Representing and Solving Hedonic Games with Ordinal Preferences and Thresholds

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ABSTRACT

We propose a new representation setting for hedonic games, where each agent partitions the set of other agents into friends, enemies, and neutral agents, with friends and enemies being ranked. Under the assumption that preferences are monotonic (respectively, anti-monotonic) with respect to the addition of friends (respectively, enemies), we propose a bipolar extension of the Bossong–Schweigert extension principle, and use this principle to derive the (partial) preferences of agents over coalitions. Then, for a number of solution concepts, we characterize partitions that necessarily (respectively, possibly) satisfy them, and identify the computational complexity of the associated decision problems. Alternatively, we suggest cardinal comparability functions in order to extend to complete preference orders consistent with the generalized Bossong–Schweigert order.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences—Economics

General Terms

Economics, Theory

Keywords

Computational Social Choice, Coalition Formation, Game Theory

1. INTRODUCTION

Hedonic games are strategic games where agents, from a set A, are free to form coalitions. Each agent has a preference relation over the set of all coalitions containing her; various solution concepts—such as individual rationality, Nash stability, individual contractual stability, core stability, and so on—have been proposed and studied. However, an important bottleneck is how the agents' preferences over all coalitions that contain them are expressed. As there are exponentially many coalitions containing agent i, it is unreasonable to expect that agent i should express explicitly a ranking (or a utility function) over all these coalitions. This issue is often addressed by assuming that only a small part of the preference relation is expressed by the agent, and that this small part is then extended into a complete preference relation over coalitions. Various assumptions about the nature of the input (what the agents express) and the preference extension have been made in the literature (for a survey, see Woeginger [23]):

1. The *individually rational encoding* [4]: Each agent ranks only the coalitions she prefers to herself being alone.

2. The *additive encoding* [21, 22, 3, 24]: Each agent gives a valuation (positive or negative) of each other agent; preferences are additively separable, and the extension principle is that the valuation of a set of agents, for agent i, is the sum of the valuations i gives to the agents in the set (and then the preference relation is derived from this valuation function).

3. The *"friends and enemies" encoding* [15, 21]: Each agent partitions the set of other agents into two sets (her friends and her enemies); under the *friend-oriented preference extension*, coalition X is preferred to coalition Y if X contains more friends than Y, or as many friends as Y and fewer enemies than Y; under the *enemy-oriented preference* extension, X is preferred to Y if X contains fewer enemies than Y, or as many enemies as Y and more friends than Y.

4. The *singleton encoding* [12, 10, 11]: Each agent ranks only single agents; under the optimistic (respectively, pessimistic) extension, X is preferred to Y if the best (respectively, worst) agent in X is preferred to the best (respectively, worst) agent in Y.

5. The *anonymous encoding* [4, 13]: Each agent specifies only a preference relation over the number of agents in her coalition (and does not care about the identities of these agents).

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6. *Hedonic coalition nets* [16]: Each agent specifies her utility function over the set of all coalitions via (more or less) a set of weighted logical formulas.

7. *Fractional hedonic games* [2]: Each agent assigns a value to each other agent (and 0 to herself); an agent's utility of a coalition is the average value she assigns to the members of the coalition. A coalition X is preferred to Y if the utility of X is greater than that of Y.

Naturally, compact representation either does not avoid exponential-size representations in the worst case (Case 1 and, to a lesser extent, Case 6), or comes with a loss of expressivity, corresponding to a demanding domain restriction, such as separable preferences (Cases 2 and 4), anonymous preferences (Case 5), or other domain restrictions that do not bear a specific name (Case 3).

In Cases 2 and 6, preferences are expressed numerically: Agents do explicitly express numbers. In all other cases, they are expressed ordinally. Advantages of ordinal preferences in social choice have been discussed many times and we want to stick here to ordinality. We do not want to make the very demanding anonymity assumption, which does not allow to distinguish between agents. The individually rational encoding is not compact in general. So there remain only the "friends and enemies" and singleton encodings. The problem with "friends and enemies" is that an agent cannot express preferences inside the friend set nor inside the enemy set: Preferences over individual agents are dichotomous (but preferences between coalitions are not, because they depend on the number of friends and enemies). The problem with the singleton encoding is that having simply a rank $\triangleright_i$ for each agent i does not tell us which agents i would like to see in her coalitions and which agents she would like not to: For instance, if $\triangleright_1$ is $2 \triangleright_1 3 \triangleright_1 4$, we know that 1 prefers 2 to 3 and 3 to 4, but nothing tells us whether 1 prefers to be with 2 (respectively, 3 and 4) to being alone, that is, if the absolute desirability of $2, 3$, and 4 is positive or negative (of course, if it is negative for 3, it is also negative for 4, etc.). So, both ways are insufficiently informative: Specifying only a partition into positive and negative agents ("friends" and "enemies") does not tell which of her friends i prefers to which other agents, and which of her enemies she wants to avoid most. On the other hand, specifying a ranking over agents does not say which agents i prefers to be with rather than being alone. Here we propose a model that integrates the models of Cases 1, 3, and 4: Each agent i first subdivides the other agents into three groups, her friends, her enemies, and an intermediate type of agents on which she has neither a positive nor a negative opinion and then specifies a ranking of her friends and enemies. Based on this representation, we consider a natural extension to a player's preference, the generalized Bossong–Schweigert extension (see [8, 14]), which is a partial order over coalitions containing the player. A related model can be found in the context of matching theory: Responsive preferences are studied in bipartite many-to-one matching markets and consider the comparison of one participant to another,[1] although not in distinction of friends or enemies (see, e.g., [19, 20]). In the following, we consider different ways of how to deal with incomparabilities within these partial orders. A first approach is to leave incomparabilities open and define notions such as "possible" and "necessary" stability concepts. A second approach is to define comparability functions in order to determine the relation between incomparable coalitions that extend the generalized Bossong–Schweigert extension to a total preference order for each player. Questions of interest include appropriate characterizations of stability concepts and a computational study of the related problems.

[1] In the context of many-to-one matching markets, an agent on the one side has *responsive preferences* over assignments of the agents on the other side if, for any two assignments that differ in only one agent, the assignment containing the most preferred agent is preferred.

2. PRELIMINARIES

Generally, a *hedonic game* is a tuple (A, P) consisting of a set of *players* (or *agents*) $A = \{1, 2, \dots, n\}$ and a profile of preference relations $P = (\succeq_1, \succeq_2, \dots, \succeq_n)$ defining for each player a weak preference order over all possible *coalitions* $C \subseteq A$ containing the player herself. For two coalitions $C, D \subseteq A$, both containing player i, we say that *i weakly prefers* C to D if $C \succeq_i D$; *i prefers* C to D, denoted by $C \succ_i D$, if $C \succeq_i D$, but not $D \succeq_i C$; and *i is indifferent between* C and D, denoted by $C \sim_i D$, if both $C \succeq_i D$, and $D \succeq_i C$. A *coalition structure* Γ for a given game (A, P) is a partition of A into disjoint coalitions, and for each player $i \in A$, $\Gamma(i)$ denotes the unique coalition in Γ containing i.

An important solution concept for the study of hedonic games is the notion of stability of a coalition structure. There are several known such stability concepts [7, 3, 1]. In this paper we focus on concepts that deal with avoiding a player to deviate to another (possibly empty) existing coalition. Relatedly, other commonly studied concepts consider group deviations, such as core stability with the goal that there is no blocking coalition. A third group of stability concepts, such as Pareto optimality and popularity, is based on a relation comparing different coalition structures. Further restrictions of games as well as properties can be found amongst others in [5].

A coalition structure Γ is called

- *perfect* if each player i weakly prefers $\Gamma(i)$ to every other coalition containing i,
- *individually rational* if each player $i \in A$ weakly prefers $\Gamma(i)$ to being alone in $\{i\}$,
- *Nash stable* if for each player $i \in A$, $\Gamma(i) \succeq_i A' \cup \{i\}$ holds for each coalition $A' \in \Gamma \cup \emptyset$,
- *individually stable* if for each player $i \in A$ and for each coalition $A' \in \Gamma \cup \emptyset$, it holds that $\Gamma(i) \succeq_i A' \cup \{i\}$ or there exists a player $j \in A'$ such that $A' \succ_j A' \cup \{i\}$,
- *contractually individually stable* if for each player $i \in A$ and for each coalition $A' \in \Gamma \cup \emptyset$, it holds that $\Gamma(i) \succeq_i A' \cup \{i\}$, or there exists a player $j \in A'$ such that $A' \succ_j A' \cup \{i\}$, or there exists a player $j' \in \Gamma(i)$ such that $\Gamma(i) \succ_{j'} \Gamma(i) \smallsetminus \{i\}$.

3. DERIVING PREFERENCES OVER COALITIONS FROM PREFERENCES OVER SINGLE FRIENDS AND ENEMIES

We define a new representation of preferences combining ordinal rankings with friend and enemy sets. We suggest deriving a player's preference over coalitions by generalizing the Bossong–Schweigert extension principle.

3.1 Ordinal Preferences with Thresholds

DEFINITION 1. *Let $A = \{1, 2, \dots, n\}$ be a set of agents. For each $i \in A$, a* weak ranking with double threshold *for agent i, denoted by $\trianglerighteq_i^{+0-}$, consists of a partition of $A \smallsetminus \{i\}$ into three sets:*

- *A_i^+ (i's friends), together with a weak order $\trianglerighteq_i^+$ over A_i^+,*
- *A_i^- (i's enemies), together with a weak order $\trianglerighteq_i^-$ over A_i^-, and*

- A_i^0 *(the neutral agents, i.e., the agents i does not care about).*

We also write $\trianglerighteq_i^{+0-}$ *as* $(\trianglerighteq_i^+ \,|\, j_1 \cdots j_k \,|\, \trianglerighteq_i^-)$ *for* $A_i^0 = \{j_1, \ldots, j_k\}$.

Not having an order of the neutral agents can be interpreted as being indifferent about them all: $j_a \sim_i j_b$ for all $j_a, j_b \in A_i^0$. Agent i strictly prefers all her friends to her neutral agents, and those to her enemies. The *weak order induced by* $\trianglerighteq_i^{+0-}$ is therefore defined via $f \triangleright_i j$, for each $f \in A_i^+$ and $j \in A_i^0$, $j_1 \sim_i j_2 \sim_i \cdots \sim_i j_k$, and $j \triangleright_i e$, for each $j \in A_i^0$ and $e \in A_i^-$.

EXAMPLE 2. *Let* $A = \{1, 2, \ldots, 11\}$. *Then,*

$$\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 4 \,|\, 567 \,|\, 8 \triangleright_1 9 \sim_1 10 \triangleright_1 11)$$

means that 1 *likes* 2*,* 3*, and* 4 *(and prefers* 2 *to both* 3 *and* 4*, and is indifferent between* 3 *and* 4*);* 1 *does not care about* 5*,* 6*, and* 7 *(and is indifferent between them); and* 1 *does not like* 8*,* 9*,* 10*, and* 11 *(but still prefers* 8 *to* 9 *and* 10*, is indifferent between* 9 *and* 10*, and prefers* 9 *and* 10 *to* 11*). The weak order* $\trianglerighteq_1$ *induced by* $\trianglerighteq_1^{+0-}$ *is* $2 \triangleright_1 3 \sim_1 4 \triangleright_1 5 \sim_1 6 \sim_1 7 \triangleright_1 8 \triangleright_1 9 \sim_1 10 \triangleright_1 11$. *Note that here the preference between a friend and a neutral player is strict, because we assume below that a coalition containing a friend instead of a neutral player is preferred. Analogously, the preference between a neutral player and an enemy is strict, because a player does not care about having a neutral player in a coalition but is less happy with having an enemy in the coalition instead.*

3.2 Generalizing Bossong–Schweigert Extensions

DEFINITION 3. *Let* $\trianglerighteq_i^{+0-}$ *be a weak ranking with double threshold for agent i. The extended order* $\succeq_i^{+0-}$ *is defined as follows: For every* $X, Y \subseteq A$, $X \succeq_i^{+0-} Y$ *if and only if the following two conditions hold:*

1. *There is an injective function* σ *from* $Y \cap A_i^+$ *to* $X \cap A_i^+$ *such that for every* $y \in Y \cap A_i^+$*, we have* $\sigma(y) \trianglerighteq_i y$.
2. *There is an injective function* θ *from* $X \cap A_i^-$ *to* $Y \cap A_i^-$ *such that for every* $x \in X \cap A_i^-$*, we have* $x \trianglerighteq_i \theta(x)$.

Finally, $X \succ_i^{+0-} Y$ *if and only if* $X \succeq_i^{+0-} Y$ *and not* $(Y \succeq_i^{+0-} X)$.

Intuitively speaking, for a fixed coalition C adding a further friend makes the coalition strictly more valuable while adding an enemy causes the opposite. When exchanging two friends, the valuation of the coalition changes depending on the relation between the exchanged players (the same holds when two enemies are exchanged). When both a friend and an enemy are added or are both removed, the original and the new coalition are incomparable with respect to the Bossong–Schweigert extension principle.

Thus, to construct the *generalized Bossong–Schweigert extension* (GBS-extension, for short) for a player i, we start with the coalition containing i and her set of friends (which is the most preferred coalition) and then construct all directly comparable coalitions by adding enemies, removing friends, or exchanging enemies or friends. For each newly obtained coalition we repeat this procedure until we reach the least preferred coalition containing all of i's enemies. Note that the elements of A_i^0 are disregarded as their adding to or removing from a coalition does not change the value of a coalition. The following examples illustrate the just presented extension principle.

EXAMPLE 4. *For* $A = \{1, 2, \ldots, 6\}$*, consider*

$$\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 4 \,|\, \,|\, 5 \triangleright_1 6).$$

The graph in Figure 1 shows the generalized Bossong–Schweigert extension of this preference, where an arc from coalition X to coalition Y implies that $X \succ_1^{+0-} Y$. *Hence, any path leading from* X' *to* Y' *implies* $X' \succ_1^{+0-} Y'$*, whereas coalitions that are not connected by a path, such as* $\{1,2,3\}$ *and* $\{1,2,3,4,5\}$*, are incomparable.*

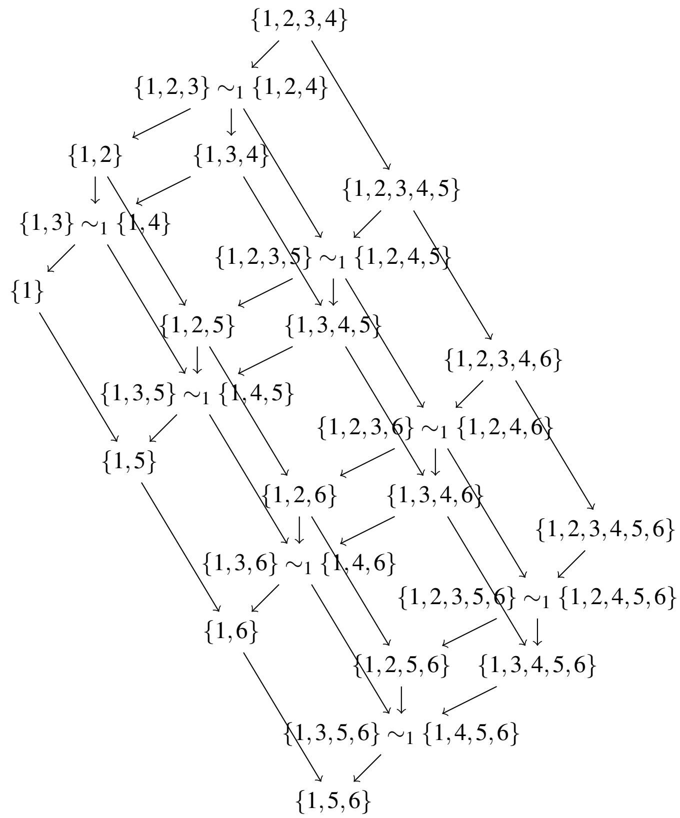


Figure 1: The generalized Bossong–Schweigert extension of $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 4 \,|\, \,|\, 5 \triangleright_1 6)$**.**

Note that if there were additional players $j > 6$ in A considered as neutral by player 1, the general picture would be the same with indifferences at each level, for any $C \subseteq \{2, \ldots, 6\}$, between each $\{1\} \cup C \cup N$ for $N \subseteq A \setminus \{1, \ldots, 6\}$.

EXAMPLE 5. *Consider* $A = \{1, 2, 3, 4, 5\}$ *and the first players' preference* $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \,|\, \,|\, 4 \triangleright_1 5)$. *The graph in Figure 2 shows the generalized Bossong–Schweigert extension of this preference using the same notation as in Example 4.*

Using the generalized Bossong–Schweigert extension principle, we can extend the given preferences of the players to a preference over the possible coalitions. However, this preference over the coalitions might be incomplete; there are coalitions that remain incomparable. We consider two possibilities to deal with these incomparabilities: Leave them open and consider every possible extension that does not conflict with transitivity; alternatively, determine the relation between incomparable coalitions by adapting the Borda scoring rule, which is well-known from voting theory.

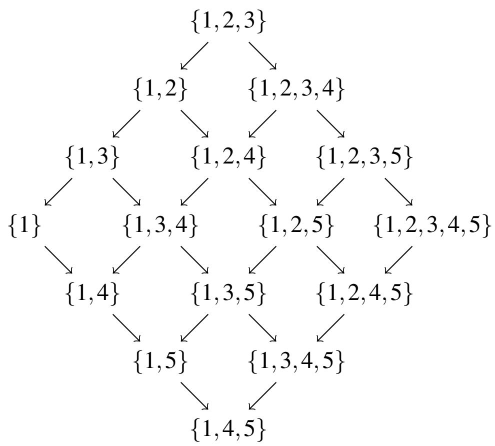


Figure 2: The generalized Bossong–Schweigert order of $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \mid \mid 4 \triangleright_1 5)$.

Intuitively, the relation between two coalitions C and D ($C \succ_i D$, $D \succ_i C$, $C \sim_i D$, or undecided) from player i's point of view can be determined by the following characterizations. These characterizations are inspired by Bouveret et al. [9] who show characterizations for the original Bossong–Schweigert order in the context of fair division.

PROPOSITION 6. *1. Let $\trianglerighteq_i^{+0-}$ be a weak ranking with double threshold for agent i, and let C and C' be two coalitions containing i. Consider the orders $f_1 \trianglerighteq_i f_2 \trianglerighteq_i \cdots \trianglerighteq_i f_\mu$ with $\{f_1, f_2, \ldots, f_\mu\} = C \cap A_i^+$ and $f'_1 \trianglerighteq_i f'_2 \trianglerighteq_i \cdots \trianglerighteq_i f'_{\mu'}$ with $\{f'_1, f'_2, \ldots, f'_{\mu'}\} = C' \cap A_i^+$, as well as $e_1 \trianglerighteq_i e_2 \trianglerighteq_i \cdots \trianglerighteq_i e_\nu$ with $\{e_1, e_2, \ldots, e_\nu\} = C \cap A_i^-$ and $e'_1 \trianglerighteq_i e'_2 \trianglerighteq_i \cdots \trianglerighteq_i e'_{\nu'}$ with $\{e'_1, e'_2, \ldots, e'_{\nu'}\} = C' \cap A_i^-$. Then, $C \succ_i^{+0-} C'$ if and only if*

(a) $\mu \geq \mu'$ and $\nu \leq \nu'$,

(b) for each k, $1 \leq k \leq \mu'$, it holds that $f_k \trianglerighteq_i f'_k$, and

(c) for each ℓ, $1 \leq \ell \leq \nu$, it holds that $e_{\nu-\ell+1} \trianglerighteq_i e'_{\nu'-\ell+1}$.

2. Say that $w_i : A \to \mathbb{R}$ is compatible with $\trianglerighteq_i^{+0-}$ if and only if

- *for each $j \in A_i^+$, we have $w_i(j) > 0$;*
- *for each $j \in A_i^-$, we have $w_i(j) < 0$;*
- *for each $j \in A_i^0$, we have $w_i(j) = 0$; and*
- *for all $j, k \in A_i^+ \cup A_i^-$, we have $j \triangleright_i k$ if and only if $w_i(j) > w_i(k)$.*

Then, $C \succ_i^{+0-} C'$ if and only if for any w_i compatible with $\trianglerighteq_i^{+0-}$, we have $\sum_{j \in C} w_i(j) > \sum_{j' \in C'} w_i(j')$.

PROOF. 1. Obviously, if (a) to (c) hold, the two injective functions $\sigma : C' \cap A_i^+ \to C \cap A_i^+$, and $\theta : C \cap A_i^- \to C' \cap A_i^-$ mapping $f'_k \mapsto f_k$ for each k, $1 \leq k \leq \mu'$, and $e_{\nu-\ell+1} \mapsto e'_{\nu'-\ell+1}$ for each ℓ, $1 \leq \ell \leq \nu$, satisfy $\sigma(f'_k) \trianglerighteq_i f'_k$ and $e_{\nu-\ell+1} \trianglerighteq_i \theta(e_{\nu-\ell+1})$, for the same range of k and ℓ. On the other hand, if there are two injective functions with the desired requirements, (a) holds. If there was a k with $f'_k \triangleright_i f_k$ (or an ℓ with $e'_{\nu'-\ell+1} \triangleright_i e_{\nu-\ell+1}$), this would imply $\sigma(f'_k) = f_j$ for a $j < k$ (or $\theta(e_{\nu-\ell+1}) = e'_{\nu-j+1}$ with $j > \ell$, respectively). This, however, implies that either a requirement is violated for f'_1 (or e_ν), or that σ (or θ) is not injective, a contradiction.

2. Assume that $C \succ_i^{+0-} C'$, that is, $C \succeq_i^{+0-} C'$ and not $C' \succeq_i^{+0-} C$. For the set of friends A_i^+, with $F = C \cap A_i^+$ and $F' = C' \cap A_i^+$, it follows that there is an injective function $\sigma : F' \to F$ such that for each $y \in F'$, we have $\sigma(y) \trianglerighteq_i y$. Hence, for each compatible w_i, $w_i(\sigma(y)) \geq w_i(y)$. Thus, since σ is injective,

$$\sum_{j \in F} w_i(j) \geq \sum_{j \in \sigma(F') \subseteq F} w_i(j) = \sum_{j' \in F'} w_i(\sigma(j')) \geq \sum_{j' \in F'} w_i(j'). \quad (1)$$

Similarly, for A_i^-, with $E = C \cap A_i^-$ and $E' = C' \cap A_i^-$, and θ injective, it holds that

$$0 \geq \sum_{j \in E} w_i(j) \geq \sum_{j \in E} w_i(\theta(j)) = \sum_{j' \in \theta(E) \subseteq E'} w_i(j') \geq \sum_{j' \in E'} w_i(j'). \quad (2)$$

Since $C' \succeq_i^{+0-} C$ does not hold, at least one of the inequalities (1) and (2) is strict, since one preference ($\sigma(j') \triangleright_i j'$ or $j \triangleright_i \theta(j)$) or one inclusion ($\sigma(F') \subset F$ or $\theta(E) \subset E'$) is strict. For each player $j \in A_i^0$, we have $w_i(j) = 0$; therefore, in total,

$$\sum_{j \in C} w_j > \sum_{j' \in C'} w_{j'}. \quad (3)$$

Now assume that for each compatible w_i, (3) holds. Thus,

$$\sum_{j \in F} w_i(j) - \sum_{j' \in E'} w_i(j') > \sum_{j' \in F'} w_i(j') - \sum_{j \in E} w_i(j).$$

Assume there were no injective function mapping from each summand from the right-hand side to one at least as large on the left hand side; then, there exists an assignment to the values of w_i compatible with $\trianglerighteq_i^{+0-}$ that does not satisfy the inequality, a contradiction. Hence, such a function must exist, and this function induces the mappings σ and θ, showing $C \succeq_i^{+0-} C'$. Additionally, because the inequality is strict in (3), $C' \succeq_i^{+0-} C$ does not hold, which completes the proof.

This completes the proof. ❑

4. POSSIBLE/NECESSARY STABILITY

As we have seen above, the generalized Bossong–Schweigert extension can leave uncertainties between two coalitions in a player's preference order.

DEFINITION 7. *A complete preference relation $\succeq_i$ over all coalitions containing i* extends *$\succeq_i^{+0-}$ if and only if it contains it; that is, if $C \succeq_i^{+0-} D$ implies $C \succeq_i D$ for all coalitions C, D. Let $\mathrm{Ext}(\succeq_i^{+0-})$ be the set of all complete preference relations extending $\succeq_i^{+0-}$.*

Now we can define games where each player has friends, enemies, and neutral co-players, and preferences over the former two sets such that we can derive each player's preference relation as introduced in the previous section.

DEFINITION 8. *An* FEN-hedonic game *is a tuple $H = \langle A, \trianglerighteq_1^{+0-}, \ldots, \trianglerighteq_n^{+0-} \rangle$, where $A = \{1, 2, \ldots, n\}$ is a set of players, and $\trianglerighteq_i^{+0-}$ gives the ordinal preferences with thresholds of player $i \in A$ as defined in Definition 1.*

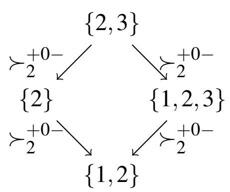
DEFINITION 9. *Let α be a stability concept for hedonic games, $\langle A, \trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-} \rangle$ be an FEN-hedonic game and Γ be a coalition structure. Γ satisfies* possible α *if and only if there exists a profile $\langle \succeq_1, \dots, \succeq_n \rangle$ in $\times_{i=1}^{n} \text{Ext}(\succeq_i^{+0-})$ such that $\langle A, \succeq_1, \dots, \succeq_n \rangle$ satisfies α. Γ satisfies* necessary α *if and only if for each $\langle \succeq_1, \dots, \succeq_n \rangle$ in $\times_{i=1}^{n} \text{Ext}(\succeq_i^{+0-})$, $\langle A, \succeq_1, \dots, \succeq_n \rangle$ satisfies α.*

EXAMPLE 10. *Let $A = \{1,2,3\}$, $\trianglerighteq_1^{+0-} = (2 \rhd_1 3 \mid \mid)$, $\trianglerighteq_2^{+0-} = (3 \mid \mid 1)$, and $\trianglerighteq_3^{+0-} = (1 \mid 2 \mid)$.*

The generalized Bossong–Schweigert orders are

$$\{1,2,3\} \succ_1^{+0-} \{1,2\} \succ_1^{+0-} \{1,3\} \succ_1^{+0-} \{1\}$$

for player 1,



for player 2, and for player 3

$$\{1,3\} \sim_3^{+0-} \{1,2,3\} \succ_3^{+0-} \{3\} \sim_3^{+0-} \{2,3\}.$$

So, two preferences are already complete, and there are three complete preferences extending $\succeq_2^{+0-}$, one setting $\{2\} \succ_2 \{1,2,3\}$, another setting $\{2\} \sim_2 \{1,2,3\}$, and the third setting $\{1,2,3\} \succ_2 \{2\}$, leaving all other relations the same.

4.1 Properties and Characterizations

Observe first that there always exists a necessarily individually rational coalition structure (namely, the coalition structure where every agent is alone).

PROPOSITION 11. *Consider an FEN-hedonic game $\langle A, \trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-} \rangle$.*

1. *A coalition structure Γ is (necessarily and possibly) perfect if and only if for each player i, $A_i^+ \subseteq \Gamma(i)$ and $A_i^- \cap \Gamma(i) = \emptyset$.*[2]
2. *A coalition structure Γ is possibly individually rational if and only if for each $i \in A$, $\Gamma(i)$ contains at least a friend of i's or only neutral agents.*
3. *A coalition structure Γ is necessarily individually rational if and only if for each $i \in A$, $\Gamma(i)$ does not contain any enemies of i's.*
4. *A coalition structure Γ is necessarily individually stable if and only if it is necessarily individually rational and no player i can join a coalition that she would possibly prefer and the members of which do not see her as an enemy.*

PROOF. 1. A coalition structure is perfect if and only if each player is in one of her favorite coalitions, that is, each player is together with all her friends and no enemies.

2. For each $i \in A$, i necessarily prefers $\{i\}$ to $\Gamma(i)$ if and only if $\Gamma(i)$ contains no friend and at least one enemy of i's.

3. For each $i \in A$, i possibly prefers $\{i\}$ to $\Gamma(i)$ if and only if $\Gamma(i)$ contains an enemy of $i's$.

4. Note that a player j possibly prefers a coalition C to $C \cup \{i\}$ if and only if j necessarily prefers C to $C \cup \{i\}$ if and only if i is an enemy of j's. Assume that Γ is necessarily individually stable. Then, for each $i \in A$, if i prefers to move to another (possibly empty) coalition C in Γ, there is a player in C that prefers player i not being in the coalition. If C is empty, there is no such player, thus, Γ has to be individually rational. Hence, C is nonempty and there has to be a player in C that sees i as an enemy. Now assume that Γ is not individually stable, that is, there is a player i and a coalition $C \in \Gamma \cup \{\emptyset\}$ such that i prefers $C \cup \{i\}$ to $\Gamma(i)$ and, for each $j \in C$, $C \cup \{i\} \succeq_j C$. If $C = \emptyset$, then Γ is not individually rational. Otherwise, each j does not see i as an enemy.

This completes the proof. ❑

[2] As a consequence, a possibly perfect coalition structure in an FEN-hedonic game is always necessarily perfect.

Note that a similar characterization holds for contractually individual stability, where additionally to the conditions of individual stability, it is required that no j in $\Gamma(i)$ considers i a friend.

EXAMPLE 12. *Consider the FEN-hedonic game from Example 10. Observe that there does not exist a (possibly) perfect coalition structure. While $\{\{1,2,3\}\}$ is possibly Nash stable, there does not exist a necessarily Nash stable coalition structure, as in each of five cases, player 1 or player 2, at least possibly, wants to move to another coalition. Coalition structure $\{\{1,2,3\}\}$ is possibly individually rational, but not necessarily due to player 2; $\{\{1,2\},\{3\}\}$ is not possibly individually rational; the other three coalition structures are necessarily individual rational.*

For $\{\{1,3\},\{2\}\}$ it holds that player 2 possibly wants to move to $\{1,3\}$ and 1 and 2 do not see 2 as an enemy, thus necessary individual stability is not satisfied. Also, since in $\{2\}$ there is no other player who considers 2 a friend, contractually individual stability is not satisfied either. Observe that this coalition structure is, however, possibly individually stable.

Coalition structure $\{\{1\},\{2,3\}\}$ is not necessarily individually stable, as player 3 wants to move to $\{1,3\}$ where 1 welcomes him. Player 2, however, considers 3 a friend, thus, as 2 does not want to move, and 1 is considered an enemy by 2 when moving to $\{2,3\}$, this coalition structure is contractually individually stable.

4.2 Complexity of Possible and Necessary Stability Problems

We are interested in axiomatic properties and characterizations of stability concepts in FEN-hedonic games. However, for some concepts no general statements can be made as to whether there exists a coalition structure satisfying a stability concept α (possibly or necessarily). In these cases we ask how hard it is to decide whether for a given FEN-hedonic game a given coalition structure possibly or necessarily satisfies α, and to decide whether there exists a coalition structure in a given FEN-hedonic game that possibly or necessarily satisfies α. Similar questions are often analyzed in the context of hedonic games [24, 3, 18]. Here, we redefine the verification and existence problems to the notions of possible and necessary existence.

Note that two interpretations of necessary existence can be distinguished, the first one asking whether there always exists a coalition structure that satisfies α, while the second one is asking whether a particular coalition structure necessarily satisfies α. Intuitively this distinction makes sense, since in the first case the setting might provide a central authority with partial knowledge of the agents' preferences and require the knowledge that whatever the possible preferences are, there is always some coalition structure satisfying α; in the second case, the choice of coalition structure is independent of the agents' possible preferences.

EXAMPLE 13. *For example, consider the following game with three players,* $A = \{1,2,3\}$, *with* $\trianglerighteq_1^{+0-} = (2 \mid 3 \mid)$, $\trianglerighteq_2^{+0-} = (1 \mid 3 \mid)$, *and* $\trianglerighteq_3^{+0-} = (1 \mid \mid 2)$. *We obtain the following generalized Bossong–Schweigert orders:* $\{1,2\} \sim_1 \{1,2,3\} \succ_1 \{1\} \sim_1 \{1,3\}$, $\{1,2\} \sim_2 \{1,2,3\} \succ_2 \{2\} \sim_2 \{2,3\}$, *and* $\{1,3\} \succ_3 \{3\} \succ_3 \{2,3\}$ *and* $\{1,3\} \succ_3 \{1,2,3\} \succ_3 \{2,3\}$, *while* 3 *is undecided between* $\{3\}$ *and* $\{1,2,3\}$. *Any coalition structure in which players* 1 *and* 2 *are not in the same coalition cannot possibly be Nash stable. On the one hand,* $\{\{1,2\},\{3\}\}$ *is Nash stable if and only if an extension provides* $\{3\} \succeq_3 \{1,2,3\}$. *On the other hand,* $\{\{1,2,3\}\}$ *is Nash stable if and only if* $\{1,2,3\} \succeq_3 \{3\}$ *in an extension. Thus, for every extension, there certainly exists a Nash stable coalition structure. However, there is no necessarily Nash stable coalition structure.*

Here, we focus on the second interpretation. Possible existence is unambiguous, asking whether there is some coalition structure satisfying α for some extension.

PROPOSITION 14. *All problems regarding perfection are in* P.

PROOF. Verfication of whether a coalition structure is possibly and necessarily perfect is easy by Proposition 11.

Existence can be decided by, e.g., the following algorithm: Start with player 1 and let $\Gamma(1) := \{1\} \cup A_1^+$. Sequentially, for each $i \in \Gamma(1)$, add A_i^+ to $\Gamma(1)$ until there are no further possible changes. Check whether, for each $i \in \Gamma(1)$, $A_i^- \cap \Gamma(1) = \emptyset$. If not, output "there is no perfect coalition structure"; if so, start over with $A \setminus \Gamma(1)$. It might be the case that a friend cannot be added, because he is already assigned to another coalition. If he is on his own, add him anyway; otherwise, output "there is no perfect coalition structure." Continue until each player is allocated to a coalition. Then, output "there is a perfect coalition structure."

Note that this algorithm works in polynomial time. ❑

All problems regarding individual rationality are in P by the characterizations in Proposition 11 and the observation preceding it.

Proposition 11 does not provide a characterization of Nash stability. Nevertheless, it can be verified in polynomial time whether a given coalition structure in a given FEN-hedonic game is necessarily Nash stable.

LEMMA 15. *The verification problem for possible Nash stability is in* P.

PROOF. Given an FEN-hedonic game and a coalition structure Γ, verify the following steps for each $i \in A$: For each (of at most n coalitions) $C \in \Gamma \cup \{\emptyset\}$, $C \neq \Gamma(i)$, determine the relation between $\Gamma(i)$ and $C \cup \{i\}$. This can be done in polynomial time by Proposition 6.1. If $C \cup \{i\} \succ_i \Gamma(i)$, output "$\Gamma$ is not Nash stable." If the relation is undecided, output "Γ is possibly not Nash stable." Otherwise, if this is not true for any player or coalition in $\Gamma \cup \{\emptyset\}$, output "$\Gamma$ is necessarily Nash stable." ❑

By the characterizations in Proposition 11, similar algorithms work for individual and contractually individual stability. Note that this cannot easily be transferred to possible Nash stability, since resolving an undecided relation might influence another relation for the same player.

THEOREM 16. *The problem of whether there exists a possibly Nash stable coalition structure in a given FEN-hedonic game is* NP*-complete.*

PROOF. The problem belongs to NP, since it is enough to check whether there exists a coalition structure of A and an extension persuing the GBS-extension such that for each player $i \in A$ and each coalition $C \in \Gamma$, $\Gamma(i) \succeq_i C \cup \{i\}$. The latter can be tested in time polynomial in $n = \|A\|$, since there are at most n coalitions in Γ and the relation between two coalitions from a common player's perspective can be decided in polynomial time by Proposition 6.1.

NP-hardness can be shown via a polynomial-time many-one reduction from EXACT-COVER-BY-THREE-SETS (X3C, see [17]): Given a set R with $3m$ elements and a family $\mathscr{S}$ of subsets $s \subseteq R$ with $\|s\| = 3$, is there an exact cover of R in $\mathscr{S}$, that is, is there a subset $S \subseteq \mathscr{S}$ such that $\cup_{s \in S} s = R$ and $\|S\| = m$? Without loss of generality it can be assumed that $m \geq 2$ and each element in R occurs at most three times in a set in $\mathscr{S}$. Given such an X3C instance, we construct the following game. This construction is inspired by the construction of the proof that it is NP-hard to decide whether there exists a Nash stable coalition structure in an additively separable hedonic game [22, Theorem 3]. Here, however, several adjustments have to be made in order to guarantee necessary preferences over coalitions.[3] Let

$$\begin{aligned} A &= \{\alpha_i \mid 1 \leq i \leq 3m-1\} \cup \{\beta_r \mid r \in R\} \\ &\cup \{\zeta_{s,k} \mid s \in \mathscr{S}, 1 \leq k \leq 3m-2\} \end{aligned}$$

and let the players' preferences be defined as follows, where in player i's preference and for a set $X = \{a_1, a_2, \ldots, a_x\}$, $X_\sim$ denotes $a_1 \sim_i a_2 \sim_i \cdots \sim_i a_x$

- $\trianglerighteq_{\alpha_i}^{+0-} = \big(\alpha_{i+1} \mid \{\alpha_j : i \neq j \neq i+1\}_\sim \mid \{\text{other players}\}_\sim\big)$, for each i, $1 \leq i \leq 3m-2$,

  $\trianglerighteq_{\alpha_{3m-1}}^{+0-} = \big(\mid \{\alpha_j : j \neq 3m-1\}_\sim \mid \{\text{other players}\}_\sim\big)$,

- $\trianglerighteq_{\beta_r}^{+0-} = \big(\{\alpha_i : 1 \leq i \leq 3m-1\}_\sim \triangleright_{\beta_r} \bigcup_{r \in s} Q_{s\sim} \triangleright_{\beta_r} \{\beta_{r'} : r' \neq r\}_\sim \mid \mid \{\text{other players}\}_\sim\big)$, for each $r \in R$,

- $\trianglerighteq_{\zeta_{s,k}}^{+0-} = \big(\zeta_{s,k+1} \mid \{\zeta_{s,k'} : k \neq k' \neq k+1\} \cup \{\beta_r : r \in s\}_\sim \mid \{\text{other players}\}_\sim\big)$, for each $s \in \mathscr{S}$, and k, $1 \leq k \leq 3m-3$,

  $\trianglerighteq_{\zeta_{s,3m-2}}^{+0-} = \big(\mid \{\zeta_{s,k'} : k' \neq 3m-2\} \cup \{\beta_r : r \in s\}_\sim \mid \{\text{other players}\}_\sim\big)$, for each $s \in \mathscr{S}$

where $Q_s = \{\zeta_{s,k} \mid 1 \leq k \leq 3m-2\}$ for each $s \in \mathscr{S}$. Moreover, let $P_s = \{\beta_r \mid r \in s\} \cup Q_s$. This profile can be constructed in polynomial time, since there are $n \leq 3m + 3m + 3m \cdot (3m-2)$ players, and each player's preference can be written in linear time in n.

We now show that $(R, \mathscr{S})$ is a positive instance for X3C if and only if there exists a possibly Nash stable coalition structure in the GBS-extension of the constructed game.

Only if: Assume there exists a solution S for $(R, \mathscr{S})$. Consider the coalition structure

$$\Gamma = \{\{\alpha_i \mid 1 \leq i \leq 3m-1\}\} \cup \{P_s \mid s \in S\} \cup \{Q_s \mid s \notin S\}.$$

[3] Consider, e.g., a coalition $\{i, f, e\}$ where player i has a positive value for f, and a negative value for e. In comparison to $\{i\}$ this coalition is preferred by player i if f has a greater absolute value than e in the additively separable representation, is considered indifferent if f and e have the same absolute value, and is less preferred otherwise. If we do not provide values but ordinal preferences and thresholds and consider f as a friend and e as an enemy of i's, $\{i, f, e\}$ and $\{i\}$ are incomparable from i's perspective; thus, all three scenarios are possible in an extension persuing GBS.

By a close look at all possibly empty coalitions in Γ it can be seen that no α_i, $1 \le i \le 3m-1$, and no $\zeta_{s,k}$, $s \in \mathscr{S}$, $1 \le k \le 3m-2$, wants to move, and each β_r, $r \in R$, possibly does not want wo move, thus, Γ is possibly Nash stable.

If: Assume there is a possibly Nash stable coalition structure Γ. Ruling out, one by one, coalitions that cannot be contained in Γ, it can be shown that for each $r \in R$, there exists an $s \in \mathscr{S}$ such that $\Gamma(\beta_r) = P_s$, which means that there is an exact cover of R in $\mathscr{S}$. ❑

By similar, but not trivially the same methods we can show that the problem of necessary Nash stable existence is NP-complete.

5. CHALLENGES

In order to give a prospect to future work we provide initial thoughts on further stability concepts as well as comparability functions in order to deal with incomparabilities.

5.1 Further Stability Concepts

So far we have focused on single-player deviations. In this section, we give a prospect to other stability concepts such as group deviations, Pareto optimality, and popularity. A coalition structure Γ is called *core stable* if for each coalition $A' \subseteq A$, there exists a player $i \in A'$ such that $\Gamma(i) \succeq_i A'$. A coalition structure Γ is called *Pareto-optimal* if for each coalition structure Δ, there exists a player $i \in A$ such that $\Gamma(i) \succ_i \Delta(i)$, or for each player $j \in A$, $\Gamma(j) \sim_j \Delta(j)$. A coalition structure Γ is called *popular* if for each coalition structure Δ, the number of players i with $\Gamma(i) \succ_i \Delta(i)$ is at least as large as the number of players j with $\Delta(j) \succ_j \Gamma(j)$. We furthermore introduce the notion of *strict popularity*. A coalition structure Γ is called *strictly popular* if it *beats* each other coalition structure Δ *in pairwise comparison*,[4] that is,

$$\|\{i \in A \mid \Gamma(i) \succ_i \Delta(i)\}\| > \|\{j \in A \mid \Delta(j) \succ_j \Gamma(j)\}\|.$$

For each extension there exists a Pareto-optimal coalition structure (perhaps a different one for different extensions). Observe that if there exists a necessarily strictly popular coalition structure, it is unique, whereas there can be more than one possibly strictly popular coalition structure.

If there exists a necessarily strictly popular coalition structure, it is necessarily Pareto optimal. If there exist possibly strictly popular coalition structures, each of them is possibly Pareto-optimal. A necessarily strictly popular coalition structure does not need to be possibly individually rational. Even if the possible core is nonempty, a necessarily strictly popular coalition structure does not need to be possibly core stable. The same holds for the concepts of Nash stability, individual stability, contractual individual stability, and strict core stability. If there exists a unique perfect partition, it is necessarily the unique necessarily strictly popular coalition structure.

With techniques related to those in the proof of Theorem 16, we can show that the questions of whether a given coalition structure is possibly strictly popular or popular or Pareto-optimal are coNP-hard, necessarily strictly popular or popular or Pareto-optimal are coNP-complete, and it is coNP-hard to decide whether there exists a strictly popular coalition structure, for both, the possible and the necessary case.

Moreover, coNP-hardness of the problems of whether a given coalition structure is core stable or strictly core stable can be shown with help of the reduction from CLIQUE to the core stability verification problem in the enemy-based representation [21]. Note that this representation is a special case of the representation with ordinal preferences and thresholds, where there are no neutral agents and only indifferences between all friends and between all enemies in a player's preference. Furthermore, note that the enemy-based-extension [15] is a possible extension in $\times_{i=1}^{n} \mathrm{Ext}(\succeq_i^{+0-})$. While a "clique" of friends is necessarily preferred by all members to a coalition containing fewer friends or even more enemies, there is not necessarily a blocking coalition in the construction if there is no such clique (for example, there is no blocking coalition in the enemy-based extension).

5.2 Breaking Incomparabilities with Borda-Like Scoring Vectors

In this section, we present a mechanism for determining the relation between coalitions that are not comparable via the ordering that the Bossong–Schweigert extension induces.

Every player has to evaluate a total preference order over all possible coalitions she might be part of, so we define a so-called *comparability function* (short CF) for a fixed player, say $i \in A$. One possibility to do so is to use scoring vectors that assign values to the players in $A \setminus \{i\}$ depending on the position they have in the weak ranking with double threshold of player i. In particular, for the notions presented in Definition 1, we define the following variants of Borda-like scoring vectors.

We define scoring vectors $w_i : A \to \mathbb{Z}$ assigning points to the players in the sets of friends, neutral agents, and enemies of agent i, according to their positions in ranking $\trianglerighteq_i^{+0-}$, compatible with $\trianglerighteq_i^{+0-}$ as in Proposition 6. In more detail, we propose the following possibilities, distinguishing between an "optimistic" and a "pessimistic" case (see also the optimistic and pessimistic scoring model for modified Borda voting, due to Baumeister et al. [6]), and for each we have a regular and a strong variant. Recall that we have n agents in total. Suppose that i's friends, A_i^+, are ordered as follows: $\trianglerighteq_i^+ = A_{i,1}^+ \rhd_i^+ A_{i,2}^+ \rhd_i^+ \cdots \rhd_i^+ A_{i,\ell}^+$, where each $A_{i,j}^+$ contains some agents i is indifferent about. Similarly, suppose that i's enemies, A_i^-, are ordered as follows: $\trianglerighteq_i^- = A_{i,1}^- \rhd_i^- A_{i,2}^- \rhd_i^- \cdots \rhd_i^- A_{i,m}^-$, where each $A_{i,j}^-$ contains agents i is indifferent about. Here, we do not explicitly define all 16 combinations of (strictly) friend/enemy-optimistic/pessimistic scoring vectors. For instance, consider the cases of a strongly friend-optimistic and a strongly enemy-pessimistic setting.

DEFINITION 17. *Let A be a set of players and $\trianglerighteq_i^{+0-}$ be player i's preference relation. Let $w_i : A \to \mathbb{Z}$, compatible with $\trianglerighteq_i^{+0-}$, assign n points to each agent in $A_{i,1}^+$, $n-1$ points to each agent in $A_{i,2}^+$, ..., and $n-\ell+1$ points to each agent in $A_{i,\ell}^+$. Moreover, let each agent in $A_{i,m}^-$ get $-n$ points, each agent in $A_{i,m-1}^-$ get $-n+1$ points, ..., and each agent in $A_{i,1}^-$ get $-(n-m+1)$ points. Then, we call w_i* strongly friend-optimistic *and* strongly enemy-pessimistic.

We now define a numerical comparability function that captures the notion of Borda-like scoring.

DEFINITION 18. *For each fixed agent $i \in A$ and for every fixed choice of scoring vectors w_i, the Borda-like CF*

$$f_{\mathrm{Borda}}^i : \{C \subseteq A \mid i \in C\} \to \mathbb{Z}$$

maps every coalition C containing i to the sum of the scores the agents in C obtain from w_i. The value of a coalition *$C \subseteq A$ is defined as $F_{\mathrm{Borda}}(C) = \sum_{i \in C} f_{\mathrm{Borda}}^i(C)$.*

[4]This notion is adapted from the voting-theoretic term of *Condorcet winner*: Such a candidate wins an election if and only if she beats each other candidate in pairwise comparison.

	$\{1,2,3,4\}$	$\{1,2,3\} \sim \{1,2,4\}$	$\{1,2,3,4,5\}$
v_1	16	11	11

Table 1: Values of some coalitions in player 1's view for the scoring vector $\mathbf{v_1} = (*, \mathbf{6}, \mathbf{5}, \mathbf{5}, -\mathbf{5}, -\mathbf{6})$

EXAMPLE 19. *Let $A = \{1,2,3,4,5,6\}$ and the preference with thresholds from Example 2: $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 4 \mid \mid 5 \triangleright_1 6)$. Figure 1 shows the graph corresponding to the Bossong–Schweigert extension of this preference. For six agents and $\trianglerighteq_1^{+0-}$, the scoring vector in the strongly friend-optimistic and strongly enemy-pessimistic setting is $v_1 = (*, 6, 5, 5, -5, -6)$.*

Table 1 shows the scores of some of the coalitions from agent 1's view with scoring vector v_1.

To determine the overall value of all coalitions, the individual scores of the other five agents have to be determined as well.

The following observation follows directly from the definitions above.

OBSERVATION 20. *For each player $i \in A$, the comparability function f^i_{Borda} preserves those rankings that are induced by the Bossong–Schweigert extension.*

Furthermore, a game that is induced by comparability function F_{Borda} (as an extension) is additively separable.

This observation allows us to use known results for the complexity of the various stability problems in general additive separable hedonic games (ASHGs, for short), which have been studied intensely (see, e.g., the work by Aziz et al. [3] for a comprehensive overview). Upper bounds can be transferred directly from known results for general ASHGs. Whether the known lower bounds also hold for our special games, however, has to be checked separately. For certain settings of scoring vectors (often all 16 combinations at once), we were able to adapt known hardness proofs for some of the stability concepts to our setting. Although the cardinaliziation of the ordinal preferences might suggest that verification and existence of a stability concept become more tractable. However, for the strongly friend-pessimistic and strongly friend-optimistic case, we obtain the same complexity results as for Nash stability: verification is decidable in P, existence NP-complete. The problem of whether there exists a core stable coalition structure in a given FEN-hedonic game is even Σ_2^p-complete.

6. CONCLUSIONS AND FUTURE WORK

In this paper we introduce a new representation of preferences in hedonic games using the Bossong–Schweigert principle to extend the players' preferences over the other players to preferences over the coalitions. This generalized Bossong–Schweigert extension principle to positive and negative items (here called friends and enemies), and neutral items, is new and it is original in itself, independently of its use in hedonic games.

We have then looked at several stability concepts in hedonic games with such preferences. The problem of remaining incomparabilities is tackled in two ways: Firstly, by letting these incomparabilities unresolved and introducing known stability concepts with respect to notions of necessity and possibility, and secondly by introducing a comparability function based on Borda-like scoring vectors.

For both approaches we analyze for the induced games the complexity of the existence and verification of well-known stability concepts. So far, with the help of these solution concepts we can verify if a coalition structure is a "good" solution, compare two coalition structures, and decide, whether there even exists such a coalition structure—sometimes at great cost in terms of complexity.

Besides completing the analysis initiated here (such as considering other solution concepts and solving remaining open problems), we suggest for future work introducing the notion of *partition correspondences* with the purpose to actually identify "good" coalition structures as an output. In contrast to the original idea of hedonic games where coalitions form in a decentralized manner, here a central correspondence is used, in order to decide which coalitions will work together. This might, for example, be the case in a setting where the head of a department has to divide a group of employees into teams. The teams should be stable, in the sense that the team members are as happy as possible with their group to create a good working atmosphere.

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