Mathematical Social Sciences 57 (2009) 304–324

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Sequential composition of voting rules in multi-issue domains ☆

Jérôme Lang [a,*], Lirong Xia [b]

[a] LAMSADE, Université Paris-Dauphine, France

[b] Department of Computer Science, Duke University, USA

ARTICLE INFO

Article history:
Received 29 May 2007
Received in revised form
29 October 2008
Accepted 10 December 2008
Available online 7 January 2009

JEL classification:
C60
D71

Keywords:
Voting
Multiple elections
Preferential independence
CP-networks

ABSTRACT

In many real-world group decision making problems, the set of alternatives is a Cartesian product of finite value domains for each of a given set of variables (or issues). Dealing with such domains leads to the following well-known dilemma: either ask the voters to vote separately on each issue, which may lead to the so-called multiple election paradoxes as soon as voters' preferences are not separable; or allow voters to express their full preferences on the set of all combinations of values, which is practically impossible as soon as the number of issues and/or the size of the domains are more than a few units. We try to reconciliate both views and find a middle way, by relaxing the extremely demanding separability restriction into this much more reasonable one: there exists a linear order $\mathbf{x}_1 > \cdots > \mathbf{x}_p$ on the set of issues such that for each voter, every issue $\mathbf{x}_i$ is preferentially independent of $\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p$ given $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$. This leads us to define a family of sequential voting rules, defined as the sequential composition of local voting rules. These rules relate to the setting of conditional preference networks (CP-nets) recently developed in the Artificial Intelligence literature. Lastly, we study in detail how these sequential rules inherit, or do not inherit, the properties of their local components.

☆ Parts of this paper have appeared in the following conference articles: Lang, J., 2007. Voting and aggregation on combinatorial domains with structured preferences, In: Proceedings of the Twentieth Joint International Conference on Artificial Intelligence, IJCAI'07, pp. 1366–1371; Xia, L., Lang, J., Ying, M., 2007. Sequential voting and multiple election paradoxes, In: Proceedings of the 11th Conference on the Theoretical Aspects of Rationality and Knowledge (TARK 2007).

* Corresponding author.
E-mail addresses: lang@irit.fr (J. Lang), xialirong@gmail.com (L. Xia).

0165-4896/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.mathsocsci.2008.12.010

1. Introduction

In many contexts, a group of voters has to make a common decision on several possibly related issues, such as in multiple referenda, or voting for committees (the issues then are the positions to be filled — see Benoit and Kornhauser (1991)). As soon as voters have preferential dependencies between issues, it is generally a bad idea to decompose a voting problem on p issues into a set of p smaller problems, each one bearing on a single issue: "multiple election paradoxes" (or "paradoxes of multiple referenda") then arise.

Since the number of possible alternatives is then exponential in the number of variables, it is not reasonable to ask voters to rank all alternatives explicitly. Consider for example that voters have to agree on a common menu to be composed of a first course, a main course, a dessert and a wine, with a choice of 6 items for each. This makes 6^4 candidates. This would not be a problem if each of the four items to be chosen were mutually independent: in this case, this vote over a set of 6^4 candidates would come down to four independent votes over 6 candidates each, and any standard voting rule could be applied without difficulty. Things become more complicated if voters express dependencies between items, such as "if the main course is meat then I prefer red wine, otherwise I prefer white wine".

As soon as voters have preferential dependencies between issues, it is generally a bad idea to decompose a voting problem on p issues into a set of p smaller problems, each one bearing on a single issue: "multiple election paradoxes" (or "paradoxes of multiple referenda") then arise. Such paradoxes have been studied in several papers, with two slightly different views. In Brams et al. (1998) and Scarsini (1998), voters can vote only Y or N on each issue; the paradox occurs when the set of propositions that wins, when votes are aggregated separately for each proposition, receives the fewest votes when votes are aggregated by combination: for instance, suppose there are 3 propositions A, B, C and three voters voting respectively for $AB\bar{C}$, $A\bar{B}C$ and $\bar{A}BC$. Propositionwise aggregation leads to ABC, whereas ABC receives support from zero voter. As argued in Saari and Sieberg (2001), "these paradoxical behaviors arise because the separation of inputs into disconnected parts can cause a concomitant loss of available and crucial information". The source of these paradoxes is the loss of available information occurring when separating the input profile into a set of profiles—one profile per each single issue. Both Brams et al. (1998) and Saari and Sieberg (2001) argue that the only way of avoiding the paradox would consist in voting for combinations of values ("bundle voting"), but they stress its practical difficulty caused by the too large number of possible bundles.

The paradox studied in Lacy and Niou (2000) is a little bit different. They show that voting issue-by-issue is feasible (to some extent) when preferences are separable, and that it generally fails when they are not (a voter's preferences are separable if her preferences on an issue does not depend on the choice to be made for other issues).[1] However, separability is an extremely strong assumption that is unlikely to be met in practice. Furthermore, even when preferences are separable, some paradoxes still arise, such as the choice of a Pareto-dominated outcome (Özkal-Sanver and Sanver, 2006; Benoit and Kornhauser, 2006).

Example 1. A common decision has to be made about whether or not to build a new swimming pool (S or $\bar{S}$) and a new tennis court (T or $\bar{T}$). Assume that the preferences of voters 1 and 2 are $S\bar{T} \succ \bar{S}T \succ \bar{S}\bar{T} \succ ST$, those of voters 3 and 4 are $\bar{S}T \succ S\bar{T} \succ \bar{S}\bar{T} \succ ST$ and those of voter 5 are $ST \succ S\bar{T} \succ \bar{S}T \succ \bar{S}\bar{T}$.

The first problem with Example 1 is that voters 1 to 4 feel ill at ease when asked to report their projected preference on $\{S, \bar{S}\}$ and $\{T, \bar{T}\}$. Only voter 5 knows that whatever the other voters' preferences about $\{S, \bar{S}\}$ (resp. $\{T, \bar{T}\}$), she can vote for T (resp. S) without any risk of experiencing regret (this is called *simple voting* in Benoit and Kornhauser (1991)). The analysis of the paradox in

[1] When the value domains are intervals of real numbers, and preferences continuous, separability is essentially equivalent to additivity (Debreu, 1960; Gorman, 1968). Later work (Hodge, 2002; Bradley et al., 2005; Hodge and Ter Haar, 2008) showed that separability is stronger than additivity in finite value domains, and explored the differences.

Lacy and Niou (2000) considers that voters report their preferences optimistically (thus voters 1–2 report a preference for S over $\bar{S}$), but this assumption, even if it has been justified by experimental studies (see Plott and Levine (1978)), remains arbitrary, and would not necessarily carry on to more complex situations such as a voter with the following preference relation: $ABC \succ \bar{A}\bar{B}\bar{C} \succ \bar{A}\bar{B}C \succ \bar{A}B\bar{C} \succ \bar{A}BC \succ A\bar{B}\bar{C} \succ A\bar{B}C \succ AB\bar{C}$: only a very optimistic voter would report a preference for A (except, of course, if some prior beliefs about the others' preferences make him believe that the common decision about B and C will be BC.)

The second problem (the paradox itself) is that under this assumption that voters report optimistic preferences, the outcome in Example 1 will be ST, which is *the worst outcome for all but one voter*, and a fortiori, is a Condorcet loser. Lacy and Niou (2000) and Benoit and Kornhauser (2006) give more complicated examples, with three issues, leading to an even worse paradox where the outcome is ranked last by everyone.

The main question is now, how can these paradoxes be avoided? Reformulating the question in a more constructive way, how should a vote on related issues be conducted? We argue that we have to choose one of the following two ways, each of which has some specific pitfalls: either work at the global level and vote for combinations of values, or work at the local level and vote separately on each issue, sequentially or simultaneously.[2]

The "global way" consists in giving up decomposing the global vote into local votes and *voting for combinations of values*. This solution is supported by Brams et al. (1997, 1998). There is some ambiguity on how the process should be conducted, thus leading to three possible methods:

1. ask voters to report their entire preference relation on the set of alternatives, and then apply an usual voting rule such as Borda.
2. ask voters to report only a small part of their preference relation and apply a voting rule that needs this information only, such as plurality;
3. limit the number of possible combinations that voters may vote for.

From a theoretical point of view, Solution 1 works: each agent specifies his preference relation *in extenso* and then any fixed voting rule is applied to the obtained profile, with no risk of a paradoxical outcome. However, as noticed in Brams et al. (1997), this solution is practically unfeasible if the number of issues is more than a small number (say, 3): the exponential number of alternatives makes it unreasonable to ask voters to rank all alternatives explicitly. In other words, implementing such a voting rule on a multi-issue domain needs an *exponential protocol*. Clearly, exponentially long protocols are not acceptable. Therefore, as soon as the number of issues is not very small, this solution is ruled out by *communication complexity* considerations.

Solution 2 requires little communication, but it is its only merit. Voting rules that are implementable by a cheap protocol make use of a very small part of the voters' preferences: if the protocol is required to have a polynomial communication complexity, then the voting rule it implements uses at most a logarithmic part of the profile. Such rules do exist: not only plurality and veto, but more generally all rules that require, for instance, the K top candidates of each voter, where K is a fixed integer. However, when the number of issues grows, these rules could give extremely bad results. For instance, using plurality when the number of issues is significant and the number of voters is small could well result in a situation where no outcome gets more than one vote, in which case plurality would give an extremely poor result.

Solution 3, sketched in Brams et al. (1997), presents the chairperson with a very problematic choice ("How to package combinations (e.g., of different propositions on a referendum, different amendments to a bill) so as not to swamp the voter with inordinately many choices – some perhaps inconsistent – is a practical problem that will not be easy to solve".) This may be feasible when issues can clearly be packaged into groups of issues such that two groups are clearly independent, but this favorable situation is far from being a general rule.

[2] In the context of assembly elections, these two families of voting rules are called assembly-based and seat-based, respectively (Benoit and Kornhauser, 1991, 2006).

The "local" way, supported by Lacy and Niou (2000) for multiple referenda, consists in sticking to a vote issue-by-issue, the outcome of the vote on one issue being revealed before the vote on other issues. They show that sequential voting (with whichever agenda) allows for escaping the worst versions of the multiple election paradoxes, namely, it avoids a Condorcet loser to be elected. However, this method still has three major drawbacks. First, the voters may still feel ill at ease when reporting their preference on an issue, when this preference depends on the value of issues not decided yet. Second, the study is based on the assumption that voters will behave optimistically, by reporting the projection of their preferred outcome, which is debatable except in some specific cases. Third, even if a sequential vote avoids the final outcome to be a Condorcet loser, the paradox remains to a large extent, as can be seen in the following example:

Example 2. We have three issues A, B, C and $2M + 1$ voters.

M voters: $AB\bar{C} \succ \bar{A}\bar{B}\bar{C} \succ \cdots \succ A\bar{B}C \succ ABC$
M voters: $A\bar{B}C \succ \bar{A}\bar{B}\bar{C} \succ \cdots \succ ABC \succ AB\bar{C}$
1 voter: $\bar{A}BC \succ \bar{A}\bar{B}\bar{C} \succ \bar{A}B\bar{C} \succ \bar{A}\bar{B}C \succ ABC \succ AB\bar{C} \succ A\bar{B}\bar{C} \succ A\bar{B}C$.

In Example 2, having voters decide first on A, then to B and then to C, and assuming they behave optimistically, will lead to ABC, which is (a) a "nearly-Condorcet loser" (it is dominated by all candidates except one) and (b) Pareto-dominated by half of the outcomes. (More acute paradoxes can be found with more issues.) Actually, the reason why the sequential process avoids a Condorcet loser to be elected is only because the *last* vote is made with a full knowledge of the values of other issues, thus this result loses his significance when the number of issues becomes bigger.

There is a well-known restriction on voter preferences that allows for such paradoxes to be avoided, that is, when all voters have *separable* preferences across the outcomes of the issues. Then, a voter's preferences on the values of an issue is independent from the values of other issues, and the elicitation process can be performed safely issue-by-issue (and even without needing to resort to sequentiality). Under the separability assumption, voting separately on each issue (either sequentially or simultaneously) enjoys good properties, including the election of a Condorcet winners when there is one. However, the separability restriction is very demanding, and unlikely to be met in practice, especially because separable preferences constitute a very tiny proportion of possible preferences on multiple issues (see Hodge (2002)).

The question is now, can this extreme separability assumption be relaxed without hampering the nice properties of sequential voting? As it stands, the answer is positive, as the method can be safely applied to far many profiles than separable profiles. Informally, the condition should be that each time a voter is asked to report his preferences on a single issue or a small set of issues, these preferences do not depend on the values of the issues *that have not been decided yet*.

Formally, this can be expressed as the following condition: there is a linear order $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ on the set of issues such that for every voter V and every j, the preferences of V on $\mathbf{x}_j$ are preferentially independent from $\mathbf{x}_{j+1}, \ldots, \mathbf{x}_p$ given $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}$. If this property is satisfied, then a simple protocol can be implemented: the voters' preferences about issue $\mathbf{x}_1$ are elicited; then a "local" voting rule is applied so as to make a decision on the value of $\mathbf{x}_1$; then this chosen value of $\mathbf{x}_1$ is communicated to the voters, who then report their preferences on the values of $\mathbf{x}_2$ given the fixed value of $\mathbf{x}_1$, and so on. Such preference profiles are called *$\mathcal{O}$-legal* and abbreviated as *legal* for $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ in this paper. This protocol generalizes to clusters of issues $I_1, \ldots, I_n$ where for each voter and each i, I_i is preferentially independent of $I_{j+1}, \ldots, I_m$ given $I_1, \ldots, I_{i-1}$, where $\{I_1, \ldots, I_m\}$ forms a partition of the set I of issues.

This domain restriction (*$\mathcal{O}$-legality*) and the resulting sequential voting rules and correspondences that are then applicable are defined in Section 4. In Section 5 we focus on the natural notion of *sequential Condorcet winner*. In Section 6 we study in detail the properties of these sequential compositions of voting rules, by relating them to the corresponding properties of the local voting rules. It turns out that while many properties expectedly transfer from local rules to their sequential composition, this is not the case for two important properties, namely neutrality and efficiency. Related and further issues are discussed In Section 7.

2. Preferences on multi-issue domains

Let $I = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of *issues*. For each $\mathbf{x}_i \in I$, D_i is the finite *value domain* of $\mathbf{x}_i$. Without loss of generality, we assume $|D_i| \geq 2$ for every i. An issue $\mathbf{x}_i$ is *binary* if $|D_i| = 2$; in this case, we use the following two notations: $D_i = \{x_i, \overline{x_i}\}$ and $D_i = \{1_i, 0_i\}$. (Note the difference between the issue $\mathbf{x}_i$ and the value x_i.) If $X = \{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_m}\} \subseteq I$, with $i_1 < \cdots < i_p$, then D_X denotes $D_{i_1} \times \cdots \times D_{i_m}$.

$\mathcal{X} = D_1 \times \cdots \times D_p$ is the set of all *alternatives* (or *candidates*). Elements of $\mathcal{X}$ are denoted by vectors $\vec{x}, \vec{x}'$ etc. and represented by concatenating the values of the issues: for instance, if $I = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $x_1\overline{x_2}x_3$ assigns x_1 to $\mathbf{x}_1$, $\overline{x_2}$ to $\mathbf{x}_2$ and x_3 to $\mathbf{x}_3$. We allow concatenations of vectors of values: for instance, let $I = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$, $Y = \{\mathbf{x}_1, \mathbf{x}_2\}$, $Z = \{\mathbf{x}_3, \mathbf{x}_4\}$, $\vec{y} = x_1\overline{x_2}$, $\vec{z} = \overline{x_3}x_4$, then $\vec{y}.\vec{z}.\overline{x_5}$ denotes the alternative $x_1\overline{x_2}\,\overline{x_3}x_4\overline{x_5}$.

A *(strict) preference relation* on $\mathcal{X}$ is a strict order (an irreflexive, asymmetric and transitive binary relation). A *vote* V is a linear preference relation on $\mathcal{X}$, i.e., a complete strict order (for any $\vec{x}$ and $\vec{y} \neq \vec{x}$, either $\vec{x} \succ \vec{y}$ or $\vec{y} \succ \vec{x}$ holds). We often note $\vec{x} \succ_V \vec{x}'$ instead of $V(\vec{x}, \vec{x}')$. An N-voter *profile* w.r.t. $\mathcal{X}$ is a collection of N individual linear preference relations over $\mathcal{X}$: $P = (V_1, \ldots, V_N)$.

Let $\{X, Y, Z\}$ be a partition of the set I and V be a linear preference relation over $\mathcal{X} = D_I$. X is *conditionally preferentially independent* of Y given Z (w.r.t. V) if and only if for all $\vec{x}_1, \vec{x}_2 \in D_X$, $\vec{y}_1, \vec{y}_2 \in D_Y$, $\vec{z} \in D_Z$,

$$\vec{x}_1.\vec{y}_1.\vec{z} \succ_V \vec{x}_2.\vec{y}_1.\vec{z} \quad \text{iff} \quad \vec{x}_1.\vec{y}_2.\vec{z} \succ_V \vec{x}_2.\vec{y}_2.\vec{z}.$$

Informally, X is *conditionally preferentially independent* of Y given Z, if for any fixed value $\vec{z}$ of Z, the preference over the possible values of X is independent from the value of Y. We use the notation $CPI_V(X, Y, Z)$ to denote that X is conditionally preferentially independent of Y given Z (w.r.t. V). When X is a singleton we simply note $CPI_V(\mathbf{x}, Y, Z)$ instead of $CPI_V(\{\mathbf{x}\}, Y, Z)$.[3] A preference relation is *separable* if for every $\mathbf{x}_i \in I$, $\mathbf{x}_i$ is preferentially independent from $I \setminus \{\mathbf{x}_i\}$.

Conditional preferential independence originates in the literature of multiattribute decision theory (Keeney and Raiffa, 1976). Unlike probabilistic independence, it is a directed notion: X may be independent of Y given Z without Y being independent of X given Z.

Conditional preference networks, or *CP-nets*, are a language for specifying preferences based on the notion of conditional preferential independence. They allow for eliciting preferences, and for storing them, as economically as possible.

Definition 1 (*CP-nets (Boutilier et al., 2004a)*). Let $I = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of variables (or issues); for each i, let D_i be the finite domain of $\mathbf{x}_i$, and let $\mathcal{X} = D_1 \times \cdots \times D_p$ be the set of alternatives. A *CP-net* $\mathcal{N}$ over I is a pair consisting of:

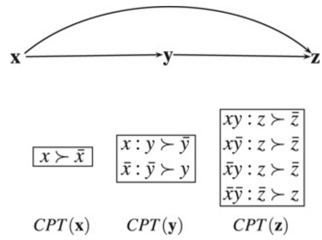
- a *directed acyclic*[4] *graph* $G = \langle I, E\rangle$ whose set of vertices is the set of issues I, and the set of edges is E. For every vertex $\mathbf{x} \in I$, $Pa_G(\mathbf{x})$ denotes the set of parents of $\mathbf{x}$ in G, that is, $\{\mathbf{y} \in I \mid (\mathbf{y}, \mathbf{x}) \in E\}$, and $NonPa_G(\mathbf{x})$ denotes the set of "non-parents" of $\mathbf{x}$ in G, defined by $NonPa_G(\mathbf{x}) = I \setminus (\{x\} \cup Pa_G(\mathbf{x}))$.
- a collection of *conditional preference tables* $CPT(\mathbf{x}_i)$ for each $\mathbf{x}_i \in I$, defined as follows: each conditional preference table $CPT(\mathbf{x}_i)$ associates a total order $\succ^i_{\vec{u}}$ on D_i with each instantiation $\vec{u}$ of $\mathbf{x}_i$'s parents $Pa_G(\mathbf{x}_i) = U$.

Intuitively, the edges of G represent preferential dependencies: for every i, $\mathbf{x}_i$ is preferentially independent from its "non-parents" given its parents.

Example 3. Let $\mathcal{N}$ be the following CP-net, whose graph G is depicted below, and the conditional preference tables below it.

[3] Conditional preference independence is also considered in Bradley et al. (2005) under the name *non-influenceability*. A related notion investigated in Bradley et al. (2005) is that of *separable subsets*: a subset S of I is separable with respect to V if S is preferentially independent from $I \setminus S$ with respect to V.

[4] The original definition of CP-nets (Boutilier et al., 2004a) allows G to contain cycles. However, in this paper we do not need to refer to this more general framework. Note that the assumption that G is acyclic is usual (see Boutilier et al. (2004a,b)).



The graph in Example 3 means that the agent has unconditional preferences over the values of **x**, that her preferences over the values of **y** are fully determined given the value of **x**, and that her preferences over the values of **z** depend both on **x** and **y**. The conditional preference table for **x** means that x is preferred to $\bar{x}$ (unconditionally). The conditional preference table for **y** means that y is preferred to $\bar{y}$ when $\mathbf{x} = x$, and $\bar{y}$ is preferred to **y** when $\mathbf{x} = \bar{x}$.

A CP-net $\mathcal{N}$ induces a partial preference relation in the following way.

Definition 2 (*Preference Relation Induced by a CP-net*). Let $\mathcal{N}$ be a CP-net over I.

- for each $\mathbf{x}_i \in I$, let $Z = Pa_G(\mathbf{x}_i)$ and $Y = NonPa(\mathbf{x}_i)$. The relation $\succ_{\mathcal{N}}^{\mathbf{x}_i}$ induced from the conditional preference table for $\mathbf{x}_i$ is defined by

  $$\succ_{\mathcal{N}}^{\mathbf{x}_i} = \{(x_i.\vec{y}.\vec{z}, x_i'.\vec{y}.\vec{z}) | \vec{y} \in D_Y, \vec{z} \in D_Z, CPT(\mathbf{x}_i) \text{ contains } \vec{z} : x_i \succ x_i'\}$$

- the primitive relation induced by $\mathcal{N}$ is defined as the union of the relations induced by the conditional preference tables:

  $$Prim(\mathcal{N}) = \bigcup \{\succ_{\mathcal{N}}^{\mathbf{x}_i} | \mathbf{x}_i \in I\}$$

- the preference relation $\succ_{\mathcal{N}}$ induced by $\mathcal{N}$ is the transitive closure of $Prim(\mathcal{N})$.

Note that $\succ_{\mathcal{N}}$ is an irreflexive and asymmetric relation that possesses a dominating element (Boutilier et al., 2004a).[5] Note that the preference relation induced by a CP-net is generally not complete. We say that a linear preference V extends $\mathcal{N}$, or $\succ_{\mathcal{N}}$, if $\succ_{\mathcal{N}} \subseteq V$, namely for any $\alpha, \beta \in \mathcal{X}$, $\alpha \succ_{\mathcal{N}} \beta$ implies $\alpha \succ_V \beta$.

Example 3, continued. The relations induced by the conditional preference tables of $\mathcal{N}$ are

$$\begin{array}{lllll}
\succ_{\mathcal{N}}^{\mathbf{x}} : & xyz \succ \bar{x}yz, & xy\bar{z} \succ \bar{x}y\bar{z}, & x\bar{y}z \succ \bar{x}\bar{y}z, & x\bar{y}\bar{z} \succ \bar{x}\bar{y}\bar{z} \\
\succ_{\mathcal{N}}^{\mathbf{y}} : & xyz \succ x\bar{y}z, & xy\bar{z} \succ x\bar{y}\bar{z}, & \bar{x}\bar{y}z \succ \bar{x}yz, & \bar{x}\bar{y}\bar{z} \succ \bar{x}y\bar{z} \\
\succ_{\mathcal{N}}^{\mathbf{z}} : & xyz \succ xy\bar{z}, & x\bar{y}z \succ x\bar{y}\bar{z}, & \bar{x}yz \succ \bar{x}y\bar{z}, & \bar{x}\bar{y}\bar{z} \succ \bar{x}\bar{y}z.
\end{array}$$

The preference relation $\succ_{\mathcal{N}}$ is the transitive closure of $\succ_{\mathcal{N}}^{\mathbf{x}} \cup \succ_{\mathcal{N}}^{\mathbf{y}} \cup \succ_{\mathcal{N}}^{\mathbf{z}}$:

$$\succ_{\mathcal{N}}: \quad xyz \begin{array}{c} \nearrow \; x\bar{y}z \; \searrow \\ \searrow \; xy\bar{z} \; \nearrow \end{array} x\bar{y}\bar{z} \to \bar{x}\bar{y}\bar{z} \to \bar{x}\bar{y}z \to \bar{x}yz \to \bar{x}y\bar{z}.$$

Let G be a directed graph over I, and $\succ$ a linear preference relation. $\succ$ is said to be *compatible with* G if and only if for each $\mathbf{x} \in I$, **x** is preferentially independent of $NonPa_G(\mathbf{x})$ given $Pa_G(\mathbf{x})$. The following fact is important:

Observation 1. *A linear preference relation V is compatible with G if and only if there exists a CP-net $\mathcal{N}$ whose associated graph is G such that V extends $\succ_{\mathcal{N}}$.*

[5] The assumption that G is acyclic is crucial for this statement to hold.

Proof. ($\Rightarrow$) Let V be a linear preference relation compatible with G: for each $\mathbf{x} \in V$, we have $CPI_V(\mathbf{x}, Y, Z)$ where $Z = Pa_G(\mathbf{x})$ and $Y = NonPa_G(\mathbf{x})$. Let us build the following CP-net $\mathcal{N}_R$ whose associated graph is G and the conditional preference table for each issue $\mathbf{x} \in I$ contains $\vec{z} : x_1 \succ x_2$ if and only if $x_1.\vec{y}.\vec{z} \succ x_2.\vec{y}.\vec{z}$ for all $\vec{y} \in D_Y$. First notice that since $CPI_V(\mathbf{x}, Y, Z)$ holds, if $x_1.\vec{y}.\vec{z} \succ_V x_2.\vec{y}.\vec{z}$ holds for *some* $\vec{y} \in D_Y$ then $x_1.\vec{y}.\vec{z} \succ x_2.\vec{y}.\vec{z}$ holds for *all* $\vec{y} \in D_Y$. Therefore, for each issue $\mathbf{x} \in I$ and each $x_1, x_2 \in D_x$, and each $\vec{z} \in D_Z$, the conditional preference table for $\mathbf{x}$ contains either $\vec{z} : x_1 \succ x_2$ or $\vec{z} : x_2 \succ x_1$ (and not both). This shows that $\mathcal{N}_R$ is a well-defined CP-net. Consider an element $(x_i.\vec{y}.\vec{z}, x_i'.\vec{y}.\vec{z})$ of $Prim(\mathcal{N})$. By construction of $\mathcal{N}$, $(x_i.\vec{y}.\vec{z}, x_i'.\vec{y}.\vec{z}) \in Prim(\mathcal{N})$ implies $x_i.\vec{y}.\vec{z} \succ_V x_i'.\vec{z}.\vec{y}$. Hence, $Prim(\mathcal{N}) \subseteq V$. Because $\succ_{\mathcal{N}}$ and V are both transitive, $\succ_{\mathcal{N}} = Prim(\mathcal{N})^* \subseteq V^* = V$, where $Prim(\mathcal{N})^*$ (resp. V^*) is the transitive closure of $Prim(\mathcal{N})$ (resp. V). Therefore, V extends $\succ_{\mathcal{N}}$.

($\Leftarrow$) Let $\mathcal{N} = (G, CPT)$ and let V such that V extends $\succ_{\mathcal{N}}$. Let $\mathbf{x} \in I$, $Z = Pa_G(\mathbf{x})$, $Y = NonPa_G(\mathbf{x})$, $\vec{z} \in D_Z$, $x_1, x_2 \in D_{\mathbf{x}}$ and assume without loss of generality that $CPT(\mathbf{x})$ contains $\vec{z} : x_1 \succ x_2$. Then for all $\vec{y} \in D_Y$, we have $x_1.\vec{y}.\vec{z} \succ_{\mathcal{N}} x_2.\vec{y}.\vec{z}$, and since V extends $\succ_{\mathcal{N}}$, we have $x_1.\vec{y}.\vec{z} \succ_V x_2.\vec{y}.\vec{z}$. This holds for all $\vec{y} \in D_Y$, henceforth $CPI_R(\mathbf{x}, Y, Z)$ holds. ■

Definition 3 (*O-legality*). Let G be an acyclic graph over I and let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ be a linear order on I. G is said to *follow* $\mathcal{O}$ if for every edge $(\mathbf{x}_i, \mathbf{x}_j)$ in G we have $\mathbf{x}_i > \mathbf{x}_j$, namely $i < j$. A linear preference relation $\succ$ is said to be $\mathcal{O}$-legal if and only if it is compatible with some acyclic graph G following $\mathcal{O}$. We denote by $Legal(\mathcal{O})$ the set of all $\mathcal{O}$-legal linear preference relations.

Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$. Clearly, $V \in Legal(\mathcal{O})$ if and only if for all $i < p$, $\mathbf{x}_i$ is preferentially independent of $\{\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p\}$ given $\{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}$ with respect to V.

Definition 4 (*Projection of a Preference Relation on an Issue*). Let $V \in Legal(\mathcal{O})$ and $\mathbf{x}_i \in I$. The *projection* of V on $\mathbf{x}_i$ given $(x_1, \ldots, x_{i-1}) \in D_1 \times \cdots \times D_{i-1}$, denoted by $\succ_V^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$, is the linear preference relation on D_i defined by: for all $x_i, x_i' \in D_i$, $x_i \succ_V^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}} x_i'$ iff $x_1 \ldots x_{i-1}x_ix_{i+1} \ldots x_p \succ_V x_1 \ldots x_{i-1}x_i'x_{i+1} \ldots x_p$ holds for all $(x_{i+1}, \ldots, x_p) \in D_{i+1} \times \cdots \times D_p$.

Due to the fact that $V \in Legal(\mathcal{O})$ and that V is a linear order, $\succ_V^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ is a well-defined linear order on D_i. Note also that if V is legal with respect to both $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ and $\mathcal{O}' = \mathbf{x}_{\sigma(1)} > \cdots > \mathbf{x}_{\sigma(k-1)} > \mathbf{x}_i (= \mathbf{x}_{\sigma(k)}) > \cdots > \mathbf{x}_{\sigma(p)}$, then $\succ_V^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ and $\succ_V^{\mathbf{x}_i|\mathbf{x}_{\sigma(1)}=x_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(k-1)}=x_{\sigma(k-1)}}$ coincide. In other words, the local preference relation on $\mathbf{x}_i$ depends only on the values of the variables that precede $\mathbf{x}_i$ in $\mathcal{O}$ and in $\mathcal{O}'$.

Example 4. Let $I = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, all three issues being binary, and let V be the following linear preference relation:

$$xyz \succ xy\bar{z} \succ x\bar{y}\bar{z} \succ x\bar{y}z \succ \bar{x}y\bar{z} \succ \bar{x}\bar{y}\bar{z} \succ \bar{x}yz \succ \bar{x}\bar{y}z.$$

Let G be the graph over I whose set of edges is $\{(\mathbf{x}, \mathbf{z}), (\mathbf{y}, \mathbf{z})\}$. The orders $\mathbf{x} > \mathbf{y} > \mathbf{z}$ and $\mathbf{y} > \mathbf{x} > \mathbf{z}$ both follow G; therefore, V is both in $Legal(\mathbf{x} > \mathbf{y} > \mathbf{z})$ and in $Legal(\mathbf{y} > \mathbf{x} > \mathbf{z})$. Fix the order to be $\mathbf{x} > \mathbf{y} > \mathbf{z}$; then we have $x \succ_V^{\mathbf{x}} \bar{x}$, $y \succ_V^{\mathbf{y}|\mathbf{x}=x} \bar{y}$, $y \succ_V^{\mathbf{y}|\mathbf{x}=\bar{x}} \bar{y}$, $z \succ_V^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y} \bar{z}$, $\bar{z} \succ_V^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=\bar{y}} z$ etc.

Lastly, for any acyclic graph G over I, we say that two linear preference relations V_1 and V_2 are G-equivalent, denoted by $V_1 \sim_G V_2$, if and only if V_1 and V_2 are both compatible with G and for any $\mathbf{x} \in I$, for any $\vec{y}, \vec{y}' \in D_{Pa_G(\mathbf{x})}$ we have $V_1^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}} = V_2^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}}$.

Observation 2. *For any linear preference relations V_1 and V_2, $V_1 \sim_G V_2$ if and only if there exists a CP-net $\mathcal{N}$ whose associated graph is G and such that V_1 and V_2 both extend $\succ_{\mathcal{N}}$.*

Proof. ($\Rightarrow$) Assume $V_1 \sim_G V_2$. This entails, by definition, that V_1 and V_2 are both compatible with G, which by Observation 1 entails that there exist two CP-nets $\mathcal{N}_1$ and $\mathcal{N}_2$ whose associated graphs are G and such that V_1 extends $\succ_{\mathcal{N}_1}$ and V_2 extends $\succ_{\mathcal{N}_2}$. Furthermore, for any $\mathbf{x} \in I$, for any $\vec{y}, \vec{y}' \in D_{Pa_G(\mathbf{x})}$ we have $V_1^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}} = V_2^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}}$, which implies that $\mathcal{N}_1 = \mathcal{N}_2$.

($\Leftarrow$) Assume there exists a CP-net $\mathcal{N}$ whose associated graph is G such that V_1 and V_2 both extend $\succ_{\mathcal{N}}$. By Observation 1, V_1 and V_2 are both compatible with G. Let $\mathbf{x} \in I, x_1, x_2 \in D_{\mathbf{x}}$ and $\vec{y}, \vec{y}' \in D_{Pa_G(\mathbf{x})}$. Because V_1 extends $\succ_{\mathcal{N}}$, we have $x_1 \succ_{V_1}^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}} x_2$ if and only if $\mathcal{N}$ contains $\vec{y} : x_1 \succ x_2$. Similarly, because V_2 extends $\succ_{\mathcal{N}}$, we have $x_1 \succ_{V_2}^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}} x_2$ if and only if $\mathcal{N}$ contains $\vec{y} : x_1 \succ x_2$. Therefore, we have $V_1^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}} = V_2^{\mathbf{x}|Pa_G(\mathbf{x})=\vec{y}}$. ■

Example 5. Let $I = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, all three issues being binary, and let V and V' be the following linear preference relations:

$$V : xyz \succ xy\bar{z} \succ x\bar{y}\bar{z} \succ x\bar{y}z \succ \bar{x}y\bar{z} \succ \bar{x}\bar{y}\bar{z} \succ \bar{x}yz \succ \bar{x}\bar{y}z$$
$$V' : xyz \succ xy\bar{z} \succ \bar{x}y\bar{z} \succ x\bar{y}\bar{z} \succ \bar{x}yz \succ \bar{x}\bar{y}\bar{z} \succ x\bar{y}z \succ \bar{x}\bar{y}z.$$

Let G be the graph over I whose set of edges is $\{(\mathbf{x}, \mathbf{z}), (\mathbf{y}, \mathbf{z})\}$. V and V' are both compatible with G. Moreover, $V \sim_G V'$, since all local preference relations coincide: $x \succ_V^{\mathbf{x}} \bar{x}$ and $x \succ_{V'}^{\mathbf{x}} \bar{x}$; $z \succ_V^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y} \bar{z}$ and $z \succ_{V'}^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y} \bar{z}$; etc. The CP-net $\mathcal{N}$ such that V and V' both extend $\succ_{\mathcal{N}}$ is defined by the following local conditional preference tables: $x \succ \bar{x}$; $y \succ \bar{y}$; $xy : z \succ \bar{z}$; $x\bar{y} : \bar{z} \succ z$; $\bar{x}y : \bar{z} \succ z$; $\bar{x}\bar{y} : \bar{z} \succ z$.

3. *G*-legal profiles

Let I be a set of issues, with $|I| \geq 2$, and $\{1, \ldots, N\}$ be a set of voters, with $N \geq 2$. We now define a crucial domain restriction for the rest of the paper:

Definition 5. Given an acyclic graph G on I, we define $Legal(G)$ as the set of all collective profiles $P = (V_1, \ldots, V_N)$ such that each V_i is compatible with G. For any order $\mathcal{O}$, we also define $Legal(\mathcal{O})$ as the set of all profiles $P = (V_1, \ldots, V_N)$ such that each V_i is in $Legal(\mathcal{O})$.

The following observation is straightforward but important:

Observation 3. *$P \in Legal(G)$ if and only if $P \in Legal(\mathcal{O})$ for all $\mathcal{O}$ that G follows*

We might wonder how strong the restriction to $\mathcal{O}$-legal profiles is.

First, this restriction is much less demanding than separability. To see this, let $G_\emptyset$ be the graph whose set of vertices is I and that contains no edge; then we have the following important fact (whose proof is obvious):

Observation 4. *The following three assertions are equivalent:*

1. *$V \in Legal(G_\emptyset)$.*
2. *for any order $\mathcal{O}$ on I, $V \in Legal(\mathcal{O})$.*
3. *V is separable.*

Thus, $\mathcal{O}$-legality is a family of domain restrictions that includes separability as a special case but contains much more profiles. More precisely, the set of $\mathcal{O}$-legal preference relations for some fixed order $\mathcal{O}$ is exponentially larger than the set of separable preference relations. We prove this in the particular case of two issues, one of which is binary and the other one contains k values (with k varying).

Let $\mathcal{X}_k = \{0, 1\} \times \{0, 1, \ldots, k-1\}$, where $D_1 = \{0, 1\}$ and $D_2 = \{0, 1, \ldots, k-1\}$. Let $Legal(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)$ denote the set of all linear orders over $\mathcal{X}_k$ that are compatible with $\mathbf{x}_1 > \mathbf{x}_2$, and $\alpha_k = |Legal(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)|$. Let $Separable(\mathcal{X}_k)$ denote the set of separable linear orders over $\mathbf{x}_k$, and $\beta_k = |Separable(\mathcal{X}_k)|$.

Proposition 1. $\lim_{k\to\infty} \frac{\alpha_k}{\beta_k} \geq \frac{2^k}{\sqrt{\pi k}}$.

Proof. We first prove that $\alpha_k = \frac{2(2k)!}{2^k}$. Let *Legal*$^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)$ be the set of all preference relations V in *Legal*$^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)$ such that $V|^{\mathbf{x}_1} = 0 \succ 1$. Equivalently, *Legal*$^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)$ is the set of linear orders compatible with $\mathbf{x}_1 > \mathbf{x}_2$, and in which $\mathbf{x}_1 = 0$ is preferred to $\mathbf{x}_1 = 1$. We note that a linear order is in *Legal*$^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)$ if and only if it satisfies the following constraints: for every $i \le k-1$, $0i \succ_V 1i$. Therefore, α_k can be counted as follows. We first select the positions of 00 and 10 (among $1, 2, \ldots, 2k$), given that 00 is in the higher position and 10 in the lower. For this there are exactly $\binom{2k}{2}$ possibilities. Then, we select the positions of 01 and 11 from the remaining positions, given that 01 is in the higher position and 11 in the lower. There are $\binom{2k-2}{2}$ possibilities in this second step. From steps 3 to k, we fix the positions for $(02, 12), \ldots, (0(k-1), 1(k-1))$, respectively. In the end we have a linear order compatible with $\mathbf{x}_1 > \mathbf{x}_2$, and every linear order compatible with $\mathbf{x}_1 > \mathbf{x}_2$ can be obtained this way. It follows that $|Legal^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)| = \binom{2k}{2} \times \binom{2k-2}{2} \times \cdots \times \binom{2}{2} = \frac{(2k)!}{2^k}$, and finally $\alpha_k = 2|Legal^*(\mathbf{x}_1 > \mathbf{x}_2)(\mathcal{X}_k)| = \frac{2(2k)!}{2^k}$.

Next, we show that $\beta_k \le 2(k!)^2$. Let *Separable*$^*(\mathcal{X}_k)$ be the set of all preference relations V in *Separable*$(\mathcal{X}_k)$ such that $V|^{\mathbf{x}_1} = 0 \succ 1$ and $V|^{\mathbf{x}_2} = 0 \succ 1 \succ \cdots \succ k-1$. Equivalently, *Separable*$^*(\mathcal{X}_k)$ is the set of separable preference relations V whose marginal preference over $\mathbf{x}_1$ is $0 \succ 1$, and the marginal preference over $\mathbf{x}_2$ is $0 \succ 1 \succ \cdots \succ k-1$. We note that a linear order is in *Separable*$^*(\mathcal{X}_k)$ if and only if it satisfies the following two constraints: (1) for every $i \le k-1$, $0i \succ_V 1i$, and (2) for $j = 0, 1$, $j0 \succ_V j1 \succ_V \ldots \succ_V j(k-1)$. Therefore, any linear order in *Separable*$^*(\mathcal{X}_k)$ can be generated in the following way: we start from the initial partial order $00 \succ 01 \succ \cdots \succ 0(k-1)$; at the first step we include 10 in the partial order in such a way that $00 \succ 10$. There are k possibilities in this step. Then, we include 11 in the partial order in such a way that $01 \succ 11$ and $10 \succ 11$. There are no more than $k-1$ possibilities in this second step. At the subsequent steps we include successively $02, \ldots, 0k-1$. For every j, there are no more than $k+1-j$ possibilities at step j. It follows that $|Separable^*(\mathcal{X}_k)| \le k!$. Finally, there are $2k!$ different way of fixing the marginal preference relations on $\mathbf{x}_1$ and $\mathbf{x}_2$ (2 for $\mathbf{x}_1$ and $k!$ for $\mathbf{x}_2$). Therefore, $\beta_k = 2k!|Separable^*(\mathcal{X}_k)| \le 2(k!)^2$.

By applying Stirling's formula, we have that $\lim_{k\to\infty} \frac{(2k)!}{\sqrt{2\pi 2k}(\frac{2k}{e})^{2k}} = 1$ and $\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}(\frac{k}{e})^k} = 1$. It follows that

$$\lim_{k\to\infty} \frac{\beta_k}{\alpha_k} \ge \frac{\frac{2(2k)!}{2^k}}{2(k!)^2} \ge \lim_{k\to\infty} = \frac{2\sqrt{2\pi 2k}(\frac{2k}{e})^{2k}}{2(\sqrt{2\pi k}(\frac{k}{e})^k)^2 2^k} = \frac{2^k}{\sqrt{\pi k}}. \quad \blacksquare$$

After noticing that $|\mathcal{X}_k| = 2k$, the above proposition tells us that the number of legal linear orders is exponentially larger (in the size of alternatives) than the number of separable votes, thus showing that $\mathcal{O}$-legality is a much less demanding domain restriction than separability. Note that a related weakening of separability is considered in Hodge and Ter Haar (2008): a preference relation V is *completely nonseparable* if there does not exist any proper subset S of I such that S is preferentially independent of $I \setminus S$ with respect to V. We say that V satisfies "noncomplete nonseparability" (NCNS) if it fails to be completely nonseparable. Obviously, $\mathcal{O}$-legality, while being weaker than separability, is stronger than NCNS.

Second, in many real-life domains it may be deemed reasonable to assume that preferential dependencies between variables coincide for all voters. For instance, in a designated-seat election process (Benoit and Kornhauser, 2006) where an assembly composed of a president, a vice-president and a secretary has to be elected, it may be intuitively reasonable to assume that voters' preferences are compatible with the order `president` $>$ `vice-president` $>$ `treasurer`.

Third, having $P_1 \in Legal(G_1)$ and $P_2 \in Legal(G_2)$ for $G_2 \ne G_1$ does not mean that a profile containing P_1 and P_2 is not G-legal for some acyclic graph G. Indeed, suppose that the linear preference relations $(\succ_1, \ldots, \succ_N)$ are compatible with the acyclic graphs $G_1, \ldots, G_N$, whose sets of edges are $E_1, \ldots, E_N$. Then they are a fortiori compatible with the graph G^* whose set of edges is $E_1 \cup \cdots \cup E_N$. Therefore, *if* G^* *is acyclic*, then the profile is admissible (of course, this is no longer true if G^* has cycles — see the last paragraph of Section 4).

Lastly, the $\mathcal{O}$-legality restriction can be generalized by partitioning the set of issues into subsets $I_1, \ldots, I_q$ such that I_i is preferentially independent of $I_{i+1} \cup \cdots \cup I_q$ given $I_1 \cup \cdots \cup I_{i-1}$. Obviously, all profiles are of this form, the worst case being $q = 1$. However, we can assume without loss of generality (and we will do so in the remainder of the paper) that each cluster consists of a single issue (if this were not the case from the beginning, then each cluster I_i can be considered as a new single issue, with domain $D_{I_i} = \prod_{\mathbf{x}_j \in I_i} D_j$).

We end this section by remarking that $\mathcal{O}$-legality generalizes the *decomposability* property of Laffond et al. (1996), that we first recall. Let $\mathcal{X}$ be a set of alternatives and V a vote on $\mathcal{X}$. A nonempty subset Y of $\mathcal{X}$ is a component of V if it verifies: $\forall (y, y') \in Y^2, \forall x \in \mathcal{X} \setminus Y, x \succ_V y \Leftrightarrow x \succ_V y'$. Let $\mathcal{D}$ be a partition of $\mathcal{X}$. A profile $P = (V_1, \ldots, V_N)$ admits $\mathcal{D} = (Y_1, \ldots, Y_q)$ as decomposition if for every $i \leq N$ and $j \leq q$, Y_j is a component of V_i. Lastly, P is decomposable if it admits a proper decomposition (*i.e.*, a decomposition different from $\{\mathcal{X}\}$ and $\{\{x\}, x \in \mathcal{X}\}$). The intuition beyond decomposability is that the components of the decomposition correspond to projects, and elements of a component correspond to variants of the same project. Decomposability is used for defining a two-step procedure, where the project is chosen first, and the variant next. To see why decomposability is a particular case of $\mathcal{O}$-legality, assume P admits $\mathcal{D}$ as decomposition and rewrite $\mathcal{X}$ as a Cartesian product $D_1 \times D_2$, with $|D_1| = |\mathcal{D}| = q$ and $|D_2| = \max_{i \leq q} |Y_i|$; that is, every alternative x is identified by the project $x_1 \in D_1$ to which it belongs, and the variant $x_2 \in D_2$ of the project. Then P is $\mathbf{x}_1 > \mathbf{x}_2$-legal, and the sequential composition of voting rules, that we describe below, intuitively corresponds to the two-step procedure of Laffond et al. (1996).

4. Sequential voting rules and correspondences

We start by recalling briefly some necessary background on voting rules and correspondences. Let $P_{N,\mathcal{X}}$ be the set of all N-voter preference profiles for the set of candidates $\mathcal{X}$. A *voting correspondence* $C : P_{N,\mathcal{X}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ maps each preference profile P of $P_{N,\mathcal{X}}$ into a nonempty subset $C(P)$ of $\mathcal{X}$. A *voting rule* $r : P_{N,\mathcal{X}} \to \mathcal{X}$ maps each preference profile P of $P_{N,\mathcal{X}}$ into a single candidate $r(P)$. When there are only two candidates $\{x, y\}$, the *majority* correspondence *maj* is defined by $maj(P) = \{x\}$ (resp. $\{y\}$) if more voters in P prefer x to y (resp. y to x), and $maj(P) = \{x, y\}$ in case of tie.

From now on, we assume that the set of candidates is a multi-issue domain $\mathcal{X} = D_1 \times \cdots \times D_p$. *Sequential voting* consists in applying "local" voting rules or correspondences on single issues, one after the other, in such an order that the local vote on a given issue can be performed only when the local votes on all its parents in the graph G have been performed (Lang, 2007).

Definition 6. Let G be an acyclic graph on I; let $P = (V_1, \ldots, V_N)$ be in $Legal(G)^N$, $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ be a linear order on I that G follows and $(r_1, \ldots, r_p)$ a collection of voting rules (one for each variable $\mathbf{x}_i$). The *sequential voting rule* $\mathrm{Seq}(r_1, \ldots, r_p)$ is defined as follows:

- $x_1^* = r_1(V_1^{\mathbf{x}_1}, \ldots, V_N^{\mathbf{x}_1})$;
- $x_2^* = r_2(V_1^{\mathbf{x}_2|\mathbf{x}_1=x_1^*}, \ldots, V_N^{\mathbf{x}_2|\mathbf{x}_1=x_1^*})$;
- …
- $x_p^* = r_p(V_1^{\mathbf{x}_p|\mathbf{x}_1=x_1^*,\ldots,\mathbf{x}_{p-1}=x_{p-1}^*}, \ldots, V_N^{\mathbf{x}_p|\mathbf{x}_1=x_1^*,\ldots,\mathbf{x}_{p-1}=x_{p-1}^*})$

Then $\mathrm{Seq}(r_1, \ldots, r_p)(P) = (x_1^*, \ldots, x_p^*)$.

Example 6. Let $N = 8$, $I = \{\mathbf{x}, \mathbf{y}\}$ with $D_{\mathbf{x}} = \{x_1, x_2, x_3\}$ and $D_{\mathbf{y}} = \{y, \bar{y}\}$, and $P = (V_1, \ldots, V_8)$ be the following 8-voter profile:

$$V_1, V_2, V_3 : \quad x_1\bar{y} \succ x_1y \succ x_2\bar{y} \succ x_2y \succ x_3y \succ x_3\bar{y}$$
$$V_4, V_5 : \quad x_2y \succ x_3y \succ x_2\bar{y} \succ x_1y \succ x_3\bar{y} \succ x_1\bar{y}$$
$$V_6 : \quad x_3\bar{y} \succ x_1\bar{y} \succ x_3y \succ x_1y \succ x_2y \succ x_2\bar{y}$$
$$V_7, V_8 : \quad x_3\bar{y} \succ x_3y \succ x_2y \succ x_2\bar{y} \succ x_1y \succ x_1\bar{y}.$$

All these preference relations are compatible with the graph G over $\{\mathbf{x}, \mathbf{y}\}$ whose single edge is $(\mathbf{x}, \mathbf{y})$; equivalently, they follow the order $\mathbf{x} > \mathbf{y}$. Hence, $P \in Legal(G)$. The corresponding conditional preference tables are:

voters 1,2,3	voters 4,5	voter 6	voters 7,8
$x_1 \succ x_2 \succ x_3$	$x_2 \succ x_3 \succ x_1$	$x_3 \succ x_1 \succ x_2$	$x_3 \succ x_2 \succ x_1$
$x_1 : \bar{y} \succ y$	$x_1 : y \succ \bar{y}$	$x_1 : \bar{y} \succ y$	$x_1 : y \succ \bar{y}$
$x_2 : \bar{y} \succ y$	$x_2 : y \succ \bar{y}$	$x_2 : y \succ \bar{y}$	$x_2 : y \succ \bar{y}$
$x_3 : y \succ \bar{y}$	$x_3 : y \succ \bar{y}$	$x_3 : \bar{y} \succ y$	$x_3 : \bar{y} \succ y$

Take $r_{\mathbf{x}}$ to be the Borda rule, and $r_{\mathbf{y}}$ to be the majority rule. The projection of P on $\mathbf{x}$, namely $P^{\mathbf{x}} = (V_1^{\mathbf{x}}, \ldots, V_8^{\mathbf{x}})$, contains three votes $x_1 \succ x_2 \succ x_3$, two votes $x_2 \succ x_3 \succ x_1$, one vote $x_3 \succ x_1 \succ x_2$ and two votes $x_3 \succ x_2 \succ x_1$, therefore, the Borda winner for $P^{\mathbf{x}}$ is $x^* = r_{\mathbf{x}}(P^{\mathbf{x}}) = x_2$. Now, the projection of P on $\mathbf{y}$ given $\mathbf{x} = x_2$, namely $P^{\mathbf{y}|\mathbf{x}=x_2} = (V_1^{\mathbf{y}|\mathbf{x}=x_2}, \ldots, V_8^{\mathbf{y}|\mathbf{x}=x_2})$, is composed of 5 votes for y and 3 for $\bar{y}$, therefore $y^* = r_{\mathbf{y}}(P^{\mathbf{y}|\mathbf{x}=x_2}) = y$. The sequential winner is now obtained by combining the $\mathbf{x}$-winner and the conditional $\mathbf{y}$-winner given $\mathbf{x} = x^*$, namely $\mathrm{Seq}(r_{\mathbf{x}}, r_{\mathbf{y}})(P) = x_2y$.

In addition to sequential voting rules, we also define *sequential voting correspondences* in a similar way: if for each i, C_i is a correspondence on D_i, then $\mathrm{Seq}(C_1, \ldots, C_p)(P)$ is the set of all outcomes $(x_1, \ldots, x_p)$ such that $x_1 \in C_1(V_1^{\mathbf{x}_1}, \ldots, V_N^{\mathbf{x}_1})$, and for all $i \geq 2$, $x_i \in C_i(V_i^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}, \ldots, V_N^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}})$.

For instance, in Example 6, if we take $C_{\mathbf{x}}$ to be the plurality correspondence (electing the candidates that are ranked first by the largest number of voters) and $X_{\mathbf{y}}$ to be the majority correspondence, then $C_{\mathbf{x}}(P^{\mathbf{x}}) = \{x_1, x_3\}$, and $\mathrm{Seq}(C_x, C_y)(P) = \{x_1y, x_1\bar{y}, x_3y\}$.

For the sake of brevity we give results for voting rules only. Unless stated otherwise, similar results hold for correspondences.

An important property of such sequential voting rules and correspondences is that the outcome does not depend on $\mathcal{O}$, provided that G follows $\mathcal{O}$. This can be expressed formally:

Observation 5. *Let $\mathcal{O} = (\mathbf{x}_1 > \cdots > \mathbf{x}_p)$ and $\mathcal{O}' = (\mathbf{x}_{\sigma(1)} > \cdots > \mathbf{x}_{\sigma(p)})$ be two linear orders on I such that G follows both $\mathcal{O}$ and $\mathcal{O}'$. Then, for any G-legal profile P,*

$$\mathrm{Seq}(r_1, \ldots, r_p)(P) = \mathrm{Seq}(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P)$$

and similarly for voting correspondences.

Proof. Assume that G follows both $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ and $\mathcal{O}' = \mathbf{x}_{\sigma(1)} > \cdots > \mathbf{x}_{\sigma(p)}$. Let $i \leq N$ and let j be such that $\sigma(j) = i$. Because G follows $\mathcal{O}$ and $\mathcal{O}'$, we have $Pa_G(\mathbf{x}_i) \subseteq \{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}$ and $Pa_G(\mathbf{x}_i) \subseteq \{\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(j-1)}\}$. Let $P = (V_1, \ldots, V_N) \in Legal(G)$. For every $i \leq N$, and for every voter k, $\mathbf{x}_i$ is preferentially independent of $NonPa_G(\mathbf{x}_i)$ given $Pa_G(\mathbf{x}_i)$ w.r.t. V_k, that is, the preference of voter k on the values of $\mathbf{x}_i$ depends only on the values of the issues in $Pa_G(\mathbf{x}_i)$. Therefore, $V_k^{\mathbf{x}_i|\mathbf{x}_1=v_1,\ldots,\mathbf{x}_{i-1}=v_{i-1}} = V_k^{\mathbf{x}_i|Pa_G(\mathbf{x}_i)=\vec{z}}$, where $\vec{z} = \langle v_l | \mathbf{x}_l \in Pa_G(\mathbf{x}_i)\rangle$. Similarly, $V_k^{\mathbf{x}_i|\mathbf{x}_{\sigma(1)}=v_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(j-1)}=v_{\sigma(j-1)}} = V_k^{\mathbf{x}_i|Pa_G(\mathbf{x}_i)=\vec{z}} = V_k^{\mathbf{x}_i|\mathbf{x}_1=v_1,\ldots,\mathbf{x}_{i-1}=v_{i-1}}$. This entails that $r_i(P^{\mathbf{x}_i|\mathbf{x}_1=x_1^*,\ldots,\mathbf{x}_{i-1}=x_{i-1}^*}) = r_i(P^{\mathbf{x}_i|\mathbf{x}_{\sigma(1)}=x_{\sigma(1)}^*,\ldots,\mathbf{x}_{\sigma(j-1)}=x_{\sigma(j-1)}^*})$. This being true for every $i \leq p$, we get $\mathrm{Seq}(r_1, \ldots, r_p)(P) = \mathrm{Seq}(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P)$. ■

Note that when all variables are binary, all "reasonable" neutral voting rules coincide with the majority rule (plus some tie-breaking mechanism). Therefore, if all variables are binary and the number of voters is odd (in which case the tie-breaking mechanism is irrelevant), then the only "reasonable" sequential voting rule is $\mathrm{Seq}(r_1, \ldots, r_n)$ where each r_i is the majority rule.[6]

It is important to note that, in order to compute $\mathrm{Seq}(r_1, \ldots, r_p)(P)$, *we do not need to know the linear preference relations $V_1, \ldots, V_N$ entirely*: everything we need is the local preference relations: for

---

6 A further issue is a characterization of sequential majority, that would generalize May's theorem to multi-issue domains. See Section 5 of Xia et al. (2007a).

instance, if $V = \{\mathbf{x}, \mathbf{y}\}$ and G contains the only edge $(\mathbf{x}, \mathbf{y})$, then we need first the unconditional linear preference relations on $\mathbf{x}$ and then the linear preference relations on $\mathbf{y}$ conditioned by the value of $\mathbf{x}$. In other words, if we know the conditional preference tables (for all voters) associated with the graph G, then *we have enough information to determine the sequential winner for this profile*, even though some of the preference relations induced from these tables are incomplete. This is expressed more formally by the following fact (a similar result holds for correspondences):

Observation 6. *Let $I = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, G an acyclic graph over V, and $P = (V_1, \ldots, V_N)$, $P' = (V'_1, \ldots, V'_N)$ be two complete preference profiles such that for all $i = 1, \ldots, N$ we have $V_i \sim_G V'_i$. Then, for any collection of local voting rules $(r_1, \ldots, r_p)$, we have*

$$\mathrm{Seq}(r_1, \ldots, r_p)(P) = \mathrm{Seq}(r_1, \ldots, r_p)(P').$$

This, together with Observation 2, means that applying sequential voting to two collections of linear preference relations corresponding to the same collection of CP-nets gives the same result. This is illustrated in the following example.

Example 7. Everything is as in Example 6, except that we do not know the voters' complete preference relations, but only their corresponding conditional preference tables. These conditional preferences contain strictly less information than P, because some of the preference relations they induce are not complete: for instance, in the preference relation for the first 3 voters induced by their conditional preference tables, x_1y and $x_2\bar{y}$ are incomparable. However, we have enough information to determine the sequential winner for this profile, even though some of the preference relations are incomplete: everything we need is the marginal preference relations (first the unconditional preference relation on $\mathbf{x}$ and then the preference relation on $\mathbf{y}$ conditioned by the value of $\mathbf{x}$). In other words, we need to know the CP-nets *and nothing else*. For instance, taking again the Borda rule for $r_{\mathbf{x}}$ and the majority rule for $r_{\mathbf{y}}$, the sequential winner is x_2y for any complete profile $P' = (V'_1, \ldots, V'_8)$ extending the incomplete preference relations induced by the 12 conditional preference tables above.

Note that the assumption that G is acyclic is crucial for the definition of sequential voting rules. If G contains cycles, then no order $\mathcal{O}$ following G can be found. Certainly, one can proceed to a sequential vote anyway, but then some voters at some stage will not be able to vote "safely". This is the case in Example 1: whatever the order chosen ($S > T$ or $T > S$), voters 1 to 4 cannot vote safely on the first issue, and may experience regret after the final decision is made.

5. Sequential Condorcet winners

Recall that $x \in \mathcal{X}$ is a *Condorcet winner* (CW) for a profile P if it is preferred to any other candidate by a majority of voters: for all $y \neq x$, $\#\{i : x \succ_i y\} > \frac{N}{2}$. A *Condorcet-consistent* rule is a voting rule r such that whenever there exists a CW x for the profile P then $r(P) = x$. We may now wonder whether a CW, when there exists one, can be computed sequentially. Sequential Condorcet winners (SCW) are defined similarly as for sequential winners for a given rule: the SCW is the sequential combination of "local" Condorcet winners.

Definition 7. Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$, and $P = (\succ_1, \ldots, \succ_N) \in Legal(\mathcal{O})^N$. $(x_1^*, \ldots, x_p^*)$ be a *sequential Condorcet winner* for P if and only if

- $\forall x'_1 \in D_1, \#\{i : x_1^* \succ_i^{\mathbf{x}_1} x'_1\} > \frac{N}{2}$;
- for every $k > 1$ and $\forall x'_k \in D_k, \#\{i : x_k^* \succ_i^{\mathbf{x}_k|\mathbf{x}_1=x_1^*,\ldots,\mathbf{x}_{k-1}=x_{k-1}^*} x'_k\} > \frac{N}{2}$.

This definition is well founded because we obtain the same set of SCWs for any $\mathcal{O}$ following G. The question is now, do SCWs and CWs coincide? Clearly, the existence of a SCW is no more guaranteed than that of a CW, and there cannot be more than one SCW. We have the following positive result:

Proposition 2. *Let G be an acyclic graph and $P = (\succ_1, \ldots, \succ_N)$ in $Legal(G)$. If $(x_1^*, x_2^*, \ldots, x_n^*)$ is a Condorcet winner for P, then it is a sequential Condorcet winner for P.*

Proof. Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ be an order on I following G. Assume that there is a CW $\vec{x}^*$ for P: for any $\vec{x}' \neq \vec{x}^*$, $\#\{i : \vec{x}^* \succ_i \vec{x}'\} > \frac{N}{2}$. Let $x_1 \in D_1$ s.t. $x'_1 \neq x^*_1$. Since $\mathbf{x}_1$ is preferentially independent of $\mathbf{x}_2, \ldots, \mathbf{x}_p$, $x^*_1 \succ_i^{\mathbf{x}_1} x'_1$ iff $(x^*_1, x^*_2, \ldots, x^*_p) \succ_i (x'_1, x^*_2, \ldots, x^*_p)$; hence, $\#\{i : x^*_1 \succ_i^{\mathbf{x}_1} x'_1\} > \frac{N}{2}$: x^*_1 is a "local" CW. Similarly, for all k, by comparing $\vec{x}^*$ to $(x^*_1, \ldots, x^*_{k-1}, x'_k, x^*_{k+1}, \ldots, x^*_p)$, we show that x^*_k is a "local" CW for $(\succ_i^{\mathbf{x}_k|\mathbf{x}_1=x^*_1,\ldots,\mathbf{x}_{k-1}=x^*_{k-1}})_{i=1,\ldots,N}$. Therefore $\vec{x}^*$ is a SCW for P. ■

This simple result generalizes a result known for separable preferences (Laslier (2004, Proposition 16)). The converse fails, that is, a SCW may not always be a CW.[7] Consider for example 2 voters with the preference relation $x\bar{y} \succ \bar{x}\bar{y} \succ xy \succ \bar{x}y$, one voter with $xy \succ x\bar{y} \succ \bar{x}y \succ \bar{x}\bar{y}$, and 2 voters with $\bar{x}y \succ \bar{x}\bar{y} \succ xy \succ x\bar{y}$. All three preference relations are separable, therefore the SCW is the combination of the local CWs for $\{\mathbf{x}\}$ and for $\{\mathbf{y}\}$, provided they exist. Since 3 voters unconditionally prefer x to $\bar{x}$, x is the local CW for $\mathbf{x}$; similarly, 3 voters unconditionally prefer y to $\bar{y}$ and y is the local CW for $\mathbf{y}$. Therefore, xy is the SCW for the given profile; *but xy is not a CW* for this profile, because 4 voters prefer $\bar{x}\bar{y}$ to xy.

We now give a condition on the preference relations ensuring that SCWs and CWs coincide. Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ be a linear order on I. We say that a linear preference relation $\succ$ on $\mathcal{X}$ is *conditionally lexicographic* w.r.t. $\mathcal{O}$ if there exist local conditional preference relations $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$, such that $\vec{x} \succ \vec{y}$ if and only if there is a $j \leq p$ such that (a) for every $k < j$, $x_k = y_k$ and (b) $x_j \succ^{\mathbf{x}_j|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{j-1}=x_{j-1}} y_j$. A profile $P = (\succ_1, \ldots, \succ_N)$ is conditionally lexicographic w.r.t. $\mathcal{O}$ if each $\succ_i$ is conditionally lexicographic w.r.t. $\mathcal{O}$.

Proposition 3. *Let $\mathcal{O}$ be a linear strict order over I. If $P = (\succ_1, \ldots, \succ_N)$ is conditionally lexicographic w.r.t. $\mathcal{O}$, then $\vec{x}$ is a sequential Condorcet winner for P if and only if it is a Condorcet winner for P.*

Proof. Let $\vec{x}^*$ be a SCW for P, and $\vec{x}' = (x'_1, \ldots, x'_p) \neq \vec{x}^*$. Let $k = \min\{i : x^*_i \neq x'_i\}$ and $I_k \subseteq P$ be the set of voters who prefer x^*_k to x'_k given $\mathbf{x}_1 = x_1, \ldots, \mathbf{x}_{k-1} = x_{k-1}$. Because $\vec{x}^*$ is a SCW, $|I_k| > \frac{N}{2}$. We have $\vec{x}^* \succ_i \vec{x}'$ for every $i \in I_k$, because $\succ_i$ is lexicographic w.r.t. $\mathbf{x}_1 > \cdots > \mathbf{x}_p$. Therefore a majority of voters prefers $\vec{x}^*$ to $\vec{x}'$. This being true for all $\vec{x}' \neq \vec{x}^*$, $\vec{x}^*$ is a CW. ■

Note that a similar restriction to (unconditionally) lexicographic preferences was used in Benoit and Kornhauser (1994) to guarantee that seat-by-seat plurality elects an efficient assembly.

An important corollary of Proposition 2 is the following:

Corollary 1. *If every r_i is Condorcet-consistent then* $\mathrm{Seq}(r_1, \ldots, r_n)$ *is Condorcet-consistent.*

Therefore, the output of a sequential voting rule will be the CW when there exists one, provided that each local rule r_i is Condorcet-consistent. This applies in particular to sequential majority on domains composed of binary issues, which was already known in the particular case when all voters have separable preferences (see Lacy and Niou (2000)). This allows us to claim that *the restriction to $\mathcal{O}$-legal profiles allows for escaping multiple election paradoxes*, at least the version of the paradox that deals with Condorcet winners failing to be elected. For the version of the paradox concerning Condorcet losers, a sequential voting rule never elects a Condorcet loser, provided that each of its local rules never does either[8]:

Proposition 4. *If there exists $i \leq p$ such that r_i never elects a Condorcet loser, then* $\mathrm{Seq}(r_1, \ldots, r_p)$ *never elects a Condorcet loser.*

Proof. If r_i never elects a Condorcet loser and for a profile P, $\mathrm{Seq}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p)$ is the Condorcet loser of P, then for any $d'_i \in D_i$, $(d_1, \ldots, d_{i-1}, d'_i, d_{i+1}, \ldots, d_p)$ Condorcet-dominates $(d_1, \ldots, d_p)$. Therefore in $P^{\mathbf{x}_i|\mathbf{x}_1=d_1\ldots\mathbf{x}_{i-1}=d_{i-1}}$, d'_i Condorcet-dominates d_i, which means

7 Even worse, there may exist a candidate unanimously preferred to the sequential Condorcet winner even when preferences are separable (see Section 4.1.2 of Laslier (2004) and Benoit and Kornhauser (2006)).

8 This holds even if the profile is not legal; see Lacy and Niou (2000) for the case of separable preferences.

d_i is the Condorcet loser in $P^{\mathbf{x}_i|\mathbf{x}_1=d_1\ldots\mathbf{x}_{i-1}=d_{i-1}}$. Since r_i does not select a Condorcet loser, $r_i(P^{\mathbf{x}_i|\mathbf{x}_1=d_1\ldots\mathbf{x}_{i-1}=d_{i-1}}) \neq d_i$. This contradicts $\text{Seq}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p)$. So $\text{Seq}(r_1, \ldots, r_p)(P)$ is not a Condorcet loser. ■

6. Properties of sequential voting rules

Apart from Condorcet-consistency, there are many other classical properties that voting rules may (or may not) satisfy. A voting rule satisfies

anonymity if it is unsensitive to any permutation of the voters;
neutrality if for any profile P and any permutation M on candidates, $r(M(P)) = M(r(P))$.
monotonicity if for any profiles $P = (V_1, \ldots, V_N)$ and $P' = (V'_1, \ldots, V'_N)$ such that each V'_i is obtained from V_i by raising only $r(P)$, we have $r(P') = r(P)$.
strong monotonicity if for any profile P, any $Y \subseteq \mathcal{X}$, and any P' obtained from P only by raising the candidates in Y while keeping their relative position unchanged, we have $r(P') \in r(P) \cup Y$.
consistency (or **reinforcement**) if for any two disjoint profiles (given by two disjoint electorates) P_1, P_2 such that $r(P_1) = r(P_2)$, we have $r(P_1 \cup P_2) = r(P_1) = r(P_2)$.
participation if for any profile P and any vote V, $r(P \cup \{V\}) \succ_V r(P)$.
efficiency (or **consensus**) if for any profile $P = (V_1, \ldots, V_N)$, there is no candidate c such that $c \succ_{V_i} r(P)$ for all $i \leq N$.

Since sequential voting rules are sequential composition of multiple local rules, we may wonder whether the properties of local rules carry on to their sequential composition, and vice versa.

In the whole section we fix $\mathcal{O} = (\mathbf{x}_1 > \cdots > \mathbf{x}_p)$, and we let $P^{\mathbf{x}_i:d_1\ldots d_{i-1}}$ be $P^{\mathbf{x}_i:\mathbf{x}_1=d_1\ldots\mathbf{x}_{i-1}=d_{i-1}}$.

6.1. From local rules to sequential rules

We first give results on whether the sequential composition of local rules inherits a given property satisfied by all local rules.

First, we need to make an important remark. Sequential compositions of voting rules are defined for $\mathcal{O}$-legal profiles only, therefore, when we say that $\text{Seq}(r_1, \ldots, r_p)$ satisfies a property involving several profiles, we mean that the property holds for all $\mathcal{O}$-legal profiles. In some cases, the restriction to $\mathcal{O}$-legal profiles renders the property much weaker. This applies especially to neutrality and monotonicity. Indeed, the usual definition of neutrality is not directly applicable to sequential voting rules, because permuting two alternatives in a $\mathcal{O}$-legal profile generally results in a profile that is not $\mathcal{O}$-legal. Therefore, the definition that we take is a straightforward generalization of s-neutrality as defined in Benoit and Kornhauser (2006): a sequential voting rule $\text{Seq}(r_1, \ldots, r_p)$ on *Legal*($\mathcal{O}$) is neutral[9] if for any permutation M on $\mathcal{X}$ and any $\mathcal{O}$-legal profile P, *if* $M(P)$ *is* $\mathcal{O}$*-legal*, then $M(\text{Seq}(r_1, \ldots, r_p)(P)) = \text{Seq}(r_1, \ldots, r_p)(M(P))$. Things are similar, and even more drastic, for monotonicity (we will come back on this later).

We start by the positive results:

Proposition 5. *If for all* $1 \leq i \leq p$, r_i *satisfies anonymity (resp. consistency, strong monotonicity), then* $\text{Seq}(r_1, \ldots, r_p)$ *also satisfies anonymity (resp. consistency, strong monotonicity).*

Proof. The proof for anonymity is straightforward.

Consistency Assume that $r_1, \ldots, r_p$ all satisfy consistency. Let P_1 and P_2 be two profiles on D such that

$$\text{Seq}(r_1, \ldots, r_p)(P_1) = \text{Seq}(r_1, \ldots, r_p)(P_2) = (d_1, \ldots, d_p),$$

[9] We choose to call this property of sequential voting rules *neutrality* rather than *s-neutrality*; it is not ambiguous, provided that $\mathcal{O}$ is fixed.

which means that for every $i = 1, \ldots, n$,

$$r_i(P_1^{\mathbf{x}_i|d_1\ldots d_{i-1}}) = r_i(P_2^{\mathbf{x}_i|d_1\ldots d_{i-1}}) = d_i. \tag{1}$$

From the consistency of r_1, together with $d_1 = r_1(P_1^{\mathbf{x}_1}) = r_1(P_2^{\mathbf{x}_1})$, we get $r_1(P_1^{\mathbf{x}_1} \cup P_2^{\mathbf{x}_1}) = r_1((P_1 \cup P_2)^{\mathbf{x}_1}) = d_1$.

Let $i \geq 2$ and suppose that $r_k(P_1 \cup P_2)^{\mathbf{x}_k|d_1\ldots d_{k-1}} = d_k$ holds for all $k \leq i$. From (1), $r_{i+1}(P_1^{\mathbf{x}_{i+1}|d_1\ldots d_i}) = r_{i+1}(P_2^{\mathbf{x}_{i+1}|d_1\ldots d_i}) = d_{i+1}$. Now, from the consistency of r_{i+1}, we have

$$r_{i+1}(P_1^{\mathbf{x}_{i+1}|d_1\ldots d_i} \cup P_2^{\mathbf{x}_{i+1}|d_1\ldots d_i}) = r_{i+1}((P_1 \cup P_2)^{\mathbf{x}_{i+1}|d_1\ldots d_i}) = d_{i+1}.$$

Therefore, $\mathrm{Seq}(r_1, \ldots, r_p)(P_1 \cup P_2) = \mathrm{Seq}(r_1, \ldots, r_p)(P_1) = \mathrm{Seq}(r_1, \ldots, r_p)(P_2)$, and $\mathrm{Seq}(r_1, \ldots, r_p)$ satisfies consistency.

Strong monotonicity For any $Y \subseteq \mathcal{X}$ and $(d_1 \ldots d_{i-1}) \in D_1 \times \cdots \times D_{i-1}$, we write

$$Y^{\mathbf{x}_i|d_1\ldots d_{i-1}} = \{x_i : \vec{x} \in Y, x_j = d_j \text{ for all } j \leq i-1\}.$$

Suppose $r_1, \ldots, r_p$ all satisfy strong monotonicity. First we prove that for any profiles P and P', if P' is obtained from P by raising candidates in Y, then

$$\mathrm{Seq}(r_1, \ldots, r_p)(P')|_{\mathbf{x}_1} \in \{\mathrm{Seq}(r_1, \ldots, r_p)(P')|_{\mathbf{x}_1}\} \cup Y^{\mathbf{x}_1} \tag{2}$$

where $\mathrm{Seq}(r_1, \ldots, r_p)(P')|_{\mathbf{x}_1}$ is the $\mathbf{x}_1$ component of $\mathrm{Seq}(r_1, \ldots, r_p)(P')$. To prove this, we only need to check that $P'^{\mathbf{x}_1}$ is obtained from $P^{\mathbf{x}_1}$ by raising $Y^{\mathbf{x}_1}$. By strong monotonicity of r_1, it suffices to check for any $V \in P$ and its counterpart $V' \in P'$, that for any $y \in Y^{\mathbf{x}_1}$ and $x \in D_1$,

$$y \succ_V^{\mathbf{x}_1} x \Rightarrow y \succ_{V'}^{\mathbf{x}_1} x.$$

If not, suppose $y \succ_V^{\mathbf{x}_1} x$ but $x \succ_{V'}^{\mathbf{x}_1} y$, and $(y, \vec{d}_2) \in Y$ for some $\vec{d}_2 \in D_2 \times \cdots \times D_p$. Then we know that $(y, \vec{d}_2) \succ_V (x, \vec{d}_2)$ and $(x, \vec{d}_2) \succ_{V'} (y, \vec{d}_2)$. Since V' is obtained from V by raising candidates in Y, for any $\vec{d} \in Y$ we have

$$\{\vec{x} : \vec{x} \succ_{V'} \vec{d}\} \subseteq \{\vec{x} : \vec{x} \succ_V \vec{d}\}.$$

Take $\vec{d} = (y, \vec{d}_2)$, it follows that $(x, \vec{d}_2) \in \{\vec{x} : \vec{x} \succ_{V'} \vec{d}\}$, and $(x, \vec{d}_2) \notin \{\vec{x} : \vec{x} \succ_V \vec{d}\}$, which leads to a contradiction.

Therefore, we know that Eq. (2) holds. Denote $w_1 = r_1(P'^{\mathbf{x}_1})$. Now there are two cases: $w_1 \neq r_1(P^{\mathbf{x}_1})$ and $w_1 = r_1(P^{\mathbf{x}_1})$. For the first case there must exist $V \in P$ such that the rank of w_1 in $V'^{\mathbf{x}_1}$ is higher than the rank of w_1 in $V^{\mathbf{x}_1}$. If not, then $V'^{\mathbf{x}_1}$ is obtained from $V^{\mathbf{x}_1}$ by raising candidates in $Y^{\mathbf{x}} \setminus \{w_1\}$ for all $V \in P$, so by strong monotonicity of r_1, $w_1 \in (\{r_1(P^{\mathbf{x}_1})\} \cup Y^{\mathbf{x}_1}) \setminus \{w_1\}$, which leads to a contradiction. Suppose there exist $V \in P$ and $y \in D_1$ such that $y \succ_V^{\mathbf{x}_1} w_1$ and $w_1 \succ_{V'}^{\mathbf{x}_1} y$. Then we know that for all $\vec{d}_2 \in D_2 \times \ldots D_p$, $(w_1, \vec{d}_2) \succ_{V'} (y, \vec{d}_2)$ and $(y, \vec{d}_2) \succ_V (w_1, \vec{d}_2)$. Therefore in V', $(w_1, \vec{d}_2)$ must be raised, which means that $\{w_2\} \times D_2 \times \cdots \times D_p \subseteq Y$. So $\mathrm{Seq}(r_1, \ldots, r_p)(P') \in Y$.

For the second case, we can move to the second step of sequential voting process, and fix $\mathbf{x}_1 = w_1$. Then following the same proof we have that $\mathrm{Seq}(r_1, \ldots, r_p)(P') \in Y$ or $r_2(P^{\mathbf{x}_2|w_1}) = r_2(P'^{\mathbf{x}_2|w_1})$. Repeating this process recursively, finally we get that $\mathrm{Seq}(r_1, \ldots, r_p)(P') \in Y$ or $\mathrm{Seq}(r_1, \ldots, r_p)(P') = \mathrm{Seq}(r_1, \ldots, r_p)(P)$, which completes the proof. ■

We now consider monotonicity. We get the seemingly strange result that the monotonicity of $\mathrm{Seq}(r_1, \ldots, r_p)$ depends *only* on the monotonicity of the last rule r_p.

Proposition 6. *If r_p satisfies monotonicity, then* $\mathrm{Seq}(r_1, \ldots, r_p)$ *also satisfies monotonicity.*

This seemingly strange result is mainly due to the fact that the restriction to $\mathcal{O}$-legal profiles considerably restricts the set of pairs consisting of two profiles P and P' such that P' is obtained from P by raising exactly one candidate. This is stated more precisely by the following Lemma:

Lemma 1. *Let V, W be two preference relations on $D_1 \times \cdots \times D_n$ such that* (1) *W is obtained from V by raising one candidate $\vec{x} = (x_1, \ldots, x_p)$, and* (2) *$V$ and W are $\mathcal{O}$-legal. Then* (a) *for every $i \leq p-1$, and every $(d_1, \ldots, d_{i-1}) \in D_1 \times \cdots \times D_{i-1}$, we have $V^{\mathbf{x}_i|d_1,\ldots,d_{i-1}} = W^{\mathbf{x}_i|d_1,\ldots,d_{i-1}}$ and* (b) *$W^{\mathbf{x}_i|x_1,\ldots,x_{i-1}}$ is obtained from $W^{\mathbf{x}_i|x_1,\ldots,x_{i-1}}$ by raising x_p.*

Proof. Let V, W be as specified, V extend $\mathcal{N}_V$ and W extend $\mathcal{N}_W$, and assume that there exist $i \leq p-1$ and $s_j \in D_j, j < i$ such that

$$W^{\mathbf{x}_i|s_1\ldots s_{i-1}} \neq V^{\mathbf{x}_i|s_1\ldots s_{i-1}}.$$

Then there exist two values $s_i, s_i' \in D_i$ such that $s_1 \ldots s_{i-1} : s_i \succ_{\mathcal{N}_V} s_i'$ and $s_1 \ldots s_{i-1} : s_i' \succ_{\mathcal{N}_W} s_i$. Choose any $\vec{v}_1, \vec{v}_2 \in D_{i+1} \times \cdots \times D_p$ such that $\vec{v}_1 \neq \vec{v}_2$. Then the relative order of two pairs: $(s_1, \ldots, s_i, \vec{v}_1)$ and $(s_1, \ldots, s_i', \vec{v}_1)$, $(s_1, \ldots, s_i, \vec{v}_2)$ and $(s_1, \ldots, s_i', \vec{v}_2)$, are exchanged when moving from V to W. Now, assumption 1 implies that if a pair of candidates is ordered differently in V and W, then the pair must contain $\vec{x}$. Now, the four candidates involved in the latter two pairs are all different from one another, therefore they cannot both contain $\vec{x}$, hence a contradiction, which proves part (a) of the lemma. Part (b) is proved in a similar way from the observation that any pair ordered differently in V and W must contain $\vec{d}$. ■

Proof of Proposition 6. Let $P = (V_1, \ldots, V_N)$ be an $\mathcal{O}$-legal profile and $Q = (W_1, \ldots, W_N)$ an $\mathcal{O}$-legal profile obtained by raising only $\mathrm{Seq}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p)$. From Lemma 1 we get:

1. $W_j^{\mathbf{x}_i|\vec{s}_i} = V_j^{\mathbf{x}_i|\vec{s}_i}$ for all $i \leq N$, $i \leq p-1$, $\vec{s}_i \in D_1 \times \cdots \times D_{i-1}$
2. $W_j^{\mathbf{x}_p|d_1\ldots d_{p-1}}$ is obtained by $V_j^{\mathbf{x}_p|d_1\ldots d_{p-1}}$ by raising d_p.

So from the definition of $\mathrm{Seq}(r_1, \ldots, r_p)$, we know that r_i selects d_i from Q for all $i \leq p-1$ and that r_p selects $r_p(W_1^{\mathbf{x}_p|d_1\ldots d_{p-1}}, \ldots, W_N^{\mathbf{x}_p|d_1\ldots d_{p-1}})$. Since r_p satisfies monotonicity, we have

$$r_p(W_1^{\mathbf{x}_p|d_1\ldots d_{p-1}}, \ldots, W_N^{\mathbf{x}_p|d_1\ldots d_{p-1}}) = d_p.$$

Hence $\mathrm{Seq}(r_1, \ldots, r_p)(Q) = (d_1, \ldots, d_p) = \mathrm{Seq}(r_1, \ldots, r_p)(P)$, which proves that $\mathrm{Seq}(r_1, \ldots, r_p)$ satisfies monotonicity. ■

On the other hand, three important properties cannot be lifted from local rules to their sequential composition: neutrality, efficiency, and participation. In the case of efficiency, this was remarked by several authors in the more specific case of multiple referenda with separable preferences. In particular, Özkal-Sanver and Sanver (2006) prove that if there are at least three binary issues (or two binary issues and an even number of voters) then the parallel composition of the majority rule is not efficient (although the majority rule is, of course, efficient). The issue was developed further in Benoit and Kornhauser (2006), who prove the following: if the number of issues is at least 3, or if there are two issues, one of which has a domain with at least 3 values, then a (parallel) composition of local voting rules satisfies efficiency if and only if it is a dictatorship (this, a fortiori, applies to sequential voting rules).

The next example shows that participation cannot be lifted from local rules to their sequential composition.

Example 8. Let $I = \{\mathbf{x}, \mathbf{y}\}$, with $D_{\mathbf{x}} = \{0_x, 1_x, 2_x\}$ and $D_Y = \{0_y, 1_y\}$. Take r_1 to be the scoring rule with score vector $(3, 2, 0)$, and r_2 to be the majority rule. r_1 and r_2 satisfy participation. Consider now the following three votes:

$$V_1, V_2: \quad 0_x1_y \succ 0_x0_y \succ 1_x0_y \succ 1_x1_y \succ 2_x0_y \succ 2_x1_y$$
$$V_3: \quad 1_x1_y \succ 2_x1_y \succ 0_x1_y \succ 1_x0_y \succ 2_x0_y \succ 0_x0_y$$

V_1, V_2 and V_3 are all in *Legal*($\mathbf{x} > \mathbf{y}$) and the associated conditional preference tables are

V_1, V_2	V_3
$0_x \succ 1_x \succ 2_x$	$1_x \succ 2_x \succ 0_x$
$0_x : 1_y \succ 0_y$	$0_x : 1_y \succ 0_y$
$1_x : 0_y \succ 1_y$	$1_x : 1_y \succ 0_y$
$2_x : 0_y \succ 1_y$	$2_x : 1_y \succ 0_y$

Let $P = \{V_1, V_2\}$. We have $\text{Seq}(r_1, r_2)(P) = 0_x 1_y$. Now, let $P' = \{V_1, V_2, V_3\}$: we have $\text{Seq}(r_1, r_2)(P') = 1_x 0_y$. However, voter 3 prefers $0_x 1_y$ to $1_x 0_y$, thus she has no interest in participating, which shows that $\text{Seq}(r_1, r_2)$ does not satisfy participation.

6.2. From sequential rules to local rules

Now we focus on the reverse direction, namely, whether the individual local rules inherit a given property from their sequential composition, or, more intuitively, whether the failure of one of the local rules to satisfy some given property implies that the sequential composition fails to satisfy the property as well.

Proposition 7. *If for some $i \in \{1, \dots, p\}$, r_i does not satisfy anonymity (resp. neutrality, consistency, participation, efficiency, strong monotonicity), then* $\text{Seq}(r_1, \dots, r_p)$ *does not satisfy anonymity (resp. neutrality, consistency, participation, efficiency, strong monotonicity).*

Proof. The proof for anonymity is straightforward.

Neutrality Assume that for some $i \le N$, r_i is not neutral. Then there exists a permutation M^i on D_i and a profile $P^i = (V_1^i, \dots, V_N^i)$ on D_i such that

$$M^i(r_i(P_i)) \neq r_i(M^i(P^i)). \tag{3}$$

Then, we construct the following separable profile Q on D: $Q = (W_1, \dots, W_N)$ where for every $k = 1, \dots, N$, $W_k^{\mathbf{x}_i} = V_k^i$ (and whatever local preferences for other variables than $\mathbf{x}_i$). Then, we define a permutation M on D such that for every $\vec{x} = (d_1, \dots, d_i) \in D$,

$$M(\vec{x}) = (d_1, \dots, d_{i-1}, M^i(d_i), d_{i+1} \dots, d_p).$$

Suppose that $\text{Seq}(r_1, \dots, r_p)(P) = (d_1, \dots, d_p)$. Then

$$\text{Seq}(r_1, \dots, r_p)(M(P)) = (d_1, \dots, d_{i-1}, r_i(M^i(P^i)), d_{i+1}, \dots, d_p).$$

Now, from Eq. (3) we have that

$$(d_1, \dots, d_{i-1}, r_i(M^i(P^i)), d_{i+1}, \dots, d_p) \neq (d_1, \dots, d_{i-1}, M^i(r_i(P_i)), d_{i+1}, \dots, d_p)$$

that is, $\text{Seq}(r_1, \dots, r_p)(M(P)) \neq M(\text{Seq}(r_1, \dots, r_p)(P))$, which shows that $\text{Seq}(r_1, \dots, r_p)$ is not neutral.

Participation Suppose that r_i does not satisfy participation, which means that there exists a profile $P^i = (V_1^i, \dots, V_N^i)$ and a vote V_{N+1}^i on D_i such that

$$r_i(P^i) \succ_{V_{N+1}^i} r_i(P^i \cup \{V_{N+1}^i\}).$$

Let us construct the following separable profile $P = (V_1, \dots, V_N, V_{N+1})$ on $\mathcal{X}$ as follows:

1. for any $k = 1, \dots, N+1$, $V_k^{\mathbf{x}_i} = V_k^i$.
2. for any $j \neq i$, the voters have whatever preferences on issue $\mathbf{x}_j$.

Then, $\text{Seq}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_{i-1}, r_i(P^i), d_{i+1}, \ldots, d_p)$ for some $(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_p) \in D_1 \times \cdots \times D_{i-1} \times D_{i+1} \times \cdots \times D_p$, whereas $\text{Seq}(r_1, \ldots, r_p)(P \cup \{V_{N+1}\}) = (d_1, \ldots, d_{i-1}, r_i(P^i \cup \{V^i_{N+1}\}), d_{i+1}, \ldots, d_p)$. Assume that voter $N + 1$ has a lexicographic preference relation, where the most important variable is $\mathbf{x}_i$ (such a preference relation is of course separable). Then, because $r_i(P^i) \succ_{V^i_{N+1}} r_i(P^i \cup \{V^i_{N+1}\})$, we have that $\text{Seq}(r_1, \ldots, r_p)(P) \succ_{V_{N+1}} \text{Seq}(r_1, \ldots, r_p)(P \cup \{V_{N+1}\})$, hence shows $\text{Seq}(r_1, \ldots, r_p)$ does not satisfy participation.

Efficiency Assume r_i does not satisfy efficiency. Then there exists a profile $P^i = (V^i_1, \ldots, V^i_N)$ on D_i and a value $d_i \in D_i$ such that for all $V^i \in P^i$, $d_i \succ_{V^i} r_i(P^i)$. We construct the following separable profile $P = (V_1, \ldots, V_N)$ on D similarly as in the proof for participation above: for any $k = 1, \ldots, N$, $V^{\mathbf{x}_i}_k = V^i_k$, and for all $j \neq i$, the voters have whatever preferences on issue $\mathbf{x}_j$. Now, assume all voters have a lexicographic preference relation, where the most important variable is $\mathbf{x}_i$. We have

$$\text{Seq}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_{i-1}, r_i(P^i), d_{i+1}, \ldots, d_p).$$

But for any $V_j \in P$,

$$(d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_p) \succ_{V_j} (d_1, \ldots, d_{i-1}, r_i(P^i), d_{i+1}, \ldots, d_p),$$

hence $\text{Seq}(r_1, \ldots, r_p)$ does not satisfy efficiency.

Consistency The proof is again very similar as for the two previous properties. Assume there exists $i \le p$ such that r_i does not satisfy consistency, then there exist two profiles

$$P^i = (V^i_1, \ldots, V^i_{N_1}), \qquad Q^i = (W^i_1, \ldots, W^i_{N_2})$$

on D_i such that $r_i(P^i) = r_i(Q^i)$ and $r_i(P^i \cup Q^i) \neq r_i(P^i_1)$. We construct two separable profiles

$$P = (V_1, \ldots, V_{N_1}), \qquad Q = (W_1, \ldots, W_{N_2})$$

such that

1. for all $k \le N_1$, $V^{\mathbf{x}_i}_k = V^i_k$ and for all $l \le N_2$, $W^{\mathbf{x}_i}_l = W^i_l$.
2. the local preference on issue $\mathbf{x}_j, j \neq i$ of the $N_1 + N_2$ voters in $P \cup Q$ all coincide: for every $j \neq i$, every $k \le N_1$ and every $l \le N_2$, $V^{\mathbf{x}_j}_j = W^{\mathbf{x}_j}_j = \succ^j$.

Now, we have that

$$\text{Seq}(r_1, \ldots, r_p)(P) = \text{Seq}(r_1, \ldots, r_p)(Q) = (d_1, \ldots, d_{i-1}, r_i(P^i), d_{i+1}, \ldots, d_p)$$

where for every $j \neq i$, $d_j = r_j(\succ^j)$. Then,

$$\text{Seq}(r_1, \ldots, r_p)(P \cup Q) = (d_1, \ldots, d_{i-1}, r_i(P^i \cup Q^i), d_{i+1}, \ldots, d_p).$$

Therefore $\text{Seq}(r_1, \ldots, r_p)(P \cup Q) \neq \text{Seq}(r_1, \ldots, r_p)(P)$. This shows that $\text{Seq}(r_1, \ldots, r_p)$ does not satisfy consistency.

Strong monotonicity Suppose that r_i does not satisfy strong monotonicity. This means that there exist two profiles P^i, Q^i on D_i such that Q^i is obtained from P^i by raising candidates in some subset Y of D_i, and $r_i(Q^i) \notin \{r_i(P_i) \cup Y\}$. Let $P^i = (V^i_1, \ldots, V^i_N)$, we construct two collections of CP-nets $(\mathcal{N}_1, \ldots, \mathcal{N}_N)$, $(\mathcal{N}'_1, \ldots, \mathcal{N}'_N)$ by lifting each linear order in P^i and Q^i to a linear order on $\mathcal{X}$ similarly as in the proof of Proposition 8. Denote P and Q the resulting profiles over $\mathcal{X}$. Then it is easy to see P' is obtained from P by raising candidates in $Y_{\mathcal{X}} = D_1 \times \ldots D_{i-1} \times Y \times D_{i+1} \times \cdots \times D_p$. Since $r_i(P'^i) \notin \{r_i(P_i) \cup Y\}$, we get

$$\text{Seq}(r_1, \ldots, r_p)(P') \notin \{\text{Seq}(r_1, \ldots, r_p)(P)\} \cup Y_{\mathcal{X}}.$$

This shows that $\text{Seq}(r_1, \ldots, r_p)$ does not satisfy strong monotonicity. ■

The following result is the converse of Proposition 6: if the *last* voting rule fails to satisfy monotonicity then so does the sequential composition.

Proposition 8. *If r_p does not satisfy monotonicity, then* $\text{Seq}(r_1, \ldots, r_p)$ *does not satisfy monotonicity.*

Proof. Assume that r_p does not satisfy monotonicity. Then there exist two profiles $P^p = (V_1^p, \ldots, V_N^p)$ on D_p and $Q^p = (W_1^p, \ldots, W_N^p)$ such that W_i^p is obtained from V_i^p by raising $r_p(P^p)$ and keeping the relative positions of other values of $\mathbf{x}_p$ unchanged, and $r_p(Q^p) \neq r_p(P^p)$,

We now construct a separable profile $V = (V_1, \ldots, V_N)$ such that for every voter $k \leq N$, $V_k^{\mathbf{x}_p} = V_k^p$ and whatever preferences on the other issues than $\mathbf{x}^p$. Denote $(d_1, \ldots, d_p) = \text{Seq}(r_1, \ldots, r_p)(V_1, \ldots, V_N)$. Next, we construct the profile W consisting of the CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_p$ in $Legal(G)$, whose common graph G contain the edges $\{(\mathbf{x}_i, \mathbf{x}_p) \mid 1 \leq i \leq p-1\}$, and such that the preference table of every voter k, for $k = 1, \ldots, \leq N$, is defined as follows:

- for every issue $\mathbf{x}_i$, $i \neq p$: $W_k^{\mathbf{x}_i} = V_k^{\mathbf{x}_i}$;
- $W_k^{\mathbf{x}_p|\mathbf{x}_1=d_1,\ldots,\mathbf{x}_{p-1}=d_{p-1}} = W_k^p$;
- for every tuple of values $(d'_1, \ldots, d'_{p-1}) \neq (d_1, \ldots, d_{p-1})$, $W_k^{\mathbf{x}_p|\mathbf{x}_1=d'_1,\ldots,\mathbf{x}_{p-1}=d'_{p-1}} = V_k^p$.

We note that W is obtained from V by raising $(d_1, \ldots, d_{p-1}, r_p(P^p))$, the relative order of other alternatives being unchanged, exactly the same as Q^p is obtained from P^p by raising $r_p(P^p)$. Now,

$$\text{Seq}(r_1, \ldots, r_p)(W_1, \ldots, W_N) = (d_1, \ldots, d_{p-1}, r_p(Q^p)),$$
$$\text{Seq}(r_1, \ldots, r_p)(V_1, \ldots, V_N) = (d_1, \ldots, d_{p-1}, r_p(P^p)),$$

and since $r_p(Q^p) \neq r_p(P^p)$, we have

$$\text{Seq}(r_1, \ldots, r_p)(W_1, \ldots, W_N) \neq \text{Seq}(r_1, \ldots, r_p)(V_1, \ldots, V_N),$$

therefore $\text{Seq}(r_1, \ldots, r_p)$ does not satisfy monotonicity. ■

For the same reasons as in Proposition 6, the failure of $r_1, \ldots, r_{p-1}$ to satisfy monotonicity does not imply that $\text{Seq}(r_1, \ldots, r_p)$ fails to satisfy monotonicity.

6.3. Summary

The following table summarizes the results of this section.

Criteria	Global to local	Local to global
Anonymity	Y	Y
Neutrality	Y	N
Monotonicity	Only r_p	Only r_p
Consistency	Y	Y
Participation	Y	N
Efficiency	Y	N
Strong monotonicity	Y	Y

We end this section with some considerations on manipulability. We know that the majority rule for 2 candidates is not manipulable. What about the sequential composition of majority rules? We know from Lacy and Niou (2000) that if *all* voters have separable preferences, then sequential majority is non-manipulable. Does this extend to legal profiles in which some voters have nonseparable preferences? Unfortunately, it does not:

Proposition 9. *Let $I = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of binary issues, with $p \geq 2$. For each $i = 1, \ldots, p$, let maj_i be the majority rule on $\{x_i, \bar{x}_i\}$ (plus some tie-breaking mechanism). Then* $\text{Seq}(maj_1, \ldots, maj_p)$ *is manipulable.*

Proof. We give a counterexample for two binary issues (it is straightforward to extend it to more than two issues). Consider two binary issues **x** and **y**, and the following 3-voters profile:

$$\begin{aligned}
\text{voter 1}: &\quad (x, y) \succ (\bar{x}, y) \succ (\bar{x}, \bar{y}) \succ (\bar{x}, \bar{y}) \\
\text{voter 2}: &\quad (x, \bar{y}) \succ (x, y) \succ (\bar{x}, y) \succ (\bar{x}, \bar{y}) \\
\text{voter 3}: &\quad (\bar{x}, y) \succ (\bar{x}, \bar{y}) \succ (x, \bar{y}) \succ (x, y).
\end{aligned}$$

The profile is in *Legal*($\mathbf{x} > \mathbf{y}$). (Note that voter 1's preference order is separable). If 1 knows the preferences of 2 and 3 then he has no interest to vote sincerely on issue **x**, *even though his preference relation is separable*: if he votes sincerely, then he votes x and then the outcome is $x\bar{y}$. If he votes for $\bar{x}$ instead, then the outcome is $\bar{x}y$, which is better to him. ■

As a corollary of this result, strategyproofness does not transfer from the local level to the global level.

See Le Breton and Sen (1999) for a more general study of strategyproofness of voting rules on combinatorial domains, under the separability assumption.

7. Discussion

We have shown that the sequential composition of local voting rules allows for escaping usual multiple election paradoxes, under a domain restriction much weaker than separability. We have established many results concerning the transfer (or the failure of transfer) of important properties from local rules to/from their sequential composition.

Our work has benefited from several previous streams of work that were almost unrelated: on the one hand, social choice, and on the other hand, conditional preferential independence, initially developed in the literature of multiattribute decision making and now widely used in Artificial Intelligence (with CP-nets).

The sequential combination of local rules advocated in this paper relies on the crucial assumption that all voters' preference relations follow a common acyclic graph, or, equivalently, that they are legal with respect to a common order of the issues. Even if this domain restriction is much less demanding than separability (as it allows for many more profiles), it is not innocuous.

As it stands, sequential voting in the way that was advocated in this paper should be applied when the choice of the structure underlying the CP-net is clear enough (such as in the case of a meal, where main dish is *obviously* the first variable to vote on, etc.). In the absence of such a graph, it should be clear that imposing to the voters an arbitrary graph (or an arbitrary order on the issues) that does not fit their preferences will not only make them feel uncomfortable, as they might be unable to vote "simply" (Benoit and Kornhauser, 1991), but also will possibly lead to a paradoxical outcome. This leads us to wonder whether we can find weaker domain restrictions, or better, to find a way of conducting multiple elections without any domain restriction.

A first possible relaxation of the G-legality restriction consists in giving up the requirement that the graph G is known from the beginning, and then generalize the (fixed-order) sequential compositions of voting rules to *ordered-independent* composition of voting rules, which apply to $\mathcal{O}$-legal profiles for *some* order $\mathcal{O}$ (not fixed from the beginning). This way is pursued in Xia et al. (2007b). However, this generalization comes with a cost: the loss of the cheap sequential protocol that elicits local preferences from the voters one issue after the other. Now, it is possible to go much further and generalize the framework developed in this article into a family of voting rules applicable to all profiles. The idea developed in Xia et al. (2008) goes along this way: the acyclicity restriction is dropped, and every voter expresses a CP-net (with whatever structural graph – the graphs needs not being common to all voters, nor do they need to be acyclic); these CP-nets are then aggregated into a common CP-net, from which a winner is determined. However, this generalization comes with a major drawback: not only the elicitation is extremely costly (voters may have report an exponentially large input), but the computation of the winner is also extremely costly. A similar solution consists in expressing voters' preferences in some other compact representation language than CP-nets (see, noticeably, Gonzales et al. (2008) for the aggregation of cardinal preferences expressed by GAI-nets), but again, in that case, most voting rules are computationally hard to apply (see Lang (2004)). In summary, there seems to be a choice to make between a strong domain restriction (such as G-legality, for G acyclic), or a wider applicability coming with a prohibitive communication and computation cost.

References

Benoit, J.-P., Kornhauser, L., 1994. Social choice in a representative democracy. American Political Science Review 88 (1), 185–192.

Benoit, J.-P., Kornhauser, L.A., 1991. Voting simply in the election of assemblies. Technical Report RR#91-32. C.V. Starr Center Working Papers.

Benoit, J.-P., Kornhauser, L.A., 2006. Only a dictatorship is efficient or neutral. Technical Report.

Boutilier, C., Brafman, R., Domshlak, C., Hoos, H., Poole, D., 2004a. CP-nets: A tool for representing and reasoning with conditional *ceteris paribus* statements. Journal of Artificial Intelligence Research 21, 135–191.

Boutilier, C., Brafman, R.I., Domshlak, C., Hoos, H., Poole, D., 2004b. Preference-based constrained optimization with CP-nets. Computational Intelligence 20 (2), 137–157.

Bradley, W.J., Hodge, J.K., Kilgour, D.M., 2005. Separable discrete preferences. Mathematical Social Science 49 (3), 335–353.

Brams, S.J., Kilgour, D.M., Zwicker, W.S., 1997. Voting on referenda: the separability problem and possible solutions. Electoral Studies 16 (3), 359–377.

Brams, S.J., Kilgour, D.M., Zwicker, W.S., 1998. The paradox of multiple elections. Social Choice and Welfare 15 (2), 211–236.

Debreu, G., 1960. Topological methods in cardinal utility theory. In: Arrow, K.J., Karlin, S., Suppes, P. (Eds.), Mathematical Methods in the Social Sciences. Stanford University Press, pp. 116–26.

Gonzales, Ch., Perny, P., Queiroz, S., 2008. Preference aggregation with graphical utility models. In: Proceedings of AAAI-08. pp. 1037–1042.

Gorman, W.M., 1968. The structure of utility functions. Review of Economic Studies 35, 367–390.

Hodge, J., 2002. Separable preference orders. Ph.D. Thesis. Western Michigan University.

Hodge, J.K., Ter Haar, M., 2008. Classifying interdependence in multidimensional binary preferences. Mathematical Social Science 55, 190–204.

Keeney, R., Raiffa, H., 1976. Decision with Multiple Objectives: Preferences and Value Tradeoffs. Wiley and Sons.

Lacy, D., Niou, E., 2000. A problem with referenda. Journal of Theoretical Politics 12 (1), 5–31.

Laffond, G., Lainé, J., Laslier, J.-F., 1996. Composition-consistent tournament solutions and social choice functions. Social Choice and Welfare 13, 75–93.

Lang, J., 2004. Logical preference representation and combinatorial vote. Annals of Mathematics and Artificial Intelligence 42 (1), 37–71.

Lang, J., 2007. Voting and aggregation on combinatorial domains with structured preferences, in: Proceedings of the Twentieth Joint International Conference on Artificial Intelligence, IJCAI'07, pp. 1366–1371.

Laslier, J.-F., 2004. Le vote et la règle majoritaire, CNRS Editions.

Le Breton, M., Sen, A., 1999. Separable preferences, strategyproofness, and decomposability. Econometrica 67 (3), 605–628.

Özkal-Sanver, İ., Sanver, R., 2006. Ensuring pareto-optimality by referendum voting. Social Choice and Welfare 27, 211–219.

Plott, C., Levine, M., 1978. A model of agenda influence on committee decisions. The American Economic Review 68 (1), 146–160.

Saari, D., Sieberg, K., 2001. The sum of the parts can violate the whole. The American Political Science Review 95 (2), 415–433.

Scarsini, M., 1998. A strong paradox of multiple elections. Social Choice and Welfare 15 (2), 237–238.

Xia, L., Conitzer, V., Lang, J., 2008. Voting on multiattribute domains with cyclic preferential dependencies. In: Proceedings of AAAI-08. AAAI Press, pp. 202–207.

Xia, L., Lang, J., Ying, M., 2007a. Sequential voting and multiple election paradoxes, in: Proceedings of the 11th Conference on the Theoretical Aspects of Rationality and Knowledge (TARK 2007).

Xia, L., Lang, J., Ying, M., 2007b. Strongly decomposable voting rules on multiattribute domains. In: Proceedings of AAAI-07. AAAI Press, pp. 776–781.