# Strategic Behavior and No-Regret Learning in Queueing Systems

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#### Abstract

This paper studies a dynamic discrete-time queuing model where at every period players get a new job and must send all their jobs to a queue that has a limited capacity. Players have an incentive to send their jobs as late as possible; however if a job does not exit the queue by a fixed deadline, the owner of the job incurs a penalty and this job is sent back to the player and joins the queue at the next period. Therefore, stability, i.e. the boundedness of the number of jobs in the system, is not guaranteed. We show that if players are myopically strategic, then the system is stable when the penalty is high enough. Moreover, if players use a learning algorithm derived from a typical no-regret algorithm (exponential weight), then the system is stable when penalties are greater than a bound that depends on the total number of jobs in the system.

# 1 Introduction

In the classical treatment of queues agents arrive at random times, wait according to a specified regime, and then get served; the service time is also random. In these models there is no room for any strategic behavior of the agents. Even when agents balk or renege, this is modeled as a random event, not a strategic choice of the agents. Starting with the seminal paper by Naor (1969), strategic elements have been included in queueing models. For instance, in Naor's model, when agents arrive, they rationally choose whether to join the queue or to balk.

The suitable tools to analyze strategic queueing models come from game theory. The literature has considered several strategic models of queueing systems under different stochastic assumptions, different service regimes, and different strategy sets for the players. Various goals have been considered, such as computing Nash equilibria of the games, studying their efficiency, and examining the system stability under various equilibria.

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One interesting class of problems, first examined by Glazer and Hassin (1983), deals with situations where players need to be serviced before a fixed deadline, and otherwise pay a steep penalty. In several variations of this model, players play the game repeatedly, for instance, they commute daily to the office and need to get there on time.

In a recent development, Gaitonde and Tardos (2020, 2021) considered a discrete-time queueing model where agents use learning algorithms to make the decision of which server to choose. The novelty of their analysis was the consideration of spillovers from one period to the other. One of their goals is to establish conditions for the system to be stable.

#### 1.1 Our contribution

Our paper draws both on the literature on queues with a fixed deadline and on the contribution on learning and studies a discrete-time model where agents at every period receive a new job that requires service from a single server and need to decide when their jobs join a queue, taking into account a trade-off between waiting costs and a stiff penalty for being late. The model includes spillovers, since the jobs that cannot be served by the deadline go back to their owners and are to be sent to the server the following period; these late jobs are then added to the incoming daily new job. From a queuing point of view, the model is deterministic: each player gets exactly one new job at every period. Randomness is due to the actions of the players, which can be mixed, and to the regime of the queue: if several jobs join the queue at the same time, the order in which they get served is uniformly random.

We consider several aspects of the model under the assumption that agents are strategic, but myopic, i.e., at every period they play an equilibrium, but do not take into account the future effect of their actions. This leads us to examine the equilibria of the single-period game. In this framework we show that the structure of equilibria depends on whether the number of jobs in the system is or is not larger than the number of times in each period; moreover it depends on whether the penalty cost for being late is or is not large enough.

When the number of jobs in the system exceeds the number of times in each period, and the penalty cost is large enough, we show that the stage game has a single coarse correlated equilibrium (hence, a single Nash equilibrium), where all players sends all their jobs to the queue as early as possible. When the number of jobs in the system does not exceed the number of times in each period, then the stage game has multiple equilibria, whose structure we study. Surprisingly, even if each player has a minmax strategy that guarantees that this player's jobs will meet the deadline, nevertheless, in equilibrium some jobs will be late with positive probability. This implies that the number of jobs in the system in the following period will be larger than in the current period.

In the second part of the paper we study a model where players use a no-regret learning algorithm to make their choices. As we mentioned before, the number of jobs that each player may vary from one period to another. To face this, we will adopt a variation of the exponential weight algorithm that takes into account the changing environment.

We describe the model using the language of game theory, but other motivations are possible. For instance, we could consider a revenue-management interpretation where at each period agents buy a priority for their jobs. There are different priority levels and their price is monotone with the priority. Jobs with high priority are served before jobs with lower priority, up to the fixed capacity for each period. When there is a priority conflict, it is resolved at random. The constraint is that a maximum of one agent with the lowest priority is served, a maximum of two agents with the lowest two priorities are served, etc.

#### **1.2** Related literature

The analysis of strategic behavior in queueing systems goes back to the seminal paper of Naor (1969), who studied an M/M/1 queue with a first-in first-out (FIFO) policy where the agents' payoff consists of the reward that they get when they get served, minus a waiting cost that is proportional to the time they spend in the queue. Once they arrive, they can decide whether to join the queue or to balk. If they play a Nash equilibrium, their behavior is socially inefficient, in the sense that it does not maximize the social payoff (the sum of all players' payoffs). This is due to the fact that one agent's selfish behavior does not take into account the externalities it creates on other agents. Hassin (1985) showed that optimality can be achieved by a last-in first-out (LIFO) policy. The literature on strategic queueing system has then exploded. The reader is referred to Hassin and Haviv (2003) and Hassin (2016) for a general treatment of the topic.

Some models have considered strategic agents who can decide when to join a queue. The seminal paper by Glazer and Hassin (1983) considered a model, called ?/M/1, where agents arrive at a facility that every day starts service at time 0, serves all customers that arrive by some time T according to a FIFO policy. Each day, agents decide whether to visit the facility or not. If they do, they pick their arrival time with the goal of minimizing their expected time in the queue. This queueing literature with a strategic choice of the arrival time has been recently surveyed by Haviv and Ravner (2021).

In the framework of transportation theory, Vickrey (1969) studied a bottleneck model where agents choose the time they leave home to reach the office and used some fluid approximation. Rivera et al. (2018) studied a discrete version of the Vickrey model and examined Nash and correlated equilibria and their efficiency. Our model is a (variation of) a repeated version of their model with spillover. Kawasaki et al. (2023) extended the study of this bottleneck model to more general preferences and provided conditions for the existence of a pure Nash equilibrium; moreover they examined the link with strong equilibria.

Recently, Gaitonde and Tardos (2020) proposed a model where several queues receive packets with a fixed, time-independent (but queue dependent) probability and must send them to a number of servers. Each server may process a packet it received with a fixed, time-independent probability. The paper studied the behavior of the system when players use no-regret learning procedures and in particular determined the conditions on the model parameters for the system to be stable. In a subsequent paper, Gaitonde and Tardos (2021) compared the behavior of no-regret, short-term, learners with players who adopt a long-run optimizing behavior. Sentenac et al. (2021) used a similar model and introduced a new cooperative and decentralized algorithm that players can use. Compared to this work, our paper does assume that players intend to cooperate, in the sense that they use a standard algorithm to maximize their own payoff.

No-regret learning procedures algorithms are a family of online reinforcement learning algorithms such that they are at least as good as any constant action selected in hindsight (*i.e.*, given that other players or the environment actions are fixed). The notion, also called Hannan or universal consistency Fudenberg and Levine (1998), was originally introduced by Hannan (1957) and is satisfied by multiple algorithms (see, for a recent review, Perchet, 2014). In particular, the exponential weight algorithm (EWA) Littlestone and Warmuth (1994); Cesa-Bianchi and Lugosi (2006) is a simple but efficient reinforcement learning procedure which adjusts weights of actions based on past experiences. However, there is no widely accepted extension of the concept of no-regret learning to systems with (endogenously changing) state variables (for instance Markov Decision Processes) such as ours (the state of the queue), so we propose a simple, multi-level, extension to EWA. Besbes et al. (2015, 2019) dealt with stochastic optimization and single-agent, multi-armed bandit problems with temporal uncertainty in the rewards. In a recent interesting paper, Crippa et al. (2022) studied—in the context of (non-stochastic) repeated games—strategies that have no-regret compared to dynamic sequences of actions whose number of changes scales sublinearly in time. There is also a large literature on time-varying games were studied by Cardoso et al. (2019); Mertikopoulos and Staudigl (2021); Duvocelle et al. (2022); Zhang et al. (2022); Anagnostides et al. (2023), among others. In a game theoretic framework, the reward functions of a player can change across time for two reasons: a change in the game and a change in the strategy of the other players. In these articles, the authors investigate different solution concepts to take this distinction into account. A key difference with our model is that the change of the game is exogeneous whereas in our model it is endogeneous. We restrict ourselves to the more classical notion of regret.

### 1.3 Outline

Section 2 introduces the model and the key concept that the paper will consider. Section 3 analyzes the behavior of myopic strategic players. The key point in this section is the analysis of the one-shot game for any job allocation to the players. Section 4 analyzes the repeated game with learning players.

# 2 The model

We consider a discrete-time game where I is the set of players and time is split into periods of equal length  $L \ge 2$ . The generic period is denoted by t. Throughout the paper we assume that there are at least  $|I| \ge 3$  players.

At the beginning of every period  $t \ge 0$  each player *i* receives one job and independently chooses an action  $a_t^i \in \{0, \ldots, L-1\}$ , which represents the time during period *t* at which *i*'s jobs join the queue. The symbol  $\mathbf{a}_t := (a_t^i)_{i \in I}$  denotes the action profile at period *t*. Since all of player *i*'s jobs join the queue at the same time, the action set *A* does not depend on the number of jobs held by each player.

The choice of  $a_t^i$  may depend on the past history of the game. Depending on the actions taken by all players, some jobs are late, i.e., they cannot exit the queue before the end of the period. Late jobs are returned to their respective owners and will join the queue once more at the following period. This produces a *spillover* effect: at every period, the number of jobs that player *i* handles is the number of *i*'s late jobs in the previous period plus one.

The state of the system at period t is the vector  $\mathbf{k}_t = (k_t^i)_{i \in I}$ , where  $k_t^i$  is the number of player i's jobs at the beginning of period t. The state space is  $S := \mathbb{N}^I$ . The total number of jobs at time t is

$$k_t := \sum_{i \in I} k_t^i. \tag{2.1}$$

Late jobs at period t incur a penalty cost  $C_{k_t}$  that depends on the total number  $k_t$  of jobs in the system at period t. At the end of period t, each player i pays a cost  $c_{k_t}^i(\boldsymbol{\sigma}_t)$  that is the sum of two components: the waiting cost, i.e., the time spent in the queue by each of i's jobs, and the penalty cost:

$$c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{a}_{t}) = k_{t}^{i}(L - a_{t}^{i}) + C_{k_{t}}\mathbb{E}\left[\text{number of } i\text{'s jobs that are late at } t\right].$$
(2.2)

Given an action profile  $a_t$ , all of player *i*'s jobs are assumed to have the same probability of being late. As a consequence

$$\mathbb{E}\left[\text{number of } i\text{'s jobs that are late at } t\right] = k_t^i p^i(\boldsymbol{a}_t), \tag{2.3}$$

which implies

$$c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{a}_{t}) = k_{t}^{i}(L - a_{t}^{i}) + C_{kt}k_{t}^{i}p^{i}(\boldsymbol{a}_{t}).$$

$$(2.4)$$

A dynamic queueing model (DQM) is specified by a period length L, a number of players I and a sequence of penalties  $(C_{k_t})_{t\in\mathbb{N}}$  that depend on the total number  $k_t$  of jobs in the system.

Given the spillover effect, it is natural to investigate the asymptotic behavior of the number of jobs  $k_t$ . In particular, if  $k_t > L$ , at period t there are more jobs than time units; therefore, at least

one job is necessarily late. Some jobs can be late even if  $k_t \leq L$ . In the rest of the paper, we will be interested in bounding the number of jobs in the system, under various players' behaviors. More formally, we let  $\mathcal{H}_t := (S \times A)^{t-1} \times S$  denote the set of histories of length t and  $\mathcal{H} := \bigcup_{t \geq 1} \mathcal{H}_t$  the set of finite histories.

**Definition 1** (Strategy). A strategy is a function  $x^i \colon \mathcal{H} \to \triangle(A)$ . When there is no risk of ambiguity, we will let  $x_t^i \coloneqq x^i(h_t)$  denote player *i*'s mixed action at period *t*.

**Definition 2** (Stability of a strategy profile). A strategy profile x in a DQM is said to be *stable* if the number of jobs  $k_t$  is almost surely bounded, i.e.,

$$\mathbb{P}_{\boldsymbol{x}}\left[\exists M, \forall t \in \mathbb{N}, k_t \leq M\right] = 1.$$

The following assumption will be in place throughout the paper.

Assumption 3. The following inequality holds:  $L \ge I$ .

Without Assumption 3, no strategy profile in a DQM could be stable, because at every period at most L jobs leave the system, and I new jobs arrive.

We study the stability of two different families of strategy profiles. In the first case, agents are myopically strategic, i.e., they focus on the situation at the current period and play in a strategic way for that period, i.e., they play at every period a correlated equilibrium. We prove that the DQM is stable under this strategy profile, provided the penalty costs  $C_{kt}$  are large enough (potentially constant). In the second case, agents use no-regret strategies that depend on their number of jobs. We provide sufficient conditions for the stability of these strategies in the DQM.

### 3 Myopic strategic players

In this section, players are assumed to be strategic and myopic. The myopic assumptions implies that at period t players only consider the costs that they may pay at the current period without taking into consideration the effect that their actions have on the future of the game. In other words, players repeatedly play one-shot games with payoff functions  $(c_{k}^i)_{i \in I}$ .

We now introduce the definitions of myopic Nash equilibrium (NE) and myopic coarse correlated equilibrium (CCE).

**Definition 4** (Myopic NE). A strategy profile  $\boldsymbol{x} \colon \mathcal{H} \to \triangle(A^I)$  is called a *myopic NE* if, for any history  $h_t$  with current state  $\boldsymbol{k}_t$ , the mixed action  $\boldsymbol{x}(h_t)$  is a Nash equilibrium of the one-shot game with jobs  $\boldsymbol{k}_t = (k_t^i)_{i \in I}$ .

**Definition 5** (Myopic CCE). A strategy distribution  $\boldsymbol{x} \colon \mathcal{H} \to \triangle(A^I)$  is called a *myopic CCE* if, for any history  $h_t$  with current state  $\boldsymbol{k}_t$ , the distribution  $\boldsymbol{x}(h_t)$  is a coarse correlated equilibrium of the one-shot static game with jobs  $\boldsymbol{k}_t = (k_t^i)_{i \in I}$ .

When the number of jobs is lower than L, any player who plays 0 is sure not to pay the penalty cost. Even in this case, there exist equilibria where some late jobs are late with positive probability. Therefore, the number of jobs may not be constant throughout the play, as detailed in Section 3.1.

We now analyze the family of one-shot games parameterized by the number of jobs owned by each player.

The rest of the section is organized as follows. We start by focusing on the one-shot game, first when  $L \ge k_t$ , and then when  $L < k_t$ . We conclude by analyzing the global queue.

### 3.1 Equilibria in the one-shot game when $L \ge k_t$

We will show that, even when  $L \ge k_t$ , in equilibrium some jobs are late with positive probability.

We first prove that only the last  $k_t$  actions are used in an equilibrium; hence, it is enough to consider the case L = k.

**Lemma 6** (Dominated actions for  $L > k_t$ ). If  $L > k_t$ , then for each player *i*, any action  $a_t^i \in \{0, \ldots, L - k_t - 1\}$  is strictly dominated by action  $L - k_t$ . Furthermore, for any (mixed) action profile  $\sigma_t$ ,

$$c_{\boldsymbol{k}_{t}}^{i}(L-k_{t},\boldsymbol{\sigma}_{t}^{-i}) = k_{t}^{i}k_{t}.$$
(3.1)

Eq. (3.1) shows that the cost  $k_t^i k_t$  that player *i* pays when playing action  $L - k_t$  is independent of the other players' actions.

Proof of Lemma 6. If player *i*'s jobs join the queue at time  $a_t^i$ , then they leave the queue no later than time  $a_t^i + l$ , where *l* is the number of jobs that the queue at time  $a_t^i$  or before. Since  $l \leq k_t$ , player *i*'s jobs leave the queue no later than time  $a_t^i + k_t$ , which is smaller or equal than *L* if  $a_t^i$  is smaller or equal than  $L - k_t$ . So, the probability  $p^i(a_t^i, \boldsymbol{\sigma}_t^{-i})$  that a specific job of player *i* is late is 0 and player *i*'s cost is  $k_t^i(L - a_t^i)$ . In particular, the cost of choosing action  $L - k_t$  is  $k_t^i k_t$ , which is strictly smaller than the cost of any  $a_t^i \in \{0, \ldots, L - k_t - 1\}$ .

Lemma 6 implies that it is enough to consider the case  $L = k_t$ . In the rest of this section, all the results will be proved under this hypothesis. The general case  $L \ge k_t$  can be obtained by renaming the actions.

The next theorem shows that, in any equilibrium, players can be divided into two groups: players in the first group choose the same mixed action and mix only on the two first non-dominated actions  $L - k_t$  and  $L - k_t + 1$ ; players in the second group do not mix on  $L - k_t$  and put strictly positive weight on  $L - k_t + 1$ . The way they mix on the remaining non-dominated actions is not specified.

The symbol  $SC(\sigma_t)$  denotes the social cost of a profile  $\sigma_t$ , that is,

$$\mathsf{SC}(\boldsymbol{\sigma}_t) \coloneqq \sum_{i \in I} c^i_{\boldsymbol{k}_t}(\boldsymbol{\sigma}_t).$$
(3.2)

**Theorem 7** (Structure of Nash equilibria). If  $L \ge k_t$ ,  $C_{k_t} > k_t^2$ , and  $\sigma_t$  is a Nash equilibrium, then:

- (i) for each player j and action  $a_t^j < L k_t$ , we have  $\sigma_t^j(a_t^j) = 0$ ;
- (ii) there exists a player i such that
  - (a)  $\sigma_t^i(L-k_t) > 0$ ,
  - (b) for all  $a_t^i > L k_t + 1$ , we have  $\sigma_t^i(a_t^i) = 0$ ;
- (iii) for every player  $j \neq i$ ,  $\sigma_t^j(L k_t + 1) > 0$ ;
- (iv) for every player j, if  $\sigma_t^j(L-k_t) > 0$ , then  $\sigma_t^j = \sigma_t^i$ ;

(v) 
$$k_t^2 - k_t + 1 \leq \mathsf{SC}(\boldsymbol{\sigma}_t) \leq k_t^2$$
.

Proof. (i) The actions before  $L - k_t$  are not chosen. By Lemma 6, we know that all actions smaller than  $L - k_t$  are strictly dominated. The result follows from the fact that Nash equilibria do not mix on dominated actions. In the rest of the proof, we will assume that  $L = k_t$ . As mentioned before, if  $L < k_t$ , then the proof can be easily adapted by translation: action 0 becomes  $L - k_t$ , 1 becomes  $L - k_t + 1$ , etc.

(ii)(a) There exists a player i such that  $\sigma_t^i(0) > 0$ . Assume ad absurdum that  $\sigma_t^j(0) = 0$  for each player j. Then at least one job is late. This implies that

$$\mathbb{E}\left[\text{number of late jobs}\right] = \sum_{j \in I} k_t^j p^j(\boldsymbol{\sigma}_t) \ge 1$$
(3.3)

and there exists i such that  $k_t^i p^i(\boldsymbol{\sigma}_t) \ge k_t^i/k_t$ . Then,

$$c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{\sigma}_{t}) \ge k_{t}^{i}C_{k_{t}}p^{i}(\boldsymbol{\sigma}_{t}) \ge C_{k_{t}}k_{t}^{i}/k_{t}.$$

So if  $C_{kt} > k_t^2$ , player *i*'s cost is strictly greater than  $k_t^i k_t$ , which is a contradiction. Therefore, one player *i* mixes on 0 and  $c_{k_t}^i(\boldsymbol{\sigma}_t) = k_t^i k_t$ .

(iii) Every other player mixes on 1. If this is not the case, then  $\sigma_t^j(1) = 0$  for some  $j \neq i$ . We now prove that *i* can profitably deviate by playing the pure action 1. Assume that, indeed, *i* chooses action 1. If *j* plays the pure action 0, then player *i*'s jobs joining the queue at 1 are not late because the number of jobs joining the queue at 1 is smaller than  $k_t = L$ , so they are processed before the end of the period. Else, if *j* does not play 0, then *j*'s action is not smaller than 2. Then the number of jobs that join the queue at 0 or 1 is at most L - 1 and these jobs are processed before the end of the period.

So, by deviating to 1, *i*'s jobs are not late and this leads to a cost  $c_{k_t}^i(1, \sigma_t^{-i}) = k_t^i(k_t - 1)$ . Hence, the deviation is profitable and  $\sigma_t$  cannot be an equilibrium. (iv) (First step) All players who mix on 0 mix on 1, and put the same the weight on action 1. Suppose that at least two players mix on 0. We first prove that they mix on 1 too. Call these two players i and j. Player i satisfies (ii)(a); therefore, by (iii), player j mixes on 1. Symmetrically, player j satisfies (ii)(a); therefore, by (iii), player i mixes on 1. This shows that both players mix on 1.

We now prove that they put the same weight on 1. A job sent by any of these players can be late only if all jobs join the queue at time 1. Indeed, if at least one job joins the queue at 0, then none of the jobs that join the queue at 1 can be late because there are L-1 units of time to process them.

Therefore, a job departing at 1 is late only if all other jobs also depart at 1, i.e., all players play action 1. If this happens, then the probability that a job is late is  $1/k_t$ . Therefore, the probability that a job owned by player *i* is late, conditionally on the fact that *i* plays 1, is equal to

$$p^{i}(1, \boldsymbol{\sigma}_{t}^{-i}) = \frac{\mathbb{P}\left[a_{j}^{i} = 1 \ \forall j \neq i\right]}{k_{t}} = \frac{\prod_{j \neq i} \sigma_{t}^{j}(1)}{k_{t}}$$

Furthermore, since in a Nash equilibrium player i is indifferent between 0 and 1, we have

$$c_{\boldsymbol{k}_{t}}^{i}(0,\boldsymbol{\sigma}_{t}^{-i}) = k_{t}^{i}k_{t} = c_{\boldsymbol{k}_{t}}^{i}(1,\boldsymbol{\sigma}_{t}^{-i}) = k_{t}^{i}\left(k_{t}-1+\frac{\prod_{j\neq i}\sigma_{t}^{j}(1)}{k_{t}}C_{k_{t}}\right).$$
(3.4)

Thus, we have

$$\prod_{j \neq i} \sigma_t^j(1) C_{k_t} = k_t. \tag{3.5}$$

This applies to any j such that  $\sigma_t^j(0) > 0$ . Hence  $\sigma_t^i(1) = \sigma_t^j(1)$ .

(ii) (a) and (iv) (second step) Players who mix on 0 do not mix on any action strictly larger than 1. If player i mixes on an action  $a_t^i > 1$ , then one of i's jobs is late whenever all other players play 1. This implies that the probability to be late playing action  $a_t^i > 1$  is greater than the probability that all other players play 1, so using (3.5) we get

$$c_{k_{t}}^{i}(a_{t}^{i},\boldsymbol{\sigma}_{t}^{-i}) \ge k_{t}^{i} \left\{ k_{t} - a^{i} + \prod_{j \neq i} \sigma_{t}^{j}(1)C_{k_{t}} \right\} = k_{t}^{i}k_{t} + k_{t}^{i}(k_{t} - a_{t}^{i}) > k_{t}^{i}k_{t}.$$

Since all players who mix on 0 only mix between 0 and 1 with the same weight on 1, they actually play the same mixed action, which proves (iv).

(v) Social cost. First we prove that the social cost is smaller than  $k_t^2$ . If  $\sigma_t$  is a Nash equilibrium, then for every player j, we have

$$c_{\boldsymbol{k}_{t}}^{j}(\boldsymbol{\sigma}_{t}) \leqslant c_{\boldsymbol{k}_{t}}^{j}(0, \boldsymbol{\sigma}_{t}^{-j}) = k_{j}^{i}k_{t}.$$
(3.6)

Then,  $SC(\sigma_t) \leq \sum_{j \in I} k_j^i k_t = k_t^2$ . We now prove that the social cost is greater than  $k_t^2 - k_t + k_t^i$ . One player *i* mixes on 0 and incurs a cost equal to  $k_t^i k_t$ , and all other players  $j \neq i$  mix on 1, so their cost is at least  $k_j^i (k_t - 1)$ . Thus,

$$SC(\sigma_t) \ge (k_t - k_t^i)(k_t - 1) + k_t^i k_t = k_t^2 - k_t + 1.$$

**Corollary 8.** For any Nash equilibrium  $\sigma$ , there is positive probability that a job is late, that is

 $\mathbb{E}[\text{number of late jobs}] > 0.$ 

*Proof.* By Theorem 7 (ii) (a), we know that there exists a player i who mixes on  $L - k_t$  with positive probability. We consider two cases, based on this probability. If i plays with probability strictly less than 1, then by Theorem 7 (iii) and Theorem 7 (iv) we know that all the players are playing simultaneously in  $L - k_t + 1$  with positive probability. Therefore, one of them is late.

Let us assume that *i* plays with probability 1 the action  $L - k_t$ . If two players mix on  $L - k_t$ , we can apply Theorem 7 to each of them and therefore by (iii), both put positive weight on  $L - k_t + 1$ . So, *i* is the only player playing  $L - k_t$  and by Theorem 7 (iv) every other player plays  $L - k_t + 1$  with positive probability. Let  $j \neq i$  and suppose that  $\sigma_t^j(L - k_t + 1) = 1$ . Then, there exists another j' different from both *i* and *j* (because  $I \geq 3$ ). Player *i* plays  $L - k_t + 1$ , so j' is not late when it plays  $L - k_t + 2$ . Therefore, j' strictly prefers  $L - k_t + 2$  to  $L - k_t + 1$ , so it cannot put positive weight on  $L - k_t + 1$ , which is a contradiction.

### **3.2** Equilibria when $L < k_t$

We now study the game when the period length is smaller than the number of jobs, i.e.,  $L < k_t$ .

**Theorem 9** (coarse correlated equilibria). If  $k_t > L$  and  $C_{k_t} > k_t^2 L$ , then there is a unique coarse correlated equilibrium, which is actually a pure equilibrium where all players play 0.

To show that the support of any CCE is the singleton  $\mathbf{0}$ , we proceed by contradiction. We assume the existence of a CCE whose support is not  $\mathbf{0}$ ; we sum the cost of unilateral deviations to 0 for each player and show that the sum is negative. As a consequence, there is at least one deviation cost which is negative, which shows that this player has a profitable deviation. The proof is not "constructive" in the sense that the dissatisfied player is not designated outright.

Proof of Theorem 9. Let  $\boldsymbol{\tau}_t$  be a CCE. We have

$$c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{\tau}_{t}) = \sum_{\boldsymbol{a}\in A^{I}}\boldsymbol{\tau}_{t}(\boldsymbol{a})c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{a}) = \sum_{\boldsymbol{a}\in A^{I}}\boldsymbol{\tau}_{t}(\boldsymbol{a})k_{t}^{i}\left\{L - a^{i} + p^{i}(\boldsymbol{a})C_{k_{t}}\right\}.$$
(3.7)

Since  $au_t$  is a coarse correlated equilibrium, the cost that *i* obtains by a unilaterally deviating to 0

is not smaller then the cost that *i* incurs under  $\boldsymbol{\tau}_t$ , that is,

$$\sum_{\boldsymbol{a}\in A^{I}}\boldsymbol{\tau}_{t}(\boldsymbol{a})k_{t}^{i}\left\{L+p^{i}(0,\boldsymbol{a}^{-i})C_{k_{t}}\right\} \geq \sum_{\boldsymbol{a}\in A^{I}}\boldsymbol{\tau}_{t}(\boldsymbol{a})k_{t}^{i}\left\{L-a^{i}+p^{i}(\boldsymbol{a})C_{k_{t}}\right\},$$
(3.8)

which leads to

$$\sum_{\boldsymbol{a}\in A^{I}}\boldsymbol{\tau}_{t}(\boldsymbol{a})k_{t}^{i}\{p^{i}(0,\boldsymbol{a}^{-i})C_{k_{t}}-a^{i}+p^{i}(\boldsymbol{a})C_{k_{t}}\} \ge 0.$$
(3.9)

A player who was originally playing 0 is actually not deviating. Therefore, the term of the sum that corresponds to players who play 0 are equal to zero. Therefore, in (3.9) we can sum over all players who do not play 0 and get

$$\sum_{\boldsymbol{a}\in A^{I}\setminus\{\boldsymbol{0}\}}\boldsymbol{\tau}_{t}(\boldsymbol{a})\left\{\sum_{i\in I}k_{t}^{i}(p^{i}(0,\boldsymbol{a}^{-i})-p^{i}(\boldsymbol{a}))C_{k_{t}}+k_{t}^{i}a^{i}\right\}\geq0.$$
(3.10)

A job that joins the queue after time 0, has a higher probability of being late:

$$p^{i}(\boldsymbol{a}) \ge p^{i}(0, \boldsymbol{a}^{-i}) \tag{3.11}$$

The following claim refines the above statement.

Claim 10. Given an action profile  $a \neq 0$ , if

$$a^j = \max_{i \in I} (a^i), \tag{3.12}$$

then

$$p^{j}(\boldsymbol{a}) \ge \frac{1}{N_{a^{j}}k_{t}} + p^{j}(0, a^{j}),$$
(3.13)

where

$$N_{a^j} \coloneqq \sum_{i \in I: \ a^i = a^j} k_t^i \tag{3.14}$$

is the number of jobs that join the queue at time  $a^j$ .

*Proof.* First of all notice that in (3.12) we have  $a^j > 0$ . Call  $M_{a^j}$  the number of jobs that join the queue at time  $a^j$  and leave the system before the deadline L. Since  $L \leq k_t$ , some jobs are late; more precisely,  $N_{a^j} - M_{a^j}$  jobs that join the queue at  $a^j$  are late. Then

$$p^{j}(\boldsymbol{a}) = \frac{N_{a^{j}} - M_{a^{j}}}{N_{a^{j}}}.$$
(3.15)

Moreover,  $p^j(0, \boldsymbol{a}^{-j}) \leq (k_t - L)/k_t$ , so

$$p^{j}(\boldsymbol{a}) - p^{j}(0, \boldsymbol{a}^{-j}) \ge \frac{N_{a^{j}} - M_{a^{j}}}{N_{a^{j}}} - \frac{k_{t} - L}{k_{t}} = \frac{-M_{a^{j}}k_{t} + LN_{a^{j}}}{N_{a^{j}}k_{t}}.$$
(3.16)

However, the maximum number of jobs that can leave the system without being late is L.

Under the action profile a,  $k_t - N_{a^j}$  jobs join the queue strictly before time  $a^j$ . To finish the proof, we consider several cases related to the value of  $k_t - N_{a^j}$ :

• If  $0 < k_t - N_{a^j} \leq L$ , no more than  $L - (k_t - N_{a^j})$  jobs joining the queue at time  $a^j$  can arrive on time, i.e.,  $M_{a^j} \leq L - (k_t - N_{a^j})$ . Using this inequality in (3.16), we obtain

$$p^{j}(\boldsymbol{a}) - p^{j}(0, \boldsymbol{a}^{-j}) \ge \frac{-(L - k_{t} + N_{a^{j}})k_{t} + LN_{a^{j}}}{N_{a^{j}}k_{t}}$$
$$= \frac{L}{k_{t}} - \frac{L - k_{t}}{N_{a^{j}}} - 1$$
$$= \{L - k_{t}\} \left\{ \frac{1}{k_{t}} - \frac{1}{N_{a^{j}}} \right\}$$
$$= \{k_{t} - L\} \frac{k_{t} - N_{a^{j}}}{k_{t}N_{a^{j}}}$$
$$\ge \frac{1}{k_{t}N_{a^{j}}},$$

because in this case  $k_t - N_{a^j} \ge 1$ .

• If  $k_t - N_{a^j} = 0$ , then all the jobs join the queue at a time greater than 1; therefore  $M_{a^j} \leq L-1$ . Since in this case  $k_t = N_{a^j}$ , after some simplifications (3.16) becomes

$$p^{j}(\boldsymbol{a}) - p^{j}(0, \boldsymbol{a}^{-j}) \ge \frac{-M_{a^{j}} + L}{k_{t}} \ge \frac{-(L-1) + L}{k_{t}} = \frac{1}{k_{t}} \ge \frac{1}{k_{t}N_{a^{j}}}.$$
 (3.17)

• if  $k_t - N_{a^j} > L$ , then no jobs can avoid being late and  $M_{a^j} = 0$ . Therefore, (3.16) becomes

$$p^{i}(\boldsymbol{a}) - p^{i}(0, \boldsymbol{a}^{-i}) \ge \frac{L}{k_{t}} \ge \frac{1}{k_{t}N_{a^{j}}}$$

This concludes the proof of the claim.

Claim 11. For any  $a \neq 0$ , we have

$$-\frac{1}{k_t} \ge \sum_{i \in I} k_t^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})).$$
(3.18)

*Proof.* We split the players into two groups: players whose jobs join the queue at time  $a^j$  as defined

in (3.12) and the remaining ones. We have

$$\sum_{i \in I} k_t^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})) = \sum_{i \in I: \ a^i \neq a^j} k_t^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})) + \sum_{i \in I: \ a^i = a^j} k^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})).$$
(3.19)

The first term is nonpositive because of (3.11) and the second term is nonpositive because of (3.13), leading to

$$\sum_{i \in I} k_t^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})) \leqslant -\sum_{i \in I: \ a^i = a^j} k_t^i \frac{1}{N_{a^j} k_t}.$$
(3.20)

The inequality in (3.20) yields

$$\sum_{i\in I} k_t^i(p^i(0, \boldsymbol{a}^{-i}) - p^i(\boldsymbol{a})) \leqslant -\frac{N_{a^j}}{N_{a^j}k_t} = -\frac{1}{k_t},$$

which proves the claim.

The combination of Claim 11 and Eq. (3.10) yields

$$0 \leq \sum_{\boldsymbol{a} \in A^{I} \setminus \{\boldsymbol{0}\}} \boldsymbol{\tau}_{t}(\boldsymbol{a}) \left\{ \sum_{i \in I} k_{t}^{i}(p^{i}(0, \boldsymbol{a}^{-i}) - p^{i}(\boldsymbol{a}))C_{k_{t}} + k_{t}^{i}a^{i} \right\}$$
  

$$0 \leq \sum_{\boldsymbol{a} \in A^{I} \setminus \{\boldsymbol{0}\}} \boldsymbol{\tau}_{t}(\boldsymbol{a}) \left\{ -\frac{1}{k}C_{k_{t}} + \sum_{i \in I} k_{t}^{i}a^{i} \right\}$$
  

$$0 \leq \sum_{\boldsymbol{a} \in A^{I} \setminus \{\boldsymbol{0}\}} \boldsymbol{\tau}_{t}(\boldsymbol{a}) \left\{ -\frac{1}{k_{t}}C_{k_{t}} + Lk_{t} \right\}.$$
(3.21)

Under the assumptions of the theorem,  $C_{k_t} > k^2 L$ , so

$$-\frac{1}{k_t} + Lk_t < 0. (3.22)$$

Therefore, (3.21) can hold only if  $\tau_t(a) = 0$  for all  $a \in A^I \setminus \{0\}$ , which implies that the support of  $\tau_t$  is **0**.

### 3.3 Global queue

The following theorem is a consequence of the previous results.

**Theorem 12** (Stability for Strategic Repetition). Consider a DQM with  $I \leq L$ . If  $\inf_k C_k > (L+I)^2 L$ , then any myopic CCE is stable.

**Corollary 13** (Stability for Strategic Repetition). Consider a DQM with  $I \leq L$ . If  $\inf_k C_k > (L+I)^2 L$ , then any myopic NE is stable.

Notice that in particular if the penalty cost is a constant  $C > (L+I)^2 L$ , then any myopic CCE is stable. The corollary is an immediate consequence of the theorem since any Nash equilibrium is also a coarse correlated equilibrium.

As will be clear in the proof of Theorem 12, the queue alternates between two possible regimes. The first regime corresponds to the case  $k_t \leq L$ . In this regime, stage equilibria have a non-trivial structure and in equilibrium some jobs may be late. As a consequence, the number  $k_t$  of jobs in the system may oscillate over time. When this number overcomes the level L, the system enters the other regime, corresponding to  $k_t > L$ . In this regime the only stage equilibrium is the pure profile **0**. Therefore, if I = L, the number  $k_t$  of jobs in the system stays constant; if I < L, the number of jobs decreases until the system goes back to the first regime.

Remark 14. If the penalty costs are small, there may exist a myopic NE that is not stable. For example, if for all  $k \in \mathbb{N}$ ,  $C_k < 1$ , the cost of preempting the other players is too large compared to the potential gain: playing at the last stage of the period is strictly dominating for each player. The unique myopic NE is for every player to wait the last stage of the period and there are  $k_t - 1$ late jobs. Hence,  $k_t$  is almost-surely unbounded.

Proof of Theorem 12. Assume that  $I \leq L$  and for all  $k, C_k > (L+I)^2 L$ . We show by induction that for every  $t, k \leq L+I$ .

The results is true at period 1. We now show that it is true at every period. From period t to period t + 1, we need to consider two cases. If  $k_t \leq L$ , then  $k_{t+1} \leq L + I$  because at the next period I new jobs arrive and at most  $k_t$  are late. If  $L < k_t \leq L + I$ , then

$$C_{k_t} > (L+I)^2 L \ge k_t^2 L. \tag{3.23}$$

By Theorem 9 there is a unique stage CCE, where all players play 0. This implies that at least L jobs leave the system. Since  $I \leq L$  and there are I new jobs in the next period, we have  $k_{t+1} \leq k_t \leq L + I$ .

### 4 Learning players

In this section we study a model where every player independently uses a no-regret strategy. The no-regret property, originally introduced by Hannan (1957), is a property of multiple algorithms used in online reinforcement learning (see, e.g., Perchet, 2014). It specifies that in hindsight, the actions taken by a player are at least (asymptotically) as good as any constant action. Formally, in the one-player case, given a sequence of cost functions  $l_t$  indexed by time t and a sequence of

actions  $a_t^i$ , player *i*'s regret is defined as

$$R_t^i = \max_{b^i \in A} \sum_{u=1}^t l_u(b^i) - l_u(a_u^i).$$
(4.1)

A strategy satisfies the *no-regret* property if  $R_t^i = o(t)$ . Well-known no-regret strategies include regret-matching Hart and Mas-Colell (2013), stochastic fictitious play Fudenberg and Levine (1998), and the exponential weight algorithm (EWA) Littlestone and Warmuth (1994); Cesa-Bianchi and Lugosi (2006), which we study below.

**Exponential Weight Algorithm** In the following, we use the *exponential weight algorithm* (EWA), which is known to have no-regret guarantees when the payoff is bounded. Unfortunately, boundedness is not satisfied here, as the number of jobs in the system could grow to infinity, resulting in an unbounded penalty. For this reason, we need to study the efficiency of the algorithm more closely.

In the context of repeated games, weights are classically defined as follows:

$$w_t^i(b^i) = \exp\left(\sum_{u=1}^{t-1} -\eta c^i(b^i, a_u^{-i})\right),$$
(4.2)

where  $\eta$  is a positive constant. Eq. (4.2) can be rewritten in a recursive fashion as

$$w_{t+1}^{i}(b^{i}) = w_{t}^{i}(b^{i}) \exp\left(-\eta c^{i}(b^{i}, \boldsymbol{a}_{t}^{-i})\right).$$
(4.3)

Then, the EWA strategy specifies that action  $b^i$  is chosen at time t with probability

$$x_t^i(b^i) = \frac{w_t^i(b^i)}{\sum_{a^i \in A} w_t^i(a^i)}.$$
(4.4)

**Multi-level EWA (MLEWA)** EWA is not designed to handle a changing environment. Here, the number of jobs held by every player changes with time. Therefore, there we need to specify how such information is used. We design a new protocol called MLEWA where each player uses several copies of EWA. This protocol is indexed by a parameter n, which we call a *level*. When the number of jobs that player i owns reaches a new level for the first time, this player starts a new EWA where the weights are initialized as a function of the past. When the number of jobs of player i equals a level that has already been visited, this player follows the recommendation given by EWA for this level and updates the weights following EWA. Notice that at any given period t different players may use an EWA at different levels.

Let  $\tau_n^i$  be the first time player *i* has *n* jobs, with the convention that  $\tau_n^i = +\infty$  if player *i* never has *n* jobs. Player *i*'s weights  $w_{t,n}^i(b^i)$  are now parameterized by two parameters: the period *t* and

ALGORITHM 1: multi-level EWA (MLEWA)	
$\forall b^i \in A$ , initialize $w_1^i(b^i)$ ;	
$\forall i \in I, k^i \leftarrow 1 ;$	// level 1
for each step $t \ge 1$ do	
for each player $i$ do	
select an action $a_t^i \sim x_{k^i}^i$ ;	// proportional to $w^i_{k^i}$
end	
for each player $i$ do	
$\forall b^i \in A, w^i_{k^i}(b^i) \leftarrow w^i_{k^i}(b^i) \exp(-\eta c^i_{k_t}(b^i, \boldsymbol{a}_t^{-i})) ;$	// number of late jobs + 1 $$
$k^i \leftarrow \tilde{k}^i + 1;$	
if $w_{k^i}^i$ is not defined then	
$\forall b^i \in A$ , initialize $w^i_{k^i}(b^i)$ ;	// level $k^i$
end	
end	
end	

the level n. The algorithm at level n is defined for all  $t \ge \tau_n^i$  by induction. We start by describing the induction step, which is given by

$$w_{t+1,n}^{i}(b^{i}) = \begin{cases} w_{t,n}^{i}(b^{i}) \exp\left(-\eta c_{k_{t}}^{i}(b^{i}, \boldsymbol{a}_{t}^{-i})\right) & \text{if } k_{t}^{i} = n, \\ w_{t,n}^{i}(b^{i}) & \text{otherwise.} \end{cases}$$
(4.5)

Eq. (4.5) implies that  $w_{t,n}^i(b^i)$  is updated if and only if  $k_t^i = n$ .

We now describe the initialization. At  $\tau_n^i$ , we start a new EWA protocol and define initial weights

$$w^{i}_{\tau^{i}_{n},n} := w^{i}_{\tau^{i}_{n},k^{i}_{\tau^{i}_{n}-1}}$$

that is, weights of a newly encountered state are defined as equal to the weights of the previously visited level.

Algorithm 1 summarizes all the steps of our procedure. Its variables are not indexed by t as there are a fixed number of variables and the algorithm does not access the whole history. Instead, it computes the new values at each step and updates the corresponding variables.

MLEWA is based on a no-regret algorithm adapted to changing states. Eq. (4.1) does not deal with changing states. This is a limitation well identified by Gaitonde and Tardos, where regret is computed assuming the state path is unchanged by a change of action. The situation becomes much more complicated when states change endogenously. What we can say about our algorithm is that it has the no-regret property state-by-state.

We can now state our main result about the stability of the system when players learn.

**Theorem 15** (Stability of joint no-regret strategies in the subcritical case). If I < L and  $C_k > 4kL$  for all  $k_t$ , then strategy profiles where all players use MLEWA are stable.

Several lemmas are needed to prove Theorem 15. First, we show that (in a static context) action 0 strictly dominates any other actions for a player who holds a large enough number of jobs. The implication of this dominance in our dynamic model is that that the weight on action 0 converges towards 1 when enough jobs are held by a player. Finally, we expose some results on reinforced random walks.

#### **4.1 Domination** by action 0

The following lemma shows that in the static case action 0 is strictly dominant for a player i who has enough jobs.

**Lemma 16** (Strict Domination by 0 when  $k_t^i > 2L^2$  in the static model). If  $k_t^i > 2L^2$ ,  $C_{k_t} > 4k_tL$ , and  $\boldsymbol{a}$  is an action profile such that  $a^i \neq 0$ , then  $c_{\boldsymbol{k}_t}^i(0, \boldsymbol{a}^{-i}) < c_{\boldsymbol{k}_t}^i(\boldsymbol{a}) - k_t^i$ .

*Proof.* Let **a** be a pure action profile such that  $a^i \neq 0$ . Call

$$k_t^{-i}(0) = \sum_{j \neq i} k_j^i \mathbf{1}_{a^j = 0}$$
(4.6)

the number of jobs that join the queue at 0 excluding player's i jobs.

Then:

$$c_{\boldsymbol{k}_{t}}^{i}(0,\boldsymbol{a}^{-i}) = k_{t}^{i} \left\{ L + \frac{k_{t}^{i} + k_{t}^{-i}(0) - L}{k_{t}^{i} + k_{t}^{-i}(0)} C_{k_{t}} \right\} = k_{t}^{i} \left\{ L + \left\{ 1 - \frac{L}{k_{t}^{i} + k_{t}^{-i}(0)} \right\} C_{k_{t}} \right\}.$$

For each job, *i* incurs the waiting cost *L* and an additional cost due to the probability of being late. At period 0,  $k_t^i + k_t^{-i}(0) > L$  jobs join the queue; therefore some jobs will surely be late. Moreover, the probability that a job does not incur the penalty is equal to the probability that this job joins the queue among the first *L* jobs, which happens with probability  $L/(k_t^i + k_t^{-i}(0))$ .

• If  $k_t^{-i}(0) \ge L$ , then under the profile **a** the queue is full from stage 0 and the jobs sent by *i* are all late, i.e.,  $c_{k_t}^i(\mathbf{a}) = k_t^i(L - a^i) + k_t^i C_{k_t}$ . Then

$$c_{\boldsymbol{k}_{t}}^{i}(0, \boldsymbol{a}^{-i}) - c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{a}) = -\frac{k_{t}^{i}L}{k_{t}^{i} + k_{t}^{-i}(0)}C_{k_{t}} + k_{t}^{i}a^{i}.$$

The assumption on  $C_{k_t}$  implies that  $C_{k_t} > k_t(L+1)/L$ , so:

$$c_{\boldsymbol{k}_{t}}^{i}(0, \boldsymbol{a}^{-i}) - c_{\boldsymbol{k}_{t}}^{i}(\boldsymbol{a}) \leq -\frac{k_{t}^{i}}{k_{t}^{i} + k_{t}^{-i}(0)}k_{t}(L+1) + k_{t}^{i}a^{i}.$$

Since, by definition,  $k_t \ge k_t^i + k_t^{-i}(0)$ , it follows that:

$$c_{k_t}^i(0, a^{-i}) - c_{k_t}^i(a) \leq -k_t^i(L+1) + k_t^i a^i \leq k_t^i(a^i - L) - k_t^i \leq -k_t^i.$$

• Consider now the case  $k_0^{-i} < L$ . Player *i* pays the waiting cost  $L - a^i$  for each job; at most  $L - a^i$  of player *i*'s jobs can leave the system without being late. Consequently the following bound holds:

$$c_{\boldsymbol{k}_t}^i(\boldsymbol{a}) \ge k_t^i(L-a^i) + (k_t^i - L + a^i)C_{k_t}.$$

Then

$$c_{k_{t}}^{i}(0, a^{-i}) - c_{k_{t}}^{i}(a) = k_{t}^{i}L + k_{t}^{i}C_{k_{t}} - \frac{Lk_{t}^{i}}{k_{t}^{i} + k_{t}^{-i}(0)}C_{k_{t}}$$

$$- k_{t}^{i}L + k_{t}^{i}a^{i} - (k_{t}^{i} - L + a^{i})C_{k_{t}},$$

$$\leq -\frac{k_{t}^{i}L}{k_{t}^{i} + k_{t}^{-i}(0)}C_{k_{t}} + k_{t}^{i}a^{i} + LC_{k_{t}} - a^{i}C_{k_{t}},$$

$$= \left(\frac{k_{t}^{-i}(0)L}{k_{t}^{i} + k_{t}^{-i}(0)} - a^{i}\right)C_{k_{t}} + k_{t}^{i}a^{i}.$$
(4.7)

Since  $k_t^i > 2L^2$ , it follows that  $k_t^i + k_t^{-i}(0) \ge 2L^2$ , so

$$\frac{k_t^{-i}(0)L}{k_t^i + k_t^{-i}(0)} \leqslant \frac{L^2}{2L^2} < \frac{1}{2}.$$

Hence,

$$c_{k_t}^i(0, a^{-i}) - c_{k_t}^i(a) \leq \left(\frac{1}{2} - a^i\right)C_{k_t} + k^i a^i < -\frac{C_{k_t}}{2} + k_t^i L,$$

because  $1 \leq a^i \leq L$ .

The assumption that  $C_{k_t} > 4kL$  implies

$$c_{k_t}^i(0, a^{-i}) - c_{k_t}^i(a) < -2k_t L + k_t^i L < -k_t L < -k_t < -k_t^i.$$

### 4.2 Action 0 is increasingly preferred when the number of jobs grows

We now use Lemma 16 to show that, when the number of jobs in the system is high enough, for strategies  $x_{t,n}^i$  defined as in Eq. (4.4), the weight  $w_{t,n}^i(0)$  of action 0 increases faster than the other weights. Since  $x_{t,n}^i$  is proportional to this weight, players are more and more prone to play action 0 when the state is visited again.

Call  $\lambda_{t,n}$  the number of times that the level of player *i* is *n*, up to time t - 1:

$$\lambda_{t,n} = \# \{ k_u^i = n \mid u \in \{0, \dots, t-1\} \}.$$
(4.8)

We have

**Lemma 17** (Preference for 0). If  $n > 2L^2$  and  $C_{k_t} > 4kL$ , then for all  $t \ge \tau_n^i$ ,

$$x_{t,n}^{i}(0) \ge \frac{x_{\tau_{n}^{i},n}^{i}(0)}{x_{\tau_{n}^{i},n}^{i}(0) + \left(1 - x_{\tau_{n}^{i},n}^{i}(0)\right) \exp\left(-\eta n\lambda_{t,n}\right)}.$$

*Proof.* We first prove by induction on t that for every  $b^i \neq 0$ , we get:

$$\frac{w_{t,n}^{i}(b^{i})}{w_{\tau_{n}^{i},n}^{i}(b^{i})} \leq \exp\left(-\eta n\lambda_{t,n}\right) \frac{w_{t,n}^{i}(0)}{w_{\tau_{n}^{i},n}^{i}(0)},\tag{4.9}$$

If  $t = \tau_n^i$  then by definition player *i* never had *n* jobs before and  $\lambda_{t,n} = 0$ .

It follows that both sides are equal to 1 and the result is true.

We now show that, if the result holds for t - 1, then it holds for t. There are two cases.

If  $k_{t-1}^i \neq n$ , then all weights are equal at stage t and t-1, i.e.  $w_{t,n}^i(b^i) = w_{t-1,n}^i(b^i)$  for all action  $b^i$ . Moreover  $\lambda_{t,n} = \lambda_{t-1,n}$ , so the inequality is the same at t and t-1 and therefore true. If  $k_{t-1}^i = n$ , then  $\lambda_{t,n} = \lambda_{t-1,n} + 1$ . By the recurrence hypothesis, we know that

$$\frac{w_{t-1,n}^{i}(b^{i})}{w_{\tau_{n}^{i},n}^{i}(b^{i})} \leq \exp\left(-\eta n\lambda_{t-1,n}\right) \frac{w_{t-1,n}^{i}(0)}{w_{\tau_{n}^{i},n}^{i}(0)}.$$
(4.10)

Using Lemma 16 and Eq. (4.5), for  $b^i \neq 0$ , we get

$$\frac{w_{t,n}^{i}(b^{i})}{w_{t-1,n}^{i}(b^{i})} = \exp\left(-\eta c_{\boldsymbol{k}}^{i}(b^{i}, \boldsymbol{a}_{t-1}^{-i})\right)$$
(4.11)

$$\leq \exp\left(-\eta n\right) \exp\left(-\eta c_{\boldsymbol{k}}^{i}(0, \boldsymbol{a}_{t-1}^{-i})\right),\tag{4.12}$$

$$= \exp(-\eta n) \frac{w_{t,n}^i(0)}{w_{t-1,n}^i(0)},$$
(4.13)

Multiplying Eqs. (4.10) and (4.13), we obtain the result for t.

Fix now t. Eq. (4.9) implies that

$$w_{t,n}^i(b^i) \leqslant \exp\left(-\eta n\lambda_{t,n}\right) \frac{w_{t,n}^i(0)}{w_{\tau_n^i,n}^i(0)} w_{\tau_n^i,n}^i(b^i).$$

Therefore,

$$\begin{split} x_{t,n}^{i}(0) &= \frac{w_{t,n}^{i}(0)}{\sum_{b^{i} \in A} w_{t,n}^{i}(b^{i})} \\ &= \frac{w_{t,n}^{i}(0)}{w_{t,n}^{i}(0) + \sum_{b^{i} \neq 0} w_{t,n}^{i}(b^{i})} \\ &\geqslant \frac{w_{t,n}^{i}(0)}{w_{t,n}^{i}(0) + \sum_{b^{i} \neq 0} \exp\left(-\eta n\lambda_{t,n}\right) \frac{w_{t,n}^{i}(0)}{w_{\tau_{n}^{i},n}^{i}(0)} w_{\tau_{n}^{i},n}^{i}(b^{i})} \\ &= \frac{w_{\tau_{n}^{i},n}^{i}(0)}{w_{\tau_{n}^{i},n}^{i}(0) + \sum_{b^{i} \neq 0} \exp\left(-\eta n\lambda_{t,n}\right) w_{\tau_{n}^{i},n}^{i}(b^{i})} \\ &= \frac{x_{\tau_{n}^{i},n}^{i}(0)}{x_{\tau_{n}^{i},n}^{i}(0) + \sum_{b^{i} \neq 0} \exp\left(-\eta n\lambda_{t,n}\right) x_{\tau_{n}^{i},n}^{i}(b^{i})} \\ &= \frac{x_{\tau_{n}^{i},n}^{i}(0)}{x_{\tau_{n}^{i},n}^{i}(0) + \sum_{b^{i} \neq 0} \exp\left(-\eta n\lambda_{t,n}\right) x_{\tau_{n}^{i},n}^{i}(b^{i})} \end{split}$$

which proves the lemma.

**Lemma 18.** There exists  $B^i > 0$  such that

$$x_{\tau_n^i,n}^i(0) \ge \frac{1}{1+B^i \exp(-\eta n)}.$$
 (4.14)

*Proof.* Let  $i \in I$ . We can define

$$Z^i \coloneqq \max_{n \leqslant 2L^2 \text{ s.t. } \tau_n^i < +\infty} \frac{1}{x_{\tau_n^i, n}^i(0)} - 1.$$

For n = 1, we initialize the algorithm uniformly so every action has initially a strictly positive weight. By definition of EWA, if an action has a strictly positive weight during the initialization then it is always played with strictly positive probability. When reaching the level n = 2, the initialization is done by copying the current distribution of the algorithm of level n = 1, hence each action has a strictly positive weight too. Induction proves that at every stage and for every level, the probability to play every action is strictly positive—and strictly lower than 1. It follows that  $Z^i$  is strictly positive as the minimum of finitely many strictly positive numbers.

By definition, for every  $n \leq 2L^2$  such that  $\tau_n^i < +\infty$ , one has

$$x^i_{\tau^i_n,n} \geqslant \frac{1}{1+Z^i} \geqslant \frac{1}{1+Z^i} \exp(\eta 2L^2) \exp(-\eta n),$$

so let  $B^i \coloneqq Z^i \exp(\eta 2L^2)$ .

We now prove that this is true also for  $n > 2L^2$ . The proof is by induction. Assume that it is

true for  $n \ge 2L^2$  and consider n + 1 such that  $\tau_{n+1}^i < +\infty$ . Since the increment in the number of jobs is at most one, this implies that  $\tau_n^i < +\infty$ .

By Lemma 17, for all  $t > \tau_n^i$ , the weight on 0 satisfies

$$x_{t,n}^{i}(0) \geq \frac{x_{\tau_{n}^{i},n}^{i}(0)}{x_{\tau_{n}^{i},n}^{i}(0) + \left(1 - x_{\tau_{n}^{i},n}^{i}(0)\right) \exp\left(-\eta n\lambda_{t,n}\right)} \\ = \frac{1}{1 + \left(\frac{1}{x_{\tau_{n}^{i},n}^{i}(0)} - 1\right) \exp\left(-\eta n\lambda_{t,n}\right)} \\ \geq \frac{1}{1 + (1 + B^{i} \exp(-\eta n) - 1) \exp\left(-\eta n\lambda_{t,n}\right)} \\ \geq \frac{1}{1 + B^{i} \exp\left(-\eta n(1 + \lambda_{t,n})\right)} \\ \geq \frac{1}{1 + B^{i} \exp\left(-\eta (n + 1)\right)}$$
(4.16)

where Eq. (4.15) follows from the recurrence hypothesis and Eq. (4.16) is implied by  $t > \tau_n^i$ , so  $\lambda_{t,n} \ge 1$ .

The initial weight when reaching n + 1 for the first time is equal to the current weight for n packages, it follows that

$$x_{\tau_{n+1}^{i},n+1}^{i}(0) = x_{\tau_{n+1}^{i},n}^{i}(0) \ge \frac{1}{1 + B^{i} \exp(-\eta(n+1))}.$$
(4.17)

This proves the result for n + 1. Hence, it concludes the induction and proves the lemma.  $\Box$ 

### 4.3 Reinforced Random Walks

Lemma 17 shows that every time the process reaches a given level, there is a reinforcement on the probability to play the action profile where every player plays 0. The next step is to understand how this reinforcement influences the system dynamic. In order to do so, we prove some results on reinforced random walks. We follow the presentation of (Menshikov et al., 2017, p. 47) of nearest neighbor one-dimension random walk. They study random walks that are non-homogeneous *in space* but homogeneous *in time*. The difference is that we suppose there is a reinforcement factor in the drift, resulting in a random walk that is non-homogeneous *in time and space*, but bounded. Furthermore, we suppose that there is more heterogeneity in the weight of our random walk, in the sense that precise probabilities of going up or down are highly dependent of the past but nevertheless bounded.

The proof of Theorem 15 requires the following lemma, whose proof can be found in Appendix B.

**Lemma 19** (Reinforced Random Walk). Let M > 0 and  $(X_t, Z_t, t \ge 0)$  a sequence of random variables in  $\mathbb{N}$  such that  $X_t/d \le Z_t \le X_t$  with d > 1,  $|X_{t+1} - X_t| \le M$  and  $\mathcal{F}_t = \sigma(X_0, Z_0, \dots, Z_t, X_t)$ . Suppose that there exists a function  $r : \mathbb{N}^2 \to \mathbb{R}^+$ , reals  $z_0$  and A > 0 such that:

- for all  $t \ge 0$  and  $Z_t \ge z_0$ ,  $\mathbb{P}[X_{t+1} > X_t | \mathcal{F}_t] \le r(Z_t, \lambda_{t,Z_t})$  almost surely, where  $\lambda_{t,z}$  is the number of occurrences of the  $Z_t = z$  event for  $t' \le t$ ,
- for  $z \ge z_0$ ,  $\sum_m r(z,m) < \frac{A}{z}$

Then  $X_t$  is almost surely bounded.

Proof of Theorem 15. Let  $X_t := Ik_t$  and  $Z_t := \max_{j \in I} (Ik_t^j + j)$  be random variables that are an encoding of  $k_t$ ,  $k_t^j$ , j where j maximizes  $k_t^j$  and is maximal among the maximizers. Indeed,  $k_t = X_t/I$ ,  $k_t^j = [Z_t/I]$  and  $j = Z_t \mod I$ .

By definition,

$$Z_t = \max_{j \in I} \left( Ik_t^j + j \right) \leq \max_{j \in I} \left( I(k_t - 1) + j \right) \leq Ik_t = X_t.$$

$$(4.18)$$

Moreover,

$$Z_t \ge \max_{j \in I} Ik_t^j \ge I \max_{j \in I} k_t^j \ge I \frac{k_t}{I} = \frac{X_t}{I}.$$
(4.19)

Let  $z_0 = 2IL^2$ . Suppose  $Z_t \ge z_0$  and let j such that  $Z_t = Ik_t^j + j$ . In the following, we write  $k^j$  for  $k_t^j$  and  $\tau^j$  for  $\tau_{kj}^j$ . Then the probability that j plays 0 is  $x_{t,kj}^j(0)$  which by Lemma 17 satisfies

$$x_{t,k^{j}}^{j}(0) \ge \frac{x_{\tau^{j},k^{j}}^{j}(0)}{x_{\tau^{j},k^{j}}^{j}(0) + \left(1 - x_{\tau^{j},k^{j}}^{j}(0)\right) \exp\left(-\eta k^{j}\lambda_{k^{j},t}\right)}$$
(4.20)

$$= \frac{1}{1 + \left(\frac{1}{x_{\tau^{j},k^{j}}^{(0)} - 1}\right) \exp\left(-\eta k^{j} \lambda_{k^{j},t}\right)}.$$
(4.21)

Moreover, by Lemma 18, there exists B > 0 such that  $x_{\tau^j,k^j}^j(0) \ge \frac{1}{1+B\exp(-\eta k^j)}$ , hence from Eq. (4.21)

$$x_{t,k^{j}}^{j}(0) \ge \frac{1}{1 + B \exp(-\eta k^{j}) \exp(-\eta k^{j} \lambda_{k^{j},t})}.$$
(4.22)

At each period, there are I new jobs. By Assumption 3, there are less than L new jobs. Since j has more than  $[z_0/I] = 2L^2$  jobs, when j plays 0, we know that at least L jobs are not late. Therefore, the number of jobs at the next period has to be smaller or equal compared to the current period. Hence,

$$\mathbb{P}\left[X_{t+1} > X_t \mid \mathcal{F}_t\right] \leq 1 - x_{t,k^j}^j(0) \leq \frac{B\exp(-\eta k^j)\exp(-\eta k^j\lambda_{k^j,t})}{1 + B\exp(-\eta k^j)\exp(-\eta k^j\lambda_{k^j,t})}$$
(4.23)

using Eq. (4.22).

This suggests the following definition,

$$r(Z_t, \lambda_{Z_t, t}) := B \exp(-\eta [Z_t/I]) \exp(-\eta [Z_t/I] \lambda_{Z_t, t})$$

$$(4.24)$$

which is equal to

 $B\exp(-\eta k^j)\exp(-\eta k^j\lambda_{Z_t,t}),$ 

because  $j = Z_t \mod I$ ,  $k^j = [Z_t/I]$  and consequently,  $\lambda_{Z_t,t}$  (the number of times  $Z_t$  was equal to the current value) is lower than  $\lambda_{k^j,t}$  (the number of times that j had the current number of jobs). Therefore,

$$\exp(-\eta k^j \lambda_{k^j,t}) \leqslant \exp(-\eta k^j \lambda_{Z_t,t}).$$
(4.25)

Using previous equations,

$$\mathbb{P}\left[X_{t+1} > X_t \mid \mathcal{F}_t\right] \leqslant B \exp(-\eta k^j) \exp(-\eta k^j \lambda_{k^j,t})$$
(4.26)

$$\leq B \exp(-\eta k^j) \exp(-\eta k^j \lambda_{Z_{t},t}) \tag{4.27}$$

$$\leq r(Z_t, \lambda_{Z_t, t}) \tag{4.28}$$

where 4.26 comes from Eq. (4.23), 4.27 from Eq. (4.25) and 4.28 from Eq. (4.24).

For all  $z \ge z_0$ , the sum on m of r(z, m) is

$$B\exp(-\eta[z/I])\frac{1}{1-\exp(-\eta[z/I])}$$

so it is bounded by A/z for some A > 0 and Lemma 19 applies, so we proved the theorem.

### 5 Conclusions

We have studied a repeated strategic queueing model with spillover from one period to another. We have focused on the stability of the system when players play learning strategies. Several problems remain open in this model.

**Multi-Level regret** We have used a multi-level EWA. Although MLEWA is based on a no-regret algorithm, to prove that it is *itself* a no-regret algorithm, we would need an appropriate definition of regret in the context of endogenously changing states. Proving the no-regret property of MLEWA with a suitable definition of regret and using it to prove the system stability would be an interesting generalization. Another promising research direction is the definition of other multi-level algorithms based on different no-regret algorithms. In particular, it would be important to see which stability properties depend on the specific algorithm used and which other properties are general and hold

for every no-regret algorithm.

**Model** Several extensions of the model are conceivable. For instance, a model with more than one server could be studied. In that case the strategy of each player would have two components: the chosen server and the chosen time at which jobs join the chosen server's queue. An apparently simple, but non-trivial generalization would involve the consideration of lower penalty costs.

**Importance of the value of**  $C_k$  Several results of our paper are based on the value of the penalty  $C_k$ . If it is large enough, then the system is stable. We conjecture that a large but constant penalty is not sufficient for the stability in the learning context; i.e., the penalty must depend on the number of jobs in the system, otherwise the number of jobs could be unbounded with a positive probability. This contrasts with the myopic strategic case, where a constant penalty cost guarantees stability, if it is large enough.

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# A Symbols and acronyms

# Symbols

- a, b Typical pure action profiles, with  $a^i, b^i$  the actions played by i. 5
- $a_t^i$  In a repeated setting, action played by *i* at time *t*. 5
- $C, C_{k_t}$  Penalty incurred by a job that arrives after the end of a period. 5
- $\eta$  Smoothness of weights updating in EWA. 15
- $h_t$  History: state-action profile pairs of previous periods. 6
- L Length of each period. 5
- $\lambda_{t,n}$  Number of times between 1 and t that  $k_t^i$  is equal to n. 18

- S State of the system, so the number of jobs that every player holds. 5
- $\sigma$  Typical mixed action profiles, with  $\sigma^i$  the mixed action played by *i* and  $\sigma^i(a^i)$  the probability for *i* to play  $a^i$ . 7
- $\mathsf{SC}(\boldsymbol{\sigma}_t)$  Social cost of action profile  $\boldsymbol{\sigma}$ , that is  $\sum_{i\in I} c^i(\boldsymbol{\sigma})$ . 7
- $\tau_n^i$  First time that player i has n jobs to dispatch. 15
- $w_t^i(b^i)$  Weight of action  $b^i$  of player i in the EWA at time t. 15
- $\boldsymbol{w}_{t,n}^i(b^i)$  Weight of action  $b^i$  of player i in level n in the MLEWA at time t. 16
- $x^i$  Strategy of player *i* in a repeated context, *i.e.*, a function  $\mathcal{H} \to \triangle(A)$ . 6
- $x_t^i$  Strategy of player *i* at time *t*, *i.e.*,  $x^i(h_t)$ . 6

### Acronyms

FIFO first-in first-out

LIFO last-in first-out

**CCE** coarse correlated equilibrium

**DQM** dynamic queueing model

EWA exponential weight algorithm

MLEWA multi-level EWA

**NE** Nash equilibrium

# B Proof of the reinforced random walk lemma

Proof of Lemma 19. The following proof is inspired by Pemantle (2007).

We first prove that the probability that  $\sup_{t\in\mathbb{N}} X_t \in [x - M, x]$  for all x such that  $x \ge x_0 + M$ , conditionally on the fact that [x - M, x] is reached, is lower bounded by something strictly greater than 0 and that does not depend on x. As we will show, this implies that almost surely there exists x such that  $\sup_{t\in\mathbb{N}} X_t = x$ .

Denote the event that [x-M, x] is reached by  $X_t$  by A(x). We now show that for all  $x \ge x_0 + M$ :

$$\mathbb{P}\left[\sup_{t\in\mathbb{N}}X_t\in[x-M,x]\mid A(x)\right] \ge \prod_{z\in\left[\frac{x-M}{d},x\right]}\prod_{m\in\mathbb{N}}(1-r(z,m))$$
(B.1)

To prove this result, we fix  $x \in \mathbb{N}$  such that  $x > dz_0 + M$  and we introduce an extended random process. Define the new state space  $\overline{\Omega} = \mathbb{N}^2 \times \mathbb{N} \times \mathbb{N}^x \times \{0, 1\}$ . The interpretation of the a state  $(x, z, n_0, ..., n_x, i)$  is the following:

- the current state is (x, z),
- the path of  $Z_t$  has gone  $n_r$  times through the state r,
- *i* is equal to 1 if and only if  $X_t$  went up from a state between x and x M.

Formally, let  $(X, Y, N_0, ..., N_x, I)_{t \ge 1}$  be the random process on  $\overline{\Omega}$ . The first coordinate is equal to  $X_t$  whereas all other coordinates are deduced from it. By construction, we know that  $X_t$  has a maximal increment of M, hence in order for the supremum to be strictly greater than x, it is necessary for a positive jump from a state between x - M and x, hence

$$\{\sup_{t\in\mathbb{N}} X_t > x\} \subset \{\exists t \ge 1, I_t = 0\}.$$

It follows that

$$\mathbb{P}\left[\sup_{t\in\mathbb{N}}X_t\in[x-M,x]\mid A(x)\right]\geqslant\mathbb{P}\left[\forall t\geqslant 1, I_t=0\mid A(x)\right].$$

Moreover,

$$\mathbb{P}\left[\forall t \ge 1, I_t = 0 \mid A(x)\right] \ge \prod_{z \in \left[\frac{x-M}{d}, x\right]} \prod_{m \in \mathbb{N}} (1 - r(z, m)).$$

Indeed, by construction of the auxiliary random process, we know that:

- for every  $r \in \{0, ..., M\}$ ,  $N_r$  is only increasing,
- conditionally on  $Z_t = z \ge \frac{x-M}{d}$ ,  $N_z = n$  and the past, the probability for *i* to stay equal to 0 is at least (1 r(z, n)),
- conditionally on  $z < \frac{x-M}{d}$ , the probability for *i* to stay equal to 0 is 1.

It follows that

$$\mathbb{P}\left[\sup_{t\in\mathbb{N}}X_t\in[x-M,x]\mid A(x)\right] \ge \mathbb{P}\left[\forall t\ge 1, I_t=0\mid A(x)\right]$$
(B.2)

$$\geq \prod_{z \in \left[\frac{x-M}{d}, x\right]} \prod_{m \in \mathbb{N}} (1 - r(z, m)).$$
(B.3)

Then, the logarithm of the right hand side is

$$\begin{split} \sum_{z \in \left[\frac{x-M}{d}, x\right]} \sum_{m \in \mathbb{N}} \log(1 - r(z, m)) &= \sum_{z \in \left[\frac{x-M}{d}, x\right]} \sum_{m \in \mathbb{N}} -\log(1 + \frac{r(z, m)}{1 - r(z, m)}) \\ &\geqslant \sum_{z \in \left[\frac{x-M}{d}, x\right]} \sum_{m \in \mathbb{N}} -\frac{r(z, m)}{1 - r(z, m)} \\ &\geqslant \sum_{z \in \left[\frac{x-M}{d}, x\right]} \sum_{m \in \mathbb{N}} -\frac{r(z, m)}{1 - \rho} \\ &= \sum_{z \in \left[\frac{x-M}{d}, x\right]} -\frac{A}{2(1 - \rho)} \\ &\geqslant -\frac{A}{1 - \rho} \sum_{z \in \left[\frac{x-M}{d}, x\right]} \frac{d}{x - M} \\ &= -\frac{A}{1 - \rho} \left(x - \frac{x - M}{d} + 1\right) \frac{d}{x - M} \\ &= -\frac{A}{1 - \rho} \frac{(d - 1)x + M + d}{x - M} \geqslant -B > 0, \end{split}$$

where B > 0 is a positive constant which does not depend on x.

It follows that

$$\mathbb{P}\left[\sup_{t\in\mathbb{N}}X_t\in[x-M,x]\mid A(x)\right]\ge\exp\left(-B\right)>0.$$
(B.4)

The probability that the upper bound of  $X_t$  belongs to [x - M, x] is therefore lower bounded by a constant independent of x conditionally on the fact that this interval is reached.

$$\begin{split} \mathbb{P}\left[\sup_{t} X_{t} < \infty\right] &= \sum_{k \geqslant 1} \mathbb{P}\left[\sup_{t} X_{t} \in \left[(k-1)M, kM\right]\right] \\ &\geqslant \sum_{k \geqslant \left[\frac{x_{0}}{M}\right]+1} \mathbb{P}\left[\sup_{t} X_{t} \in \left[(k-1)M, kM\right] \mid A(kM)\right] \mathbb{P}\left[A(kM)\right]. \end{split}$$

However, if  $X_t$  is unbounded, then A(kM) happens, so  $\mathbb{P}[A(kM)] \ge \mathbb{P}[\sup_t X_t = +\infty]$ . Therefore, using (B.4):

$$\mathbb{P}\left[\sup_{t} X_t < \infty\right] \ge \sum_{k \ge \left[\frac{x_0}{M}\right]} \exp\left(-B\right) \mathbb{P}\left[\sup_{t} X_t = +\infty\right].$$

The right hand side is equal to  $\infty$  if  $\mathbb{P}[\sup_t X_t = +\infty] > 0$ , so necessarily  $\mathbb{P}[\sup_t X_t = +\infty] = 0$ .  $\Box$