition certain ?

We are now giving a few technical details. The basic notions attached to the classical ATMS are generalized in the following way. Let Σ be a set of necessity-valued clauses and E a set of assumptions ; the following definitions are useful :

–) let E be an environment, i.e. a set of assumptions considered as certainly true (i.e. weighted by 1). E is said to be an α-environment of the literal p if and only if $\Sigma \cup E \models (p\ \alpha)$ with α maximal, i.e. $\forall\ \alpha' > \alpha$, $E \cup \Sigma \not\models (p\ \alpha')$.

–) E is an α–contradictory environment, or *α–nogood* if and only if Incons $(E \cup \Sigma) = \alpha$. It is said to be minimal if there is no β–nogood E' such that $E \supset E'$ and $\alpha \leq \beta$ (at least one of the two relations being strict).

In order to define the label of a proposition p, we consider only non-weighted assumptions (i.e. they will have the implicit weight 1). It can be shown that it is useless to weight the assumptions inside the labels (this remark also holds for the base of nogoods). The label of the proposition p, $L(p) = \{(E_i, \alpha_i), i \in I\}$ is the unique *fuzzy* subset of environments for which the four following properties hold (see (Dubois, Lang and Prade, 1990a, b) for more details) :

– *(weak) consistency* : $\forall\ (E_i, \alpha_i) \in L(p)$, Incons $(E_i \cup \Sigma) < \alpha_i$.
– *soundness* : L(p) is sound if and only if $\forall\ (E_i, \alpha_i) \in L(p)$ we have $E_i \cup \Sigma \models (p\ \alpha_i)$.
– *completeness* : L(p) is complete if and only if for every environment E' such that $E' \cup \Sigma \models (p\ \alpha')$ non trivially, then $\exists\ i \in I$ such that $E_i \subset E'$ and $\alpha_i \geq \alpha'$.
– *minimality* : L(p) is minimal if and only if it does not contain two different weighted environments (E_1, α_1) and (E_2, α_2) such that $E_1 \subset E_2$ and $\alpha_1 \geq \alpha_2$.

Ranking environments according to their weight in the label of each proposition provides a way of limiting the consequences of combinatorial explosion : indeed when a label contains too many environments, the possibilistic ATMS can help the user by giving the environments with the greatest weight(s) only.

A possibilistic ATMS extends Cayrol and Tayrac (1989)'s generalized ATMS, where each piece of information is represented by a (general) propositional clause, which enables

– a uniform representation for all pieces of knowledge (no differenciated storage and treatment between justifications and disjunctions of assumptions) ;
– the capability of handling negated assumptions as assumptions, i.e. environments and nogoods may contain negations of assumptions ;
– a simple and uniform algorithm for the computation of labels and nogoods, based on resolution.

An application of possibilistic ATMS to diagnosing faults under uncertainty is developed in (Dubois, Lang and Prade, 1990a). See (Benferhat et al., 1991) for implementation issues.

A possibilistic ATMS offers a simple way of managing inconsistency in a possibilistic knowledge base that is more refined than the inconsistency-tolerant deduction in possibilistic logic. More specifically, it enables to compute the strongly maximal consistent sub-bases of a possibilistic knowledge base $\mathcal{F}$, i.e. deleting from $\mathcal{F}$ only the minimally weighted formulas involved in the inconsistency. The obtained revised knowledge base $\mathcal{F}'$ may contain formulas $(\varphi\ \alpha)$ with $\alpha <$ Incons $(\mathcal{F})$ that would have been inhibited by the inconsistency-tolerant deduction from $\mathcal{F}$. The revision procedure consists in finding all the minimal inconsistent subsets of $\mathcal{F}$; this can be done by means of the possibilistic ATMS as follows : attach a specific assumption H_φ to each formula $(\varphi\ \alpha)$ in $\mathcal{F}$ (changing $(\varphi\ \alpha)$ into $(H_\varphi \rightarrow \varphi\ \ \alpha)$) and let $\mathcal{F}_H$ be the obtained knowledge base. Find all the nogoods in $\mathcal{F}_H$ using the possibilistic ATMS; each nogood in $\mathcal{F}_H$ corresponds to a minimally inconsistent subset of $\mathcal{F}$. Then roughly speaking the strategy consists in deleting the least weighted formula from each nogood (see Dubois, Lang and Prade (1991b) for details).

The handling of inconsistency in a knowlege-based by means of an ordering of formulas is more generally considered in Cayrol (1992), and Cayrol, Royer and Saurel (1992), following ideas initiated by Brewka (1989) and Poole (1988).

5.2. Discrete optimisation

So far, possibility and necessity measures have been considered as degrees of uncertainty linked to the partial absence of information. It makes sense to interpret them in a different way in the scope of constraint-based reasoning. $(\varphi\ (N\ \alpha))$ can be viewed as declaring a constraint φ with a degree of imperativeness equal to $N(\varphi) = \alpha$. When $\alpha = 1$, φ cannot be violated ; when $\alpha = 0$, φ can be dropped. In that case $\Pi(\neg\varphi) = 1 - \alpha$ evaluates to what extent φ is allowed to be violated. N and Π are thus given a deontic interpretation ; N stands for evaluating more or less compulsory constraints, while Π describes whether something is allowed or not. More specifically, let us interpret the properties of N and Π in this framework :

- $N(\top)$ $(= \Pi(\top)) = 1$ indicates that tautologies are imperative ;
- Since $(\varphi\ (N\ \alpha)) \wedge (\neg\varphi\ (N\ 1)) \vdash (\bot\ (N\ \alpha))$, contradictions are tolerated, i.e. partially feasible solutions where $(\varphi\ (N\ \alpha))$ is violated. $1 - \alpha$ thus denotes the degree of feasibility of such solutions ;
- $N(\varphi \wedge \psi) = \min(N(\varphi),N(\psi))$ is equivalent to $\Pi(\neg\varphi \vee \neg\psi) = \max(\Pi(\neg\varphi),\Pi(\neg\psi))$. It expresses that if $N(\varphi) = \alpha$ and $N(\psi) = \beta$, violating one of the two constraints can be allowed while preserving a level of feasibility at most equal to $1 - \min(\alpha,\beta)$;
- The possibility distribution $\pi_{\mathcal{F}}$ induced by a set of N-valued constraints represents the fuzzy feasibility domain, subnormalization indicating that some constraints which are not fully imperative must be violated.

The use of min and max operators suggests that the precise values of the necessity (or possibility) degrees is less important than the ordering on the formulas induced by them : thus, necessity degrees may be seen as *priority degrees*, where $N(\varphi) > N(\psi)$ expresses that the satisfaction of φ is more important than the satisfaction of ψ.

As we have seen in Section 3, the inconsistency degree of a set of necessity-valued formulas $\mathcal{F} = \{(\varphi_1\ \alpha_1), \ldots, (\varphi_n\ \alpha_n)\}$, verifies the equality

$$\mathrm{Incons}(\mathcal{F}) = 1 - \max_{\omega \in \Omega} \pi_{\mathcal{F}}(\omega)$$

and computing the best model(s) of $\mathcal{F}$ comes down to find the interpretations ω maximizing $\pi_{\mathcal{F}}(\omega)$, where $\pi_{\mathcal{F}}(\omega) = \min\{1 - \alpha_i \mid \omega \models \neg\varphi_i, i = 1, ..., n\}$. The best models correspond to the optimal (most feasible) solutions to a given problem. In a more compact way, it reduces to the discrete optimisation problem

$$\max_{\omega \in \Omega} \min\{1 - \alpha_i \mid \omega \models \neg\varphi_i, i = 1, \ldots, n\}$$

or equivalently to this other one

$$\min_{\omega \in \Omega} \max\{\alpha_i \mid \omega \models \neg\varphi_i, i = 1,\ldots, n\}$$

So, computing $\mathrm{Incons}(\mathcal{F})$ and the best model(s) of $\mathcal{F}$ is a min-max discrete optimisation problem ; hence, problems of the same nature, which have the general form

$$\min_{x \in X} \max_{y \in Y} f(x,y)$$

where X and Y are finite, can be translated into necessity-valued logic and solved by resolution or semantic evaluation ; moreover, if semantic evaluation is used, the set of best models of $\mathcal{F}$ will give the set of optimal solutions for the min-max discrete optimisation problem.

Of course, the problem of computing the inconsistency degree of $\mathcal{F}$ is NP-complete (see Proposition 8) ; thus, resolution and semantic evaluation are (in the case where we use non-Horn clauses) exponential[6] and it is clear that for a given problem, there generally exists a specific algorithm whose complexity is at least as good as (often better than) the complexity of necessity-valued semantic evaluation. Thus, we do not claim to give, for the problems we shall deal with, a more efficient algorithm than already existing ones ; however, we think that translation into necessity-valued logic is useful, for several reasons :

6 However, their *average* complexity may be polynomial in some particular cases.

- the search method is independent from the problem ;
- the pruning properties (in the search tree) of the semantic evaluation procedure can confer to the algorithm a good average complexity (even polynomial, in some cases) (see Lang, 1990) ;
- necessity-valued logic enables a richer representation capability in the formulation of a problem (one can specify complex constraints not easy to express in the language requested by a specific algorithm).

Thus, necessity-valued logic appears to be a logical framework for expressing in a declarative way some min-max discrete optimisation problems. An typical example of such a problem is the min-max assignment problem (also called "bottleneck assignment problem") formulated as follows : n tasks must be assigned to n machines (one and only one task per machine) ; if machine i is assigned to task j, the resulting cost is a_{ij}. Then the total cost of the global assignment is not the sum, but the maximum of the costs of the elementary assignments.

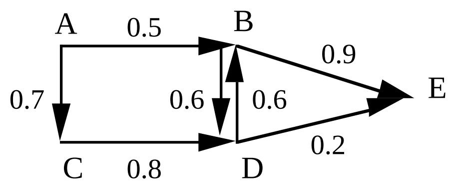
More generally, min-max discrete optimisation problems may come from constraint satisfaction problems, where the constraints are weighted by necessity degrees measuring their priority, and where the constraint set is "partially" inconsistent, in the sense of Section 3. Solving such a "prioritized" constraint satisfaction problem consists in finding the solution minimizing the degree of the most important constraint among those which are violated. Again, necessity-valued logic offers a general logical framework for representing and solving these problems. See (Lang, 1991b) for a detailed example.

5.3. Logic programming

The basic idea of logic programming (e.g. Lloyd, 1984) is to use logic as a programming language : in that sense, it is much more than automated theorem proving. In an algorithm, there are two disjoint components : the logical description of the problem and of what is to be proved (or solved), and the control part, i.e. how the problem has to be solved. In an idealistic programming language, the user has only to take care of the logical part. Since a problem may contain uncertain knowledge, possibilistic logic seems to be a nice tool for designing a logic programming language well-adapted for dealing with uncertainty. Moreover, possibilistic logic programming can be used for solving problems with min-max criteria, as said in section 5.2. It is clear that such problems can be solved using an ordinary logic programming language, but the user then must handle himself the numerical part of the program : computing the weight of a proof path, and then combining the weights of all proof paths to a goal, which leads to a more complex formulation, and also to a less efficient solving of the problem, except if the user also takes care of the central part of the program (which is long and tedious). In the following we make use of Kowalski's notation $A \leftarrow B_1 \ldots B_n$ for $\neg B_1 \vee \ldots \vee \neg B_n \vee A$, and for necessity-valued Horn clauses we use the syntax c [α] instead of (c α).

Example : least unsure path(s) in a graph

Let us consider a fuzzy graph $\tilde{G}$ viewed as a pair $(\mathcal{V}, \tilde{E})$ where $\mathcal{V}$ is a set of vertices and $\tilde{E}$ a fuzzy relation on $\mathcal{V} \times \mathcal{V}$; given two vertices v and v', we are interested in finding the weight of the least unsure path between v and v', i.e. the path with maximal weight, the weight of a path being the minimum of the weights of the edges composing it ; for instance, in the fuzzy graph



the least unsure path between A and B is (ACDBE) ; it has the weight 0.6. This weight can be found by a possibilistic logic program ; for the above example, it is the following :

1. path($x,$y) ← edge($x,$y) [1] ;
2. path($x,$y) ← path($x,$z) edge($z,$y) [1] ;
3. edge(A,B) ← [0.5] ;
4. edge(A,C) ← [0.7] ;
5. edge(B,D) ← [0.6] ;
6. edge(D,B) ← [0.6] ;
7. edge(C,D) ← [0.8] ;
8. edge(B,E) ← [0.9] ;
9. edge(D,E) ← [0.2] .

If we add the goal ← path (A,E) [1], the only optimal answer substitution is {} [0.6], i.e. the weight of the least unsure path between A and E is 0.6. If we add the goal ← path ($x,$y) [1], then we shall find all weights of the least unsure paths between two vertices of the graph, i.e. we shall have the max-min transitive closure of the graph. It is easy to modify the program in order to obtain the complete optimal paths and not only the weights (it is sufficient to introduce a third variable in the predicate ‘path’ in order to collect the list of the edges of the current path). This example exhibits the ability of a "possibilistic Prolog" to handle max-min optimisation problems ; indeed with a classical Prolog interpreter, the programmer would have to take care of the numerical aspects of the example.

More formal details and results about declarative and procedural semantics of possibilistic logic programs can be found in (Dubois, Lang and Prade, 1991c). They are connected to results obtained by Subrahmanian (1990) who generalizes Van Emden's (1986) quantitative logic programming.

6. Conclusion

Possibilistic logic appears to be a natural extension of classical logic where the notion of total ordering on formulas is embedded in the logic. It embodies the basic structure of preferential-model-based non-monotonic logics because a possibility distribution is an easy way of encoding a preference ordering on interpretations. A comparison on a tutorial example of possibilistic logic with other formalisms for reasoning under incomplete knowledge can be found in Lea Sombé (1990). Possibilistic logic thus contrasts with Ruspini (1991)'s view of fuzzy logic, based on a similarity relation between interpretations (or possible worlds) rather than on an ordering relation. In Ruspini's view, instead of focusing on the preferred models of φ in the preferential entailment $\varphi \mid\approx \psi$, we rather enlarge the set of models of ψ by considering other models which are close to a model of ψ. Moreover possibilistic logic possesses an inference machinery which is a direct extension of refutation by resolution. This fact suggests that cumulative-like non-monotonic logics studied by Kraus et al. (1990) and by Gärdenfors and Makinson (1991) can be efficiently implemented. Moreover it might turn out that possibilistic logic appears as a special case of "labelled deductive systems" studied by Gabbay (1991), since a possibilistic logic formula is a pair made of a classical formula and a weight where weights can encode uncertainties and/or contexts.

However, these results tend to make us forget that the origins of possibility theory belong to the field of fuzzy sets introduced by Zadeh (1965). This latter fact suggests that possibilistic logic might have a basic role to play in the development of fuzzy information systems where fuzzy predicates must be explicitly handled. The so-called "fuzzy logic-based controllers" introduced by Mamdani (1977) and now widely applied in Japan (e.g. Sugeno, 1985) are but very simple examples of such fuzzy information systems containing only some fuzzy rules working in parallel, and performing interpolation. Interestingly the "logic" of fuzzy controllers can be completely interpreted in the framework of possibility theory (Dubois and Prade, 1991d). A future task is to define a genuine logic handling fuzzy predicates, i.e. define its syntax and its inference rules. Viewed in the light of non-monotonic reasoning, a fuzzy proposition can be semantically interpreted as inducing itself a preference ordering on possible worlds in which this proposition is true. Hence the link established, via possibility theory, between "fuzzy logic" and non-monotonic reasoning might be worth studying further. Another line of interest for further research is the handling of weights attached to subformulas in order to express in the language pieces of knowledge such that "if p is certain and if q is at least somewhat possible, then r is almost certain". It might lead to a logical formalization, of the approach suggested by Yager (1987) for default reasoning. Besides, it has been recently shown that a set of defaults rank-ordered by system Z (Pearl, 1990) can be encoded in possibilistic logic (Benferhat, Dubois and Prade, 1992).

Lastly, possibilistic logic bears obvious analogies with probabilistic logic. However they do not seem to be tailored for the same purposes. Probabilistic logic seems to be well adapted to the structuration and processing of statistical knowledge (as in Bayesian networks, Pearl (1988)), including when this statistical knowledge is incomplete (Kyburg, 1974 ; Bacchus, 1990). Probabilistic logic has also been construed as a theory of induction (Carnap, 1950), or a logic of subjective belief (Halpern, 1989). However in both cases, the same mathematical tools as in statistics are adopted, so that computations are based on counting rather than comparing. As a consequence, probabilistic logic is much more complex than possibilistic logic, especially if conditional probabilities must be accounted for in the language. Possibilistic logic aims at reasoning with the most reliable part of a knowledge base, i.e. by means of the most "entrenched" formulas (to borrow from Gärdenfors (1988)). It strongly departs from the type of inference made in probabilistic logic where a high number of very unreliable proof paths producing a conclusion may lead to the complete certainty of this conclusion (as it is the case with the lottery paradox). Hence it might be misleading to consider possibilistic logic as a surrogate of probabilistic logic. It is neither a generalisation nor a special case of probabilistic logic. Nevertheless it turns out that possibility measures can be viewed as a very special family of upper and lower probabilities (Dubois and Prade, 1988). And Spohn (1988)'s theory of ordinal conditional functions has moreover given birth to an interpretation of possibility measures in terms of infinitesimal probabilities (e.g. Dubois and Prade, 1991b). The state of fact suggests that despite their distinct and contrasted specificities, it may be of interest to search for formal connections between probabilistic and possibilistic logics.

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Annex

In this annex we give the proofs of most of the results given in Sections 3 and 4 of the paper. Most of them can be found in (Lang, Dubois and Prade, 1991) and (Lang, 1991a).

Proofs of results from Section 3 (possibilistic logic PL1)

Proposition 1 :

Let $\mathcal{F} = \{(\varphi_1\ \alpha_1), ..., (\varphi_n\ \alpha_n)\}$ be a set of necessity-valued formulas and let us define the possibility distribution $\pi_{\mathcal{F}}$ by

$$\pi_{\mathcal{F}}(\omega) = \inf\{ 1 - \alpha_i \mid \omega \vDash \neg\varphi_i, i = 1, ..., n\}$$
$$= 1 \text{ if } \omega \vDash \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$$

then for any possibility distribution π on Ω, $\pi \vDash \mathcal{F}$ if and only if $\pi \le \pi_{\mathcal{F}}$, i.e. $\forall\ \omega \in \Omega, \pi(\omega) \le \pi_{\mathcal{F}}(\omega)$.

<u>Proof</u> : $\pi \vDash \mathcal{F}$ iff $(\forall\ i = 1, ..., n)\ \pi \vDash (\varphi_i\ \alpha_i)$
iff $(\forall\ i = 1, ..., n)\ N(\varphi_i) \ge \alpha_i$ (where N is the necessity measure induced by π)
iff $(\forall\ i = 1, ..., n)\ \inf\{1 - \pi(\omega) \mid \omega \vDash \neg\varphi_i\} \ge \alpha_i$
iff $(\forall\ i = 1, ..., n)\ (\forall\ \omega \vDash \neg\varphi_i)\ \pi(\omega) \le 1 - \alpha_i$
iff $\pi(\omega) \le \inf\{1 - \alpha_i \mid \omega \vDash \neg\varphi_i, i = 1, ..., n\}$
iff $\pi(\omega) \le \pi_{\mathcal{F}}(\omega)$. □

Proposition 3 :

$\text{Incons}(\mathcal{F}) = \inf\{N(\bot) \mid \pi \vDash \mathcal{F}\} = \sup\{\alpha, \mathcal{F} \vDash (\bot\ \alpha)\}$ where N denotes the necessity distribution induced by π.

<u>Proof</u> :

(i) $\text{Incons}(\mathcal{F}) = \inf_{\omega\in\Omega} (1 - \pi_{\mathcal{F}}(\omega)) = \inf_{\pi\le\pi_{\mathcal{F}}} \inf_{\omega\in\Omega} 1 - \pi(\omega) = \inf_{\pi\le\pi_{\mathcal{F}}} N(\bot) = \inf\{N(\bot), \pi \vDash \mathcal{F}\}$.

ii) $\sup\{\alpha, \mathcal{F} \vDash (\bot\ \alpha)\} = \sup\{\alpha, (\forall\pi, \pi \vDash \mathcal{F}, N(\bot) \ge \alpha)\} = \inf\{N(\bot), \pi \vDash \mathcal{F}\}$. □

Proposition 4 :

The least upper bound in the computation of $\text{Incons}(\mathcal{F})$ is reached, i.e. there exists (at least) one interpretation ω^* such that $\pi_{\mathcal{F}}(\omega^*) = \sup_{\omega\in\Omega} \pi_{\mathcal{F}}(\omega)$.

<u>Proof</u> : In the propositional case, this result is trivial, since Ω is finite. In the general case, since there are a finite number of necessity-valued formulas (and hence a finite number of valuations

α_i), the definition of $\pi_{\mathcal{F}}$ implies that $\pi_{\mathcal{F}}(\omega)$ takes only a finite number of values when ω ranges along the (infinite in the first-order case) set of interpretations Ω. Hence the result. □

Proposition 5 :

The inconsistency degree of an inconsistent possibilistic knowledge base $\mathcal{F}$ is the smallest weight of possibilistic formulas in any strongly minimal inconsistent subset $\mathcal{F}'$ of $\mathcal{F}$. More precisely, if Incons ($\mathcal{F}$) = $\alpha > 0$ then there exists at least one formula $(\varphi\ \alpha) \in \mathcal{F}'$ and $\forall\ (\varphi'\ \beta) \in \mathcal{F}', \beta \geq \alpha$.

<u>Proof</u> : Assume $\mathcal{F}' = \{(\varphi_i\ \alpha_i), i = 1,m\}$ is a strongly minimal inconsistent subset of $\mathcal{F}$. By definition of $\mathcal{F}'$ we have

$$\text{Incons}(\mathcal{F}') = \text{Incons}(\mathcal{F}) = \alpha = 1 - \sup_{\omega\in\Omega} \pi_{\mathcal{F}}(\omega)$$

Assume $\alpha_1 = \min_{i=1,m} \alpha_i$. Let us prove that $\alpha_1 = \alpha$. π satisfies $\mathcal{F}'$ if and only if $\forall i$, $\forall\ \omega \models \neg\varphi_i, \pi(\omega) \leq 1 - \alpha_i$; in other words, $\forall\pi, \pi \models \mathcal{F}'$ implies $\forall\ \omega \models \neg\varphi_1 \vee \neg\varphi_2 \vee \ldots\vee \neg\varphi_m, \pi(\omega) \leq \max_i (1 - \alpha_i) = 1 - \alpha_1$. Hence, since $\neg\varphi_1 \vee \neg\varphi_2 \vee \ldots\vee \neg\varphi_m$ is a tautology (otherwise $\mathcal{F}'$ would not be inconsistent), $\forall\ \omega \in \Omega, \pi(\omega) \leq 1 - \alpha_1$ is a consequence of $\pi \models \mathcal{F}'$. Hence $\alpha \geq \alpha_1$. Now let π be defined by $\pi(\omega) = 1 - \alpha_1$ if $\omega \models \varphi_2 \wedge \varphi_3 \wedge \ldots\wedge \varphi_m$, $\pi(\omega) \leq 1 - \alpha_i$ if $\omega \models \neg\varphi_i$. Now $\varphi_2 \wedge \varphi_3 \wedge\ldots \wedge \varphi_m \neq \bot$ due to the minimality of $\mathcal{F}'$, so that $\exists\omega, \pi(\omega) = 1 - \alpha_1$, and $\pi \models \mathcal{F}$. Hence $\alpha = \alpha_1$. □

Proposition 6 :

Let $\mathcal{F}$ be a set of possibilistic formulas and let Incons($\mathcal{F}$) = inc ; then

(i) $\mathcal{F}$ is semantically equivalent to $\mathcal{F}_{\text{inc}}$ and to $\mathcal{F}_{\overline{\text{inc}}} \cup \{(\bot\ \text{inc})\}$

(ii) $\mathcal{F}_{\overline{\text{inc}}}$ is consistent

(iii) if $\mathcal{F} \models (\psi\ \alpha)$ non trivially (i.e. with $\alpha >$ inc) then $\mathcal{F}_{\overline{\text{inc}}} \models (\psi\ \alpha)$.

<u>Proof of (i)</u> : Let us show that $\mathcal{F} \models \mathcal{F}_{\text{inc}}$, that $\mathcal{F}_{\text{inc}} \models \mathcal{F}_{\overline{\text{inc}}} \cup \{(\bot\ \text{inc})\}$ and that $\mathcal{F}_{\overline{\text{inc}}} \cup \{(\bot\ \text{inc})\} \models \mathcal{F}$.

(1) $\mathcal{F} \models \mathcal{F}_{\text{inc}}$ is obvious since $\mathcal{F}$ contains $\mathcal{F}_{\text{inc}}$.

(2) $\mathcal{F}_{\text{inc}} \models \mathcal{F}_{\overline{\text{inc}}}$ is obvious since $\mathcal{F}_{\text{inc}}$ contains $\mathcal{F}_{\overline{\text{inc}}}$;
$\mathcal{F}_{\text{inc}} \models \{(\bot\ \text{inc})\}$ is an immediate consequence of Proposition 5.
Hence $\mathcal{F}_{\text{inc}} \models \mathcal{F}_{\overline{\text{inc}}} \cup \{(\bot\ \text{inc})\}$

(3) $\mathcal{F}_{\overline{\text{inc}}} \cup \{(\bot\ \text{inc})\} \models \mathcal{F}$ is less obvious :

Let π be a possibility distribution satisfying $(\perp\ inc) \wedge \mathcal{F}_{\overline{inc}}$; let us prove that π satisfies $\mathcal{F}$. For any necessity-valued formula $(\varphi_i\ \alpha_i)$ of $\mathcal{F}$: if $\alpha_i > inc$ then $\pi \models (\varphi_i\ \alpha_i)$, since π satisfies $\mathcal{F}_{\overline{inc}}$; and if $\alpha_i \leq inc$ then $N(\varphi_i) \geq N(\perp) \geq inc$ (since π satisfies $(\perp\ inc)) \geq \alpha_i$, hence $\pi \models (\varphi_i\ \alpha_i)$. Thus we have proved that π satisfies $\mathcal{F}$. Hence the result. □

<u>Proof of (ii)</u> : Let us suppose that $\mathcal{F}_{\overline{inc}}$ is inconsistent; then, from Proposition 5 it comes immediately that Incons $(\mathcal{F}_{\overline{inc}})$ is equal to the valuation of a formula of $\mathcal{F}_{\overline{inc}}$, i.e. $Incons(\mathcal{F}_{\overline{inc}}) > inc$. Then, since $\mathcal{F}$ contains $\mathcal{F}_{\overline{inc}}$, we have $inc = Incons(\mathcal{F}) \geq Incons(\mathcal{F}_{\overline{inc}}) > inc$, which is contradictory. Hence, $\mathcal{F}_{\overline{inc}}$ is consistent.

<u>Proof of (iii)</u> : $\mathcal{F} \models (\psi\ \alpha)$ non trivially means $\mathcal{F} \models (\psi\ \alpha)$ with $\alpha > inc$. Let $\mathcal{F} = \{(\varphi_i\ \alpha_i), i = 1, \ldots, n\}$; using the definition of a necessity measure induced by a possibility distribution, $\mathcal{F} \models (\psi\ \alpha)$ non-trivially, is equivalent to

$$\forall\ \omega \models \neg\psi, \exists i, \omega \models \neg\varphi_i \text{ and } 1 - \alpha_i \leq 1 - \alpha\ (< 1 - inc)$$

This implies $\forall\ \omega \models \neg\psi, \exists i, \omega \models \neg\varphi_i$ and $\alpha_i > inc$. Hence

$$\forall\ \omega \models \neg\psi, \min\{1 - \alpha_i \mid \omega \models \neg\varphi_i, \alpha_i > inc\} \leq 1 - \alpha, \text{ i.e. } \mathcal{F}_{\overline{inc}} \models (\psi\ \alpha). \quad \square$$

Proposition 7 **:**

(1) (Dubois and Prade, 1987) Let $\mathcal{F}$ be a set of necessity-valued formulas; then Incons $(\mathcal{F}) = 0$ if and only $\mathcal{F}^*$ is consistent in the classical sense.

(2) $Incons(\mathcal{F}) = \sup \{\alpha \mid \mathcal{F}_\alpha^* \text{ inconsistent}\}$
$= \inf \{\alpha \mid \mathcal{F}_{\overline{\alpha}}^* \text{ consistent}\}$

and these two bounds are reached.

<u>Proof</u> :

(1) (⇒) Let $\mathcal{F} = \{(\varphi_i\ \alpha_i), i = 1, ..., n\}$. According to its definition, Incons $(\mathcal{F}) = 0$ if and only if $\pi_{\mathcal{F}}$ is normalized, i.e. iff $\exists\ \omega^* \in \Omega$ such that $\pi_{\mathcal{F}}(\omega^*) = 1$. This implies $\omega^* \models \varphi_i, \forall i$. Hence $\mathcal{F}^*$ is consistent.

(⇐) if $\mathcal{F}^*$ is consistent then it has a model $\bar{\omega}$; then $\pi_{\mathcal{F}}(\bar{\omega}) = \text{Inf } \{1\text{-}\alpha_i \mid \bar{\omega} \models \neg\varphi_i\} = 1$ since $\forall i, \bar{\omega} \models \varphi_i$. So, $\pi_{\mathcal{F}}$ is normalized and Incons $(\mathcal{F}) = 0$.

(2) Straightforward from (1) and points (i) and (ii) of Proposition 6. □

Proposition 8 **:**

Determining the inconsistency degree of a propositional necessity-valued knowledge base is a NP-complete problem.

Proof : Let us denote as (I) the problem of the computation of Incons($\mathcal{F}$) where $\mathcal{F}$ is a propositional necessity-valued knowledge base, and (S) the satisfiability problem in classical propositional logic. It is immediate that (I) must be at least as complex as (S). We are going to prove that the complexity of these two problems are of the same nature, by showing that (I) can be reduced into at most $[1+\log_2 n]$ problems (S), where n stands for the number of formulas in $\mathcal{F}$.

Following Proposition 7 we have Incons($\mathcal{F}$) = Sup $\{\alpha \mid \mathcal{F}_\alpha{}^*$ inconsistent$\}$. Let A be an algorithm for (S) ; using (A) we define an algorithm (A') for (I) computing Incons($\mathcal{F}$) by dichotomy :

Begin

Let $\mathcal{F} = \{(\varphi_i\ \alpha_i), 1 \le i \le n\}$ and let $\alpha'_1, \ldots \alpha'_m$ be the distinct valuations appearing in $\mathcal{F}$ (so $m \le n$ and $\{\alpha'_1, \ldots, \alpha'_m\}$ is included in $\{\alpha_1, \ldots, \alpha_n\}$), ranked increasingly, i.e. $0 < \alpha'_1 < \alpha'_2 < \ldots < \alpha'_m \le 1$.

lower ← 1

upper ← m

while lower < upper **do** $\{\mathcal{F}_{\alpha_{lower}}{}^*$ is inconsistent and $\mathcal{F}_{\overline{\alpha_{upper}}}{}^*$ is consistent$\}$

r ← [(lower + upper) / 2] ;

Apply A to $\mathcal{F}_{\alpha_r}{}^*$;

if $\mathcal{F}_{\alpha_r}{}^*$ is consistent

then upper ← r – 1

else lower ← r

end {while} {Incons($\mathcal{F}$) = α_r}

End

Clearly, this algorithm computes effectively Incons($\mathcal{F}$) (following Proposition 7) and its complexity order is Comp(A') = $[1 + \log_2 m]$ Comp(A) ≤ $[1 + \log_2 n]$ Comp(A), i.e. (I) comes down to solve at most $[1 + \log_2 n]$ satisfiability problems in propositional classical logic. Hence (I) is a NP-complete problem, like (S). □

Proposition 9 *(deduction theorem)* **:**

$$\mathcal{F} \cup \{(\varphi\ \ 1)\} \vDash (\psi\ \ \alpha) \quad \text{iff} \quad \mathcal{F} \vDash (\varphi \to \psi\ \ \alpha)$$

Proof :

(⇒) $\mathcal{F} \cup \{(\varphi\ \ 1)\} \vDash (\psi\ \ \alpha)$

⇒ $N_{\mathcal{F} \cup \{(\varphi\ 1)\}}(\psi) \ge \alpha$ (by Corollary 2)

⇒ $\inf\{1 - \pi_{\mathcal{F} \cup \{(\varphi\ 1)\}}(\omega), \omega \vDash \neg\psi\} \ge \alpha$

⇒ $\forall\ \omega \vDash \varphi \wedge \neg\psi, \pi_{\mathcal{F}}(\omega) \le 1 - \alpha$, since $\pi_{\mathcal{F} \cup \{(\varphi\ 1)\}}(\omega) = \pi_{\mathcal{F}}(\omega)$ for any $\omega \vDash \varphi$

$\Rightarrow$ $N_{\mathcal{F}}(\varphi \to \psi) \geq \alpha$ since $\varphi \to \psi$ is equivalent to $\neg(\varphi \wedge \neg\psi)$

$\Rightarrow \mathcal{F} \vDash (\varphi \to \psi \ \ \alpha)$ (again Corollary 2).

$(\Leftarrow)$ $\mathcal{F} \vDash (\varphi \to \psi \ \ \alpha)$

$\Rightarrow$ $\forall\, \pi \vDash \mathcal{F}$, $N(\varphi \to \psi) \geq \alpha$

$\Rightarrow$ $\forall\, \pi \vDash \mathcal{F}$, $N(\varphi) = 1$ implies $N(\psi) \geq \alpha$ since $N(\psi) \geq \min\,[N(\varphi), N(\varphi \to \psi)]$

$\Rightarrow$ $\forall\, \pi \vDash \mathcal{F} \cup \{(\varphi \ \ 1)\}$, $N(\psi) \geq \alpha$

$\Rightarrow$ $\mathcal{F} \cup \{(\varphi \ \ 1)\} \vDash (\psi \ \ \alpha)$. □

Proposition 10 *(refutation theorem)* **:**

$$\mathcal{F} \vDash (\varphi \ \ \alpha) \ \text{ iff } \ \mathcal{F} \cup \{(\neg\varphi \ \ 1)\} \vDash (\bot \ \ \alpha)$$

or equivalently :

$$\text{Val}(\varphi,\mathcal{F}) = \text{Incons}(\mathcal{F} \cup \{(\neg\varphi \ \ 1)\})$$

Proof : let us just apply Proposition 9, replacing φ by $\neg\varphi$ and ψ by $\bot$:

$\mathcal{F} \cup (\neg\varphi \ \ 1) \vDash (\bot \ \ \alpha)$ iff $\mathcal{F} \vDash (\neg\varphi \to \bot \ \ \alpha)$, i.e. $\mathcal{F} \cup \{(\neg\varphi \ \ 1)\} \vDash (\bot \ \ \alpha)$ iff $\mathcal{F} \vDash (\varphi \ \ \alpha)$. □

Proposition 11 :

Let $\mathcal{F}$ be a possibilistic knowledge base and $(\varphi \ \ \alpha)$ a necessity-valued formula. Then

$$\mathcal{F} \vDash (\varphi\ \alpha) \text{ if and only if } \mathcal{F}_{\alpha} \vDash (\varphi\ \alpha)$$

Proof : According to Proposition 10, $\mathcal{F} \vDash (\varphi\ \alpha)$ is equivalent to $\text{Incons}(\mathcal{F} \cup \{(\neg\varphi \ \ 1)\}) \geq \alpha$; then, from Proposition 6 we get that $\text{Incons}\ (\mathcal{F}_{\alpha} \cup \{(\neg\varphi \ \ 1)\}) \geq \alpha$, i.e. $\mathcal{F}_{\alpha} \vDash (\varphi\ \alpha)$ by applying again Proposition 10. The converse is obvious because $\mathcal{F}_{\alpha} \subseteq \mathcal{F}$. □

Proposition 12 :

The proposed formal system is sound and complete with respect to the inconsistency-tolerant semantics of possibilistic logic, i.e. for any set of possibilistic formulas $\mathcal{F}$ we have

$$\mathcal{F} \vDash (\varphi \ \ \alpha) \ \text{ if and only if } \ \mathcal{F} \vdash (\varphi \ \ \alpha)$$

where $\mathcal{F} \vdash (\varphi \ \ \alpha)$ means : "$(\varphi \ \ \alpha)$ can be derived from $\mathcal{F}$ in the system given in Section 6".

Proof : We need the following lemma :

Lemma 12.1 :
Let $\mathcal{F}$ be a set of necessity-valued formulas and $(\varphi\ \alpha)$ a necessity-valued formula. Then

$$\mathcal{F} \models (\varphi\ \alpha) \text{ if and only if } \mathcal{F}_\alpha^* \models \varphi \text{ in the classical sense}$$

<u>Proof</u> :

$\mathcal{F} \models (\varphi\ \alpha) \Leftrightarrow \mathcal{F}_\alpha \models (\varphi\ \alpha)$ (Proposition 11)

$\Leftrightarrow$ $\text{Incons}[\mathcal{F}_\alpha \cup \{(\neg\varphi\ 1)\}] \geq \alpha$ (Proposition 10)

$\Leftrightarrow$ $\mathcal{F}_\alpha \cup \{\neg\varphi\}$ is inconsistent in the classical sense (Proposition 7 (i))

$\Leftrightarrow$ $\mathcal{F}_\alpha^* \models \varphi$ (classical entailment property). □

<u>Proof of Proposition 12</u> :
$(\Leftarrow)$ (soundness)
By induction on the derivation steps, the proof is straightforward.
$(\Rightarrow)$ (completeness)
Using lemma 12.1, $\mathcal{F} \models (\psi\ \alpha)$ is equivalent to $\mathcal{F}_\alpha^* \models \psi$. Then, since the formal system formed by the non-weighted part of the axioms schemata and of the inference rules (except (S) whose non-valued part is trivial) is well-known to be a sound and complete Hilbert formal system for classical first-order logic, then there exists a proof of ψ from $\mathcal{F}_\alpha^*$ by this classical formal system. Then, considering again the valuations, the proof obtained by the previous one is a proof of $(\psi\ \gamma)$ from $\mathcal{F}_\alpha$ by the given formal system, with $\gamma \geq \alpha$. Lastly, using (S) we obtain a proof of $(\psi\ \alpha)$ from $\mathcal{F}_\alpha$, and a fortiori from $\mathcal{F}$. □

Proposition 13 : $\text{Incons}(\mathcal{C}) = \text{Incons}(\mathcal{F})$
<u>Proof</u> :

$\text{Incons}(\mathcal{F}) = \alpha$

$\Leftrightarrow$ $\mathcal{F}_\alpha^*$ is inconsistent and $\mathcal{F}_{\bar\alpha}^*$ is consistent (from Proposition 7)

$\Leftrightarrow$ $\mathcal{C}(\mathcal{F}_\alpha^*)$ is inconsistent and $\mathcal{C}(\mathcal{F}_{\bar\alpha}^*)$ is consistent, where $\mathcal{C}(\mathcal{F}_\alpha)^*$ and $\mathcal{C}(\mathcal{F}_{\bar\alpha}^*)$ are respectively clausal forms of $\mathcal{F}_\alpha^*$ et $\mathcal{F}_{\bar\alpha}^*$ (from the equivalence of the inconsistencies of a formula and that of its clausal forms in classical logic)

$\Leftrightarrow$ $\mathcal{C}(\mathcal{F}_\alpha)^*$ is inconsistent and $\mathcal{C}(\mathcal{F}_{\bar\alpha})^*$ is consistent

$\Leftrightarrow$ $\mathcal{C}_\alpha^*$ is inconsistent and $\mathcal{C}_{\bar\alpha}^*$ is consistent

$\Leftrightarrow$ $\text{Incons}(\mathcal{C}) = \alpha = \text{Incons}(\mathcal{F})$ (again from Proposition 7). □

Proposition 14 (soundness of rule (R)) :
Let $\mathcal{C}$ be a set of possibilistic clauses, and C a possibilistic clause obtained by a finite number of successive applications of (R) to $\mathcal{C}$; then $\mathcal{C} \models$ C.

Proof : Let $C_1 = (c_1\ \alpha_1)$, $C_2 = (c_2\ \alpha_2)$, the application of rule R yields $C' = (R(c_1,c_2)\ \min(\alpha_1,\alpha_2))$. Then $\forall\pi$ satisfying $C_1 \wedge C_2$ we have $N(c_1) \geq \alpha_1$ and $N(c_2) \geq \alpha_2$, and then $N(c_1 \wedge c_2) = \min(N(c_1),N(c_2)) \geq \min(\alpha_1,\alpha_2)$ and finally $N(R(c_1,c_2)) \geq N(c_1 \wedge c_2) \geq \min(\alpha_1,\alpha_2)$ (since $c_1 \wedge c_2 \models R(c_1,c_2)$). Thus rule R is sound. Then by induction, any possibilistic clause obtained by a finite number of successive applications of (R) to $\mathcal{C}$ is a logical consequence of $\mathcal{C}$. □

Proposition 15 (soundness and completeness of refutation by resolution in PL1) ***:***
Let $\mathcal{F}$ be a set of *necessity-valued* first-order formulas and $\mathcal{C}$ the set of necessity-valued clauses obtained from $\mathcal{F}$; then the valuation of the optimal refutation by resolution from $\mathcal{C}$ is the inconsistency degree of $\mathcal{F}$.

Proof : It is very similar to the proof of Proposition 12. Using lemma 12.1 applied with ψ being the contradiction $\bot$, $\mathcal{C} \models (\bot\ \alpha)$ if and only if $\mathcal{C}_\alpha{}^*$ is inconsistent in the classical sense. Then, the resolution principle being complete for refutation in first-order classical logic, the inconsistency of $\mathcal{C}_\alpha{}^*$ implies that there exists a refutation by resolution from the clauses of $\mathcal{C}_\alpha{}^*$. Considering again the valuations in this refutation, we obtain a refutation from $\mathcal{C}_\alpha$ (and a fortiori from $\mathcal{C}$) whose valuation is $\geq \alpha$ (since only clauses of $\mathcal{C}_\alpha$, i.e. with a valuation $\geq \alpha$, are used).
Thus, we have proved that the valuation of the optimal refutation by resolution from $\mathcal{C}$ is greater or equal to Incons ($\mathcal{C}$); the soundness of the possibilistic resolution rule (Proposition 14) forbids this valuation to be strictly greater than Incons($\mathcal{C}$); thus it is equal to Incons($\mathcal{C}$), and also to Incons($\mathcal{F}$), according to Proposition 13. □

Proofs of the main results from Section 4.1 (possibilistic logic PL2)

Proposition 5' *(Lang et al., 1991)* ***:***
The inconsistency degree of an inconsistent possibilistic knowledge base $\mathcal{F}$ is the smallest weight of possibilistic formulas in any strongly minimal inconsistent subset $\mathcal{F}'$ of $\mathcal{F}$. Especially, if Incons($\mathcal{F}$) = $(\Pi\ \beta)$ $(\beta > 0)$ then there is a unique possibility-valued formula in $\mathcal{F}'$ of the form $(\varphi\ (\Pi\ \beta))$.

Proof : Let us consider the case where Incons($\mathcal{F}$) = $(\Pi\ \beta)$. It is obvious that any strongly minimal inconsistent subset $\mathcal{F}'$ contains at least one possibility-valued formula. Let us show that it is unique. Let $\mathcal{F}' = \{(\varphi_i\ (N\ \alpha_i)),\ i = 1,m\} \cup \{(\varphi_j\ (\Pi\ \beta_j)),\ j = m + 1,\ n\}$. The inconsistency degree is now of the form :

$\beta = \inf \hat{\pi}(\omega_\perp)$

under the constraints

$$\begin{cases} N(\varphi_i) \geq \alpha_i, \ i = 1,m \\ \max(\hat{\pi}(\omega_\perp), \Pi(\varphi_j)) \geq \beta_j, \ j = m+1,\ldots, n \end{cases}$$

Since $\beta > 0$, $\forall \ \hat{\pi} \hat{\models} \mathcal{F}'$, $\exists k$ such that $\Pi(\varphi_k) < \beta_k$ (otherwise $\mathcal{F}$ would not be inconsistent), and Incons($\mathcal{F}'$) = β_k for some β_k. In order to minimize this value, let us maximize $\hat{\pi}$ over Ω, so as to make the set $\{j \mid \Pi(\varphi_j) < \beta_j\}$ as small as possible. Let $\hat{\pi}_0$ be defined by $\hat{\pi}_0(\omega) = \min\{1 - \alpha_i, \omega \models \neg\varphi_i, \omega \neq \omega_\perp\}$. Clearly, $\hat{\pi}_0 \models \{(\varphi_i \ (N \ \alpha_i)), \ i = 1,m\}$, and $\exists \ \omega \in \Omega$, $\hat{\pi}_0(\omega) = 1$ (since there is no inconsistency among the N-valued formulas), and $\forall \pi'$, $\hat{\pi}' \hat{\models} \{(\varphi_i \ (N \ \alpha_i)), \ i = 1,m\} \Rightarrow \forall \ \omega \in \Omega$, $\hat{\pi}'(\omega) \leq \hat{\pi}_0(\omega)$. The only parameter left is $\hat{\pi}_0(\omega_\perp)$. Let $\beta_k = \max\{\beta_j \mid \Pi_0(\varphi_j) < \beta_j\}$ where Π_0 is based on $\hat{\pi}_0$. Note that the maximality of $\hat{\pi}_0$ over Ω minimizes the number of $(\varphi_j \ (\Pi_0 \ \beta_j))$ with $\Pi_0(\varphi_j) < \beta_j$.
For simplicity assume $\beta_k = \beta_{m+1}$. Let us put $\hat{\pi}_0(\omega_\perp) = \beta_{m+1}$. Then clearly, $\hat{\pi}_0 \hat{\models} \mathcal{F}'$, since $\forall j$, $\max(\beta_{m+1}, \Pi_0(\varphi_j)) \geq \beta_j$ by construction. Thus Incons($\mathcal{F}'$) $\leq \beta_{m+1}$. Now, $\forall \varphi_j$ such that $\Pi_0(\varphi_j) \geq \beta_j$, Incons($\mathcal{F}' - \{(\varphi \ (\Pi \ \beta_j))\}$) = Incons($\mathcal{F}'$) ; the same thing is true for all φ_j such that $\Pi_0(\varphi_j) < \beta_j < \beta_{m+1}$. If there is another formula $(\varphi_i \ (\Pi \ \beta_i))$ such that $\beta_i = \beta_{m+1}$, dropping one of these formulas still requires $\hat{\pi}_0(\omega_\perp) = \beta_{m+1}$ for ensuring $\hat{\pi}_0 \hat{\models} \mathcal{F}'$. Hence, if $\mathcal{F}'$ is really minimal it contains only one possibility-valued formula, i.e. $(\varphi_{m+1} \ (\Pi \ \beta_{m+1}))$ and Incons($\mathcal{F}'$) = $(\Pi \ \beta_{m+1})$. □

Proof of Proposition 7' is done after the proof of Proposition 12'.

Proof of Proposition 12' :

The proposed formal system for possibilistic logic involving possibility- and necessity-qualified formulas is sound and complete with respect to the inconsistency-tolerant semantics of possibilistic logic, i.e. for any set of possibilistic formulas $\mathcal{F}$ we have

$$\mathcal{F} \hat{\models} (\psi \ w) \text{ if and only if } \mathcal{F} \vdash (\psi \ w)$$

where $\mathcal{F} \vdash (\psi \ w)$ means : "$(\psi \ w)$ can be derived from $\mathcal{F}$ in the above system".

The restriction of this proposition to PL1 has already been proved (Proposition 12). In order to extend the result to PL2, we first prove the following lemma :

Lemma 12'.2
Let $\mathcal{F}$ be a set of possibilistic formulas and $(\psi\ (\Pi\ \alpha))$ with $\alpha > 0$ a possibility-valued formula such that $\mathcal{F} \hat{\models} (\psi\ (\Pi\ \alpha))$ with α maximal (i.e. $\forall\ w > (\Pi\ \alpha)$, we do not have $\mathcal{F} \hat{\models} (\psi\ w)$). Then there exists a possibility-valued formula $(\varphi_k\ (\Pi\ \alpha))$ in $\mathcal{F}$ such that

(i) $\mathcal{F}_N \cup \{(\varphi_k\ (\Pi\ \alpha))\} \hat{\models} (\psi\ (\Pi\ \alpha))$
(ii) $\mathcal{F}_N \hat{\models} (\neg\varphi_k \vee \psi\ (N\ \beta))$ with $\beta > 1 - \alpha$.

Proof : Let $\mathcal{G} = \mathcal{F} \cup \{(\neg\psi\ (N\ 1))\}$. Then, according to the generalisation of Proposition 10 to PL2 (its proof being in (Lang, Dubois, Prade 1991)) and to the maximality of α, we get $\mathrm{Incons}(\mathcal{G}) = (\Pi\ \alpha)$. Then, using Proposition 5' ; $\mathcal{G}'$ being with a subset of $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}$, we have $(\Pi\ \alpha) = \mathrm{Incons}(\mathcal{G}') \leq \mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}) \leq \mathrm{Incons}(\mathcal{F} \cup \{(\psi\ (N\ 1))\}) = (\Pi\ \alpha)$, i.e. $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}) = (\Pi\ \alpha)$. Using again Proposition 5', we get $\mathcal{F}_N \cup \{(\varphi_k\ (\Pi\ \alpha))\} \hat{\models} (\psi\ (\Pi\ \alpha))$, which proves (i).
Let us prove (ii). (i) is equivalent to $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}) = (\Pi\ \alpha)$. Let us prove first that for any possibility distribution $\hat{\pi}$ on Ω satisfying $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ we have $\Pi(\varphi_k) < \alpha$; indeed, let us suppose that there is a possibility distribution $\hat{\pi}_0$ satisfying $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ such that $\Pi_0(\varphi_k) \geq \alpha$. If $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ were inconsistent, then according to Proposition 5, it would be the case that $\mathcal{F}_N \hat{\models} (\psi\ (N\ \gamma))$ with $\gamma > 0$, which would contradict the assumption $\mathcal{F} \hat{\models} (\psi\ (\Pi\ \alpha))$ with α being maximal. So $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ is consistent, i.e. the least specific possibility distribution π^* on Ω associated to $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ according to the corollory of lemma 1, is normalized. According to lemma 1, π_0 satisfies $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ is equivalent to $\pi_0 \leq \pi^*$ (where π_0 is the restriction of $\hat{\pi}_0$ to Ω). Thus, $\Pi^*(\varphi_k) \geq \Pi_0(\varphi_k) \geq \alpha$, i.e. if we extend π_0 to $\Omega_\perp$ by $\hat{\pi}_0(\omega_\perp) = 0$, then we have $\hat{\pi}_0 \hat{\models} \mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}$, and then $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}) < (\Pi\ \alpha)$, which contradicts $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (\Pi\ \alpha))\}) = (\Pi\ \alpha)$. So, every possibility distribution $\hat{\pi}$ on $\Omega_\perp$ satisfying $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\}$ verifies $\Pi(\varphi_k) < \alpha$, i.e. $N(\neg\varphi_k) > 1 - \alpha$, which means that $\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \hat{\models} (\neg\varphi_k\ (N\ \beta))$ with $\beta > 1 - \alpha$. Using Proposition 10, this is equivalent to $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi\ (N\ 1))\} \cup \{(\varphi_k\ (N\ 1))\}) \geq (N\ \beta)$, i.e. to $\mathrm{Incons}(\mathcal{F}_N \cup \{(\neg\psi \wedge \varphi_k\ (N\ 1))\}) \geq (N\ \beta)$; using again Proposition 10, it gives $\mathcal{F}_N \hat{\models} (\psi \vee \neg\varphi_k\ (N\ \beta))$ with $\beta > 1 - \alpha$, which proves (ii). □

Proof of Proposition 12' :
According to the above lemma, $\mathcal{F}_N \hat{\models} (\neg\varphi_k \vee \psi\ (N\ \beta))$ with $\beta > 1 - \alpha$; then, using Proposition 12, there is a deduction of $(\neg\varphi_k \vee \psi\ (N\ \beta))$ from $\mathcal{F}_N$ (a fortiori from $\mathcal{F}$), using the necessity-valued part of the given formal system. Lastly, using (GMP) from $(\neg\varphi_k \vee \psi\ (N\ \beta))$ and $(\varphi_k\ (\Pi\ \alpha))$ we infer $(\psi\ (\Pi\ \alpha))$. Hence we have found a deduction of $(\psi\ (\Pi\ \alpha))$

from $\mathcal{F}$ using the given formal system. The completeness follows. The soundness is again an obvious matter. □

Proposition 7' :

Incons($\mathcal{F}$) = ($\prod$ α) if and only if $\mathcal{F}_N$* is consistent and α = max{β, ∃ (φ ($\prod$ β)) ∈ $\mathcal{F}_{\prod}$, $\mathcal{F}_N$* ∪ {φ} inconsistent}.

<u>Proof</u> : Firstly, Incons($\mathcal{F}$) = ($\prod$ α) means that there is no ε > 0 such that $\mathcal{F} \hat{\models}$ (⊥ (N ε)) and thus that the necessity-valued part $\mathcal{F}_N$ of $\mathcal{F}$ is consistent, which entails (by Proposition 7) the consistency of $\mathcal{F}_N$*. Secondly, according to Proposition 12', Incons ($\mathcal{F}$) = ($\prod$ α) entails that there is a formal deduction of (⊥ ($\prod$ α)) in the formal system given previously. As seen in the proof of Proposition 12', only one deduction step uses a possibility-valued clause (otherwise the weight of the deduced formula would be ($\prod$ 0)), whose weight is equal to the weight attached to ⊥ (i.e., ($\prod$ α))) at the last step of the deduction. Then, this is also a deduction of (⊥ ($\prod$ α)) from $\mathcal{F}_N$ ∪ {(φ ($\prod$ α))}, where (φ ($\prod$ α)) is that possibility-valued formula. Then, considering the (classical) deduction obtained from this one by ignoring the valuations, we get a deduction of ⊥ from $\mathcal{F}_N$* ∪ {φ} ; hence, $\mathcal{F}_N$* ∪ {φ} is inconsistent. Now, suppose that there exists a formula (ψ ($\prod$ γ)) with γ > α such that $\mathcal{F}_N$* ∪ {ψ} be inconsistent:then it would be the case that $\mathcal{F}_N$ ∪ {(ψ ($\prod$ γ))} ⊢ (⊥ ($\prod$ γ)), i.e. $\mathcal{F}_N$ ∪ {(ψ ($\prod$ γ))} $\hat{\models}$ (⊥ ($\prod$ γ)) by Proposition 12', which would contradict the assumption Incons($\mathcal{F}$) = ($\prod$ α). Hence α = max{β, ∃ (φ ($\prod$ β)) ∈ $\mathcal{F}_{\prod}$, $\mathcal{F}_N$* ∪ {φ} inconsistent}. □