Chapter 10

Polyhedral Approaches

Several problems from various areas reduce to maximizing (or minimizing) a linear function with linear constraints with bivalent variables. These problems, said to be combinatorial optimization problems, are generally NP-hard. Effective methods have therefore been developed to formulate and solve this type of problem. In particular, polyhedral approaches have proved to be powerful for optimally solving these problems. These consist of transforming the problem into a linear program by describing the convex hull of its solutions by a system of linear inequalities. The equivalence established between separation and optimization on a polyhedron, and the evolution of computational tools, have given an important boost to these methods. In fact, using these techniques we can develop polynomial algorithms and min–max relationships between combinatorial structures. These approaches have been successfully applied to several combinatorial optimization problems in the last few years. In this chapter, we will discuss these methods and introduce some applications to the maximum cut and network design problems.

10.1. Introduction

During the last three decades, combinatorial optimization has developed considerably, as much on the theoretical side as on the application side. Several techniques have been developed, and have been shown to be effective in formulating and solving hard combinatorial problems from areas as diverse as transport and telecommunications, biology, VLSI circuits, and statistical physics [GRA 95].

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Given that the number of solutions of a combinatorial optimization problem is finite, in order to solve the problem we might think of using, for example, an enumerative method consisting of enumerating all the solutions of the problem, calculating the value of each solution, and choosing the best of them. However, even though the number of solutions is finite, it can be exponential, and hence this method can rapidly reach its limits, even for small problems. If we consider this method, for instance for the traveling salesman problem for 30 towns, and using the most powerful computer currently available, more than 10^{10} centuries would be needed to find an optimal solution. This shows that such naïve methods cannot be applied to combinatorial optimization problems.

As a consequence, other, more powerful tools have proved to be necessary for tackling this kind of problem. Linear programing and integer programming are at the root of such tools. A combinatorial optimization problem can always be formulated as an integer program (generally in 0–1). Therefore any relaxation of the problem, obtained by relaxing the integrality constraints and by considering a subset of constraints, is no more than a linear program. If, by solving this program, we obtain a feasible integer solution of the problem, then it is optimal.

At the end of the 1940s, Dantzig [DAN 51, DAN 63] introduced the first algorithm, the simplex method, for solving linear programming problems. This method not only proved to be practically effective, but it constitued a very important basic tool for combinatorial optimization. Indeed, an optimal solution of a given linear program, found using the simplex algorithm, always corresponds to an extreme point of the polyhedron given by the inequalities of the program. In this way, this algorithm allows us to solve any combinatorial optimization problem whose set of solutions corresponds to the vertices of a polyhedron whose description is given by a system of linear inequalities. It follows from this that, given a combinatorial optimization problem where every solution can be represented by a vector of integers, if we can describe the convex hull of these points using a linear system, we thus reduce the problem to solving a simple linear program.

This transformation from optimization on a finite discrete set, which can have a very large number of solutions, to optimization on a convex domain, has been at the origin of an important evolution in combinatorial optimization. Indeed, this reduction has allowed us to introduce a new approach, known as the *polyhedral* approach, for combinatorial optimization problems. This consists of reducing such a problem to the resolution of a linear program by describing the convex hull of its solutions by a linear inequality system. This method, initiated by Edmonds [EDM 65] for the matching problem, was later revealed to be very powerful for optimally solving these problems. In particular, this method allows us to solve effectively a combinatorial optimization problem even if we only have a partial description of the convex hull of its solutions and even if the latter contains an exponential number of constraints. It also allows us to obtain min–max relations and to devise polynomial-time algorithms.

integral systems, and blocking and antiblocking polyhedra. In section 10.6, we introduce cut, and branch-and-cut, methods. We also discuss the relationship between separation and optimization. In sections 10.7 and 10.8, we introduce some applications of these techniques to the maximum cut and survivable network design problems.

The rest of this section is devoted to some definitions and notations. We consider non-directed graphs. A graph will be expressed by $G = (V, E)$, where V is the set of *vertices* and E is the set of *edges*. If e is an edge between two vertices u and v, then we denote it by $e = uv$. A path between two vertices u and v in G is a sequence of vertices and edges $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{l-1}, e_l, v_l)$, where $u = v_0$, $v = v_l$ and $e_i = v_{i-1}v_i$ for $i = 1, \ldots, l$ and $v_0, \ldots, v_l$ are distinct vertices of V. Vertices u and v are said to be the *extremities* of the path. A path will be given by its set of edges $(e_1, \ldots, e_l)$. Two paths between two vertices u and v are said to be *edge-disjoint* (resp. *vertex-disjoint*) if they do not have any edges (resp. vertices, except for u and v) in common. A *cycle* is a path whose extremities coincide. If $F \subseteq E$, then $V(F)$ will refer to the set of vertices of the edges of F. If $S \subseteq V$, we will denote by $E(S)$ the set of edges having their extremities in S.

Let $E = \{e_1, \ldots, e_n\}$ be a finite set. If $F \subseteq E$ and $x = (x(e),\ e \in E) \in \mathbb{R}^E$, then we express by $x(F)$ the sum $\sum_{e \in F} x(e)$. If a and x are vectors of $\mathbb{R}^E$, we denote the sum $\sum_{e \in E} a(e)x(e)$ by ax. Therefore the inequality $\sum_{e \in E} a(e)x(e) \leq \alpha$ is written as $ax \leq \alpha$.

10.2. Polyhedra, faces and facets

In this section, we present some definitions and basic properties of polyhedral theory. In particular, we discuss the description of a polyhedron by its facets.

10.2.1. *Polyhedra, polytopes and dimension*

DEFINITION 10.1.– *A polyhedron $P \subseteq \mathbb{R}^n$ is the set of solutions of a finite system of linear inequalities, that is:*

$$P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$$

where A is a matrix $m \times n$, $b \in \mathbb{R}^m$, and m and n are two positive integers.

A bounded polyhedron is known as a *polytope*. In other words, a polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if $x^1, x^2 \in \mathbb{R}^n$ exists such that $x^1 \leq x \leq x^2$ for all $x \in P$.

Note that any polyhedron P is *convex*, that is if x^1 and x^2 are any two points of P, then $\lambda x^1 + (1 - \lambda)x^2$ is also a point of P for any $0 \leq \lambda \leq 1$.

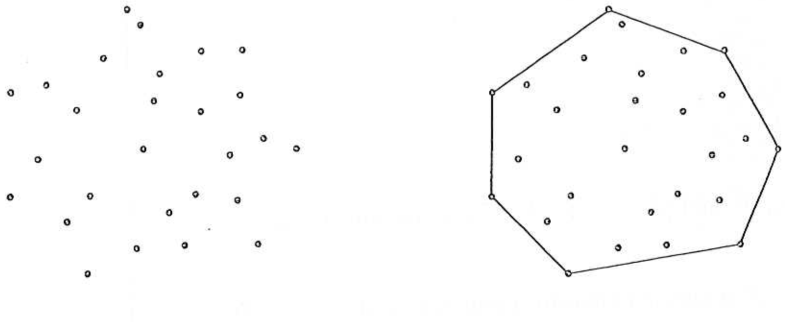

Figure 10.1. *Convex hull*

The following proposition establishes the relationship between optimizing on S and optimizing on the convex hull of S.

PROPOSITION 10.2.– *Let $S \subseteq \mathbb{R}^n$ be a set of points and ω a vector of $\mathbb{R}^n$. Thus*

$$\max\{\omega x \ : \ x \in S\} = \max\{\omega x \ : \ x \in conv(S)\}$$

Proof. Let $\bar{x} \in S$ and $x^* \in conv(S)$ such that $\omega\bar{x} = \max\{\omega x \ : \ x \in S\}$ and $\omega x^* = \max\{\omega x \ : \ x \in conv(S)\}$. Since $\bar{x} \in S$, and therefore $\bar{x} \in conv(S)$, then $\omega\bar{x} \leq \omega x^*$. Furthermore, using linear programming, we can assume that x^* is an extreme point of $conv(S)$, and consequently $x^* \in S$. This means that $\omega x^* \leq \omega\bar{x}$, and so $\omega x^* = \omega\bar{x}$. ∎

Proposition 10.2 establishes the link between combinatorial optimization and linear programming. An optimal solution of a combinatorial optimization problem can be obtained by solving the linear program induced by the convex hull of its solutions.

Let us now consider a polyhedron $P \subseteq \mathbb{R}^n$, and let us assume that P is described by a system of m_1 inequalities and m_2 equations, that is P is expressed as:

$$P = \left\{ \begin{array}{lll} x \in \mathbb{R}^n : & A_i x \leq b_i, & i = 1, \ldots, m_1 \\ & B_j x = d_j, & j = 1, \ldots, m_2 \end{array} \right\} \quad [10.2]$$

This implies that for any inequality $A_i x \leq b_i$, $i \in \{1, \ldots, m_1\}$, there exists a solution $\tilde{x}$ of P such that $A_i\tilde{x} < b_i$. In what follows, we will denote by A and B the matrices whose rows are A_i, $i = 1 \ldots, m_1$ and B_j, $j = 1, \ldots, m_2$, respectively, and

DEFINITION 10.8.– *If $ax \leq \alpha$ is a valid inequality, then the polyhedron:*

$$F = \{x \in P : ax = \alpha\}$$

is called a face of P, and we say that F is defined by the inequality $ax \leq \alpha$.

By convention, the empty set and the polytope P itself are considered as faces of P. A face of P is said to be *proper* if it is non-empty and different from P (see Figure 10.2).

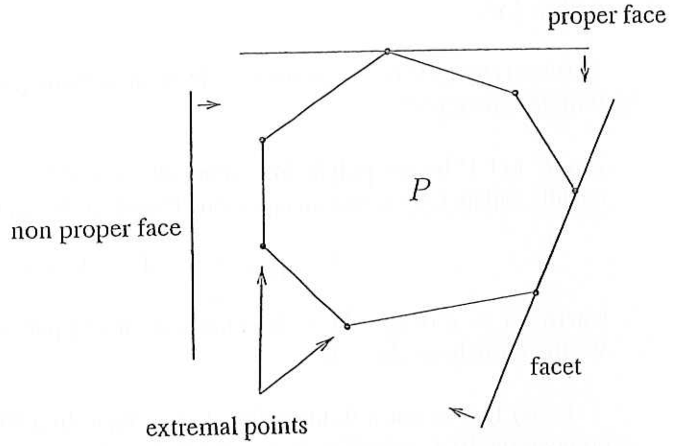

Figure 10.2. *Faces and facets*

PROPOSITION 10.5.– *A non-empty subset F of P is a face of P if and only if a subsystem $A'x \leq b'$ of $Ax \leq b$ exists such that $F = \{x \in P : A'x = b'\}$.*

Proof. ($\Longrightarrow$) Let us assume that F is a face of P. So an inequality $ax \leq \alpha$, valid for P, exists such that $F = \{x \in P : ax = \alpha\}$. Let us consider the linear program

$$\max\{ax : x \in P\} \qquad [10.3]$$

The optimal solutions of the program [10.3] are precisely the elements of F. Let (y^1, y^2) be a dual optimal solution of the program [10.3], where y^1 and y^2 are the dual vectors corresponding to the systems $Ax \leq b$ and $Bx = d$, respectively. Let $A'x \leq b'$ be the subsystem of $Ax \leq b$ whose dual variables have a strictly positive value. From the conditions of complementary slackness in linear programming, we have $F = \{x \in P : A'x = b'\}$.

must exist a point $x^* \in \mathbb{R}^n \setminus P$ such that:

$$A_j x^* \leq b_j, \quad \forall\, j \in \{1, \ldots, m_1\} \setminus \{i\}$$
$$A_i x^* > b_i$$
$$B x^* = d$$

Since $P \neq \emptyset$, from proposition 10.3, P contains an interior point, let us say $\hat{x}$. Therefore $A_i \hat{x} < b_i$. Let z be a point on the segment between x^* and $\hat{x}$ such that $A_i z = b_i$. So $z = \lambda \hat{x} + (1 - \lambda) x^*$ with $0 < \lambda < 1$. Furthermore, we have:

$$A_j z < b_j, \quad \forall\, j \in \{1, \ldots, m_1\} \setminus \{i\}$$
$$A_i z = b_i$$
$$B z = d$$

This implies that z belongs to the set $F = \{x \in P \ : \ A_i x = b_i\}$, the face of P defined by $A_i x \leq b_i$. Note that the system given by the equations of F is $\{A_i x = b_i,\ Bx = d\}$. Since $\hat{x} \in P \setminus F$, A_i is linearly independent of the rows of B. So, $\text{rank}\begin{pmatrix} A_i \\ B \end{pmatrix} = \text{rank}(B) + 1$. Hence $\dim(F) = \dim(P) - 1$, and, consequently, F is a facet of P. ∎

In what follows, we need theorem 10.1, known as Farkas' lemma, which is one of the fundamental results in mathematical programming. For the proof, see [COO 98].

THEOREM 10.1.– (Farkas' lemma for inequalities). *Given a matrix $m \times n$ A and a vector $b \in \mathbb{R}^m$, the system $Ax \leq b$ has a solution if and only if there does not exist a vector $y \geq 0$ of $\mathbb{R}^m$ such that $yA = 0$ and $yb < 0$.*

COROLLARY 10.1.– (Farkas' lemma). The system $Ax = b$ allows a solution (resp. a positive solution) if and only if there does not exist a vector y such that $yA = 0$ and $yb < 0$ (resp. $yA \geq 0$ and $yb < 0$).

See Complementary Slackness.

The following proposition shows that a facet of P must be defined by at least one of the inequalities of P.

PROPOSITION 10.8.– *For each facet F of P, one of the inequalities defining F is necessary in the description of P.*

Proof. Let us assume that F is defined by each of the inequalities $A_i x \leq b_i$, for $i \in I$, where I is a subset of $\{1, \ldots, m_1\}$. Let $\tilde{P}$ be the polyhedron obtained from P by removing all the inequalities $A_i x \leq b_i$, $i \in I$. We will show that $\tilde{P} \setminus P \neq \emptyset$, does not have a solution. Thus by corollary 10.1, a solution $\bar{x} \in \mathbb{R}^n$ exists such that:

$$\begin{aligned} a_1\bar{x} &= 0 \\ B\bar{x} &= 0 \\ a_2\bar{x} &> 0 \end{aligned}$$

Let $I \subset \{1, \ldots, m_1\}$ such that $A_i x \leq b_i$ defines F for every $i \in I$. Note that from proposition 10.8, $I \neq \emptyset$. Since F is a facet, and, consequently, $F \neq \emptyset$, from proposition 10.3, F contains an interior point, say x^*. Therefore $A_k x^* < b_k$ for every $k \in \{1, \ldots, m_1\} \setminus I$. Let $\epsilon > 0$ such that $A_k x^* + \epsilon A_k \bar{x} \leq b_k$ for every $k \in \{1, \ldots, m_1\} \setminus I$. Let $\hat{x} = x^* + \epsilon\bar{x}$. Since $a_2\bar{x} > 0$ and $a_2 x^* = \alpha_2$, we have $a_2\hat{x} = a_2 x^* + \epsilon a_2\bar{x} > \alpha_2$. As a consequence $\hat{x} \notin P$. Since $P \neq \emptyset$, let y be an interior point of P. We therefore have $a_1 y < \alpha_1$, and so $y \notin F$. Let z be the point of F on the segment between y and $\hat{x}$. Then $z = \nu y + (1-\nu)\hat{x}$ for a certain $0 < \nu < 1$. Furthermore, we have $a_1 z = \nu a_1 y + (1-\nu) a_1\hat{x} < \nu\alpha_1 + (1-\nu)\alpha_1 = \alpha_1$, which is impossible.

As a consequence, system [10.4] has a solution. As has been shown above, this solution also satisfies system [10.5], and we have$\lambda > 0$. ∎

From proposition 10.9, for every facet of P, one and only one inequality that defines F is necessary in the system that describes P. As a consequence of propositions 10.7, 10.8 and 10.9, we have theorem 10.2.

THEOREM 10.2.– *System [10.2] that defines P is minimal if and only if the rows of B are linearly independent and every inequality $A_i x \leq b_i$, $i = 1, \ldots, m_1$, defines a distinct facet of P.*

If P is a full-dimensional polyhedron, then, from proposition 10.9, two constraints induce the same facet if and only if one is a positive multiple of the other. We therefore have corollary 10.2.

COROLLARY 10.2.– If P is a full-dimensional polyhedron, then a unique minimal linear system (apart from multiplications by positive scalars) exists that describes P. Furthermore, every constraint of this system defines a distinct facet of P.

Another important class of faces is that of *minimal* faces, that is the faces that strictly do not strictly contain another face. We have the following result.

PROPOSITION 10.10.– (Hoffman and Kruskal [HOF 56]). *A non-empty subset F of P is a minimal face of P if and only if a subsystem $A'x \leq b'$ of $Ax \leq b$ exists such that $F = \{x \in \mathbb{R}^n \ : \ A'x = b', \ Bx = d\}$.*

Let $a = -y$. Since $ax^i \leq \alpha$ for $i = 1, \ldots, p$, from proposition 10.2 it follows that $ax \leq \alpha$ for every $x \in conv(S)$. Since $ax^0 > \alpha$, constraint $ax \leq \alpha$ then separates x^0 and $conv(S)$. □

DEFINITION 10.12.– *A cone is the set of solutions of a homogeneous finite system, that is a system in the form $Ax \leq 0$. A cone C is said to be generated by a set of points $\{x^1, \ldots, x^k\}$ if every point x of C can be expressed in the form $\sum_{i=1}^k \lambda_i x_i$ with $\lambda_i \geq 0$ for $i = 1, \ldots, k$.*

THEOREM 10.3.– (Minkowski [MIN 96], Weyl [WEY 50]). *A set $C \subseteq \mathbb{R}^n$ is a cone if and only if C is generated by a finite set of points.*

Proof. ($\Longrightarrow$) Let us assume that C is the set of solutions of a finite system $Ax \leq 0$. We will show that C can be generated by a finite set of points. The proof is by induction on the number of constraints in the system that defines C. If the system has no constraints (that is $C = \mathbb{R}^n$), then the points given by the unit vectors and the vector whose components are all equal to -1 generate C. Let us now assume that we have a finite set S' of points that generate $C' = \{x \in \mathbb{R}^n \ : \ A'x \leq 0\}$ and let us assume that $Ax \leq 0$ is obtained from $A'x \leq 0$ by adding a constraint $ax \leq 0$. Let S'_0, S'_+, S'_- be the subsets of points x of S' such that $ax = 0, ax > 0$ and $ax < 0$, respectively. Note that S'_0, S'_+, S'_- form a partition of S'. For every pair (x, x'), such that $x \in S'_+$ and $x' \in S'_-$, let us consider the vector:

$$y_{x,x'} = (ax)x' - (ax')x \qquad [10.7]$$

Therefore $ay_{x,x'} = 0$ and $A'y_{x,x'} = (ax)A'x' - (ax')A'x \leq 0$. Let $S = S'_- \cup S'_0 \cup \{y_{x,x'} \ : \ x \in S'_+, x' \in S'_-\}$. It then follows that $S \subseteq \{x \in \mathbb{R}^n \ : \ A'x \leq 0, ax \leq 0\}$. We will show in what follows that S generates C. For this, let us consider a solution x^* of C. Since $C \subseteq C'$, then $x^* = \sum_{x \in S'} \lambda_x x$, where $\lambda_x \geq 0$ for every $x \in S'$. The solution x^* can also be written as:

$$x^* = \textstyle\sum_{x \in S'_+} \lambda_x x + \sum_{x \in S'_0} \lambda_x x + \sum_{x \in S'_-} \lambda_x x \qquad [10.8]$$

If $\lambda_x = 0$ for every $x \in S'_+$, since $S'_0 \cup S'_- \subseteq S$, x^* is therefore generated by elements of S. If $x \in S'_+$ with $\lambda_x > 0$ exists, then $\tilde{x} \in S'_-$ with $\lambda_{\tilde{x}} > 0$ must exist. Otherwise, we would have $ax^* > 0$, which would contradict the fact that $x^* \in C$. From expression [10.7], we have:

$$(ax)\tilde{x} + (-a\tilde{x})x - y_{x,\tilde{x}} = 0 \qquad [10.9]$$

From proposition 10.2, the optimal value of [10.10] is equal to that of the problem:

$$\max\{\omega x \ : \ x \in conv(S)\} \qquad [10.11]$$

Theorem 10.4 shows that the two problems are equivalent.

THEOREM 10.4.– *A (non-empty) set of points $P \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a set of points S such that $P = conv(S)$.*

Proof. ($\Longrightarrow$) Let us assume that P is a polytope. Let S be the set of its extreme points. It is clear that $conv(S) \subseteq P$. Let us assume that there exists a point $x^0 \in P \setminus conv(S)$. From proposition 10.11, an inequality $ax \leq \alpha$ exists such that $ax \leq \alpha$ for every $x \in conv(S)$ and $ax^0 > \alpha$. Let $\alpha^* = \max\{ax \ : \ x \in P\}$, and let $F = \{x \in P \ : \ ax = \alpha^*\}$. Note that F is a non-empty face of P. Since $x^0 \in P$, we must have $\alpha < \alpha^*$. But this implies that F does not contain any extreme points of P, which would contradict corollary 10.3.

($\Longleftarrow$) Let $S = \{x^1, \ldots, x^p\}$ be a finite set of points of $\mathbb{R}^n$ and let $P = conv(S)$. We will show that P is the set of solutions of a finite linear system. To do this, let us consider the set $T \subseteq \mathbb{R}^{n+1}$ given by the points $(\lambda, y) \in \mathbb{R} \times \mathbb{R}^n$ such that:

$$\begin{aligned} &-1 \leq \lambda \leq 1 \\ &-\mathbf{1} \leq y^T \leq \mathbf{1} \\ &yx^i \leq \lambda, \quad \text{for } i = 1, \ldots, p \end{aligned}$$

where $\mathbf{1}$ denotes the vector of $\mathbb{R}^n$ whose components are all equal to 1. It is clear that T is a polytope. Let $(\lambda_1, y^1), \ldots, (\lambda_t, y^t)$ be the extreme points of T. From the first part of this proof, $T = conv(\{(\lambda_1, y^1), \ldots, (\lambda_t, y^t)\})$. We will show that P is the set of solutions of the system:

$$y^i x \leq \lambda_i, \quad \text{for } i = 1, \ldots, t. \qquad [10.12]$$

To do this, we first show that every point of P is a solution of system [10.12]. Indeed, if $\bar{x} \in P$, then $\bar{x} = \mu_1 x^1 + \ldots + \mu_p x^p$ for certain scalars $\mu_1, \ldots, \mu_p \geq 0$ such that $\sum_{i=1}^p \mu_i = 1$. Therefore $y^i\bar{x} = \mu_1 y^i x^1 + \ldots + \mu_p y^i x^p \leq \mu_1 \lambda_i + \ldots + \mu_t \lambda_i = \lambda_i$ for $i = 1, \ldots, t$. This implies that $\bar{x}$ is a solution of system [10.12].

Let us now consider a solution $\tilde{x}$ of system [10.12]. If $\tilde{x} \notin P$, then from proposition 10.11, an inequality $ax \leq \alpha$ exists such that $ax \leq \alpha$ for every $x \in P$ and $a\tilde{x} > \alpha$. By dividing the inequality by an appropriate coefficient, we can assume that $\mathbf{-1} \leq a^T \leq \mathbf{1}$ and $-1 \leq \alpha \leq 1$. Therefore $(\alpha, a) \in T$, and consequently (α, a)

It is clear that $P \subseteq \text{dom}(P)$ and $\text{dom}(P)$ is unbounded. We have the following interesting algorithmic property.

COMMENT 10.3.– If $\omega \in \mathbb{R}^n_+$, then $\min\{\omega x \ : \ x \in P\} = \min\{\omega x \ : \ x \in \text{dom}(P)\}$.

Consequently, given a combinatorial optimization problem of the form $\min\{\omega x \ : \ x \in P\}$, where P is its associated polyhedron, instead of studying P, we can consider its dominant. This latter is generally simpler to characterize.

Given a set of points $S \in \mathbb{R}^n$, we say that an inequality $ax \leq \alpha$ is valid for S if it is satisfied for every point of S.

PROPOSITION 10.12.– *An inequality $ax \leq \alpha$ is valid for S if and only if it is valid for $conv(S)$.*

Proof. Since $S \subset conv(S)$, $ax \leq \alpha$ is valid for S if it is valid for $conv(S)$. Let us assume that $ax \leq \alpha$ is valid for S, and let us consider a solution $x \in conv(S)$. Therefore $x = \sum_{i=1}^{k} \lambda_i x^i$, where $x^i \in S$, $\lambda_i \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^{k} \lambda_i = 1$ for a certain integer k. Consequently, $ax = \sum_{i=1}^{k} \lambda_i (ax^i) \leq \sum_{i=1}^{k} \lambda_i \alpha = \alpha$. ∎

PROPOSITION 10.13.– *If F is a non-empty $(p-1)$-dimensional face of $conv(S)$, then there are p affinely independent points in $S \cap F$.*

Proof. By definition, there are p affinely independent points $x^1, \ldots, x^p$ in F. If $x^i \in S$ for $i = 1, \ldots, p$, then the assertion is proven. Let us assume that $x^1 \notin S$. Then $x^1 = \sum_{j=1}^{k} \lambda_j \bar{x}^j$, where $\bar{x}^j \in S$, $\lambda_j \geq 0$ for $j = 1, \ldots, k$, and $\sum_{j=1}^{k} \lambda_j = 1$ for a certain integer k. Furthermore, the points $\bar{x}^j$, $j = 1, \ldots, k$ are all in F. Indeed, this is clear if $F = conv(S)$. If F is defined by a constraint $ax \leq \alpha$ valid for $conv(S)$, then $ax^1 = \alpha$. Since $a\bar{x}^j \leq \alpha$ for $j = 1, \ldots, k$, it follows that $a\bar{x}^j = \alpha$ for $j = 1, \ldots, k$, and thus $\bar{x}^j \in F$ for $j = 1, \ldots, k$. Since $x^1, \ldots, x^p$ are affinely independent, $t \in \{1, \ldots, k\}$ exists such that $\bar{x}^t, x^2, \ldots, x^p$ are affinely independent. Now the proof can be completed by repeating this process for every point $x^i \notin S$. ∎

From propositions 10.12 and 10.13, it is enough to consider the points of S to establish the validity of a constraint for $conv(S)$ or the dimension of a face of $conv(S)$.

10.3.2. *Extreme points and extreme rays*

Let us consider a polyhedron $P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$, where A is an $m \times n$ matrix and b a vector of $\mathbb{R}^m$. Let us note that there may be inequalities in $Ax \leq b$ that are equally satisfied by the solutions of P.

DEFINITION 10.13.– *Let $P^0 = \{r \in \mathbb{R}^n \ : \ Ar \leq 0\}$. The points of $P^0 \setminus \{0\}$ are called rays of P.*

be an interior point of F. Since $\dim(F) > 0$, there must exist a further point $y \in F$. Consider the straight line Δ going through the points $\hat{x}$ and y. It contains the points $z(\lambda) = \lambda y + (1 - \lambda)\hat{x}$ for $\lambda \in \mathbb{R}$. Assume that Δ intersects one of the hyperplanes $\{x \in \mathbb{R}^n : A_i x = b_i\}$, $i \in I$. Let $\lambda^* = \min\{|\lambda^i| : i \in I, A_i z(\lambda^i) = b_i\}$, and let $i^* \in I$ such that $\lambda^* = |\lambda^{i^*}|$. Since $\hat{x}$ is an interior point of F, and, consequently, $A_{i^*}\hat{x} < b_{i^*}$, it follows that $\lambda^* \neq 0$. Consider the set $F^* = \{x \in F : A_{i^*}x = b_{i^*}\}$. It is clear that F^* is a face of P. Also, since $z(\lambda^{i^*}) \in F^* \setminus F$, F^* is a face of lower dimension, a contradiction.

Consequently, Δ does not intersect any of the hyperplanes $\{x \in \mathbb{R}^n : A_i x = b_i\}$, $i \in I$. This implies that $\Delta \subseteq P$, that is $A(\lambda y + (1-\lambda)\hat{x}) \leq b$ for every $\lambda \in \mathbb{R}$. As $A\hat{x} \leq b$, it follows that $A(y - \hat{x}) = 0$ for every $y \in F$. So, $F = \{y \in P : Ay = A\hat{x}\}$. Since $\text{rank}(A) = n - k$, by proposition 10.4, we have $\dim(F) = k$. ∎

THEOREM 10.7.– *Suppose that rank*$(A) = n$*, and the problem:*

$$\max\{\omega x : x \in P\} \qquad [10.16]$$

has a finite optimal solution. Then there exists an optimal solution of [10.16] that is an extreme point of P.

Proof. The set of optimal solutions of [10.16] is a non-empty face $F = \{x \in P : \omega x = \omega_0\}$. Since $\text{rank}(A) = n$, by theorem 10.6, F contains a 0-dimensional face. By proposition 10.6, it follows that F contains an extreme point. ∎

THEOREM 10.8.– *If rank*$(A) = n$ *and* $\max\{\omega x : x \in P\}$ *is unbounded, then* P *has an extreme ray* r^* *such that* $\omega r^* > 0$.

Proof. Since the linear program $\max\{\omega x : x \in P\}$ does not have a finite optimal solution, by duality in linear programming, the system $\{yA = \omega, \ y \geq 0\}$ does not have a solution. By Farkas' lemma, $r \in \mathbb{R}^n$ exists such that $Ar \leq 0$ and $\omega r > 0$. Consider the linear program:

$$\max\{\omega r : Ar \leq 0, \ \omega r \leq 1\}. \qquad [10.17]$$

The optimal value of [10.17] is then equal to 1. Since this value is bounded and $\text{rank}(A) = n$, by theorem 10.7, [10.17] has an optimal solution that is an extreme point of $\{r : Ar \leq 0, \ \omega r \leq 1\}$. Let r^* be such a point. It is clear that r^* is a point of $P^0 \setminus \{0\}$, and therefore r^* is a ray of P. Furthermore, since r^* is the unique solution of n equations of the system $\{Ar \leq 0, \ \omega r \leq 1\}$, the subsystem of $Ax \leq 0$, equally satisfied by r^*, must be of rank $n - 1$. Otherwise r^* would be zero, which a strong min–max relationship between the optimal solutions of the primal problem $\max\{\omega x : x \in P\}$ and its dual. Furthermore, the use facet defining inequalities in the framework of a cutting-planes method for the primal problem can enable us to speed up the resolution of the problem. Consequently, if P is the polytope associated with a combinatorial optimization problem, given a valid inequality for P, a fundamental question that arises is to determine if this inequality defines a facet of P. Moreover, given a system of inequalities valid for P, it would be interesting to see if the system completely describes P. In this section, we discuss certain proof techniques for these questions.

10.4.1. *Facet proof techniques*

10.4.1.1. *Proof of necessity*

A first technique for proving that a valid constraint $ax \leq \alpha$ defines a facet of P is to show that $ax \leq \alpha$ is *essential* in describing P. In other words, if $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, and if $ax \leq \alpha$ is one of the constraints of $Ax \leq b$, then we must show that there exists a point $\bar{x}$ such that $a\bar{x} > \alpha$, and $\bar{x}$ satisfies all the other inequalities of $Ax \leq b$. This implies that the constraint $ax \leq \alpha$ is necessary to describe P. If P is full-dimensional, from corollary 10.2, it follows that $ax \leq \alpha$ defines a facet. If P is not full-dimensional, then we must also show that $ax \leq \alpha$ is not an equation in the system $Ax \leq b$. To do this, it is enough to establish a solution $\hat{x} \in P$ such that $a\hat{x} < \alpha$.

This technique is generally used for simple constraints. We illustrate this here for the trivial inequalities of the stable set polytope.

A *stable set* in a graph $G = (V, E)$ is a subset of non-adjacent pairwise vertices. If each vertex v of V has a weight $\omega(v)$, *the stable set problem* is to determine a stable set S such that $\sum_{v \in S} \omega(v)$ is maximum. The stable set problem is **NP**-hard even when the weights are all equal to 1.

If $S \subseteq V$ is a subset of vertices, let $x^S \in \mathbb{R}^V$ be the incidence vector of S given by:

$$x^S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if not} \end{cases}$$

Let

$$P(G) = conv\{x^S \ : \ S \subseteq V \text{ is a stable set}\}$$

be the convex hull of the incidence vectors of all the stable sets of G. $P(G)$ is called *the stable sets polytope* of G. The stable set problem in G is therefore equivalent to the linear program $\max\{\omega x : x \in P(G)\}$. The stable set polytope has been widely studied and reported in the literature. Since the problem is **NP**-hard, the polytope $P(G)$ is known explicitly only for certain particular classes of graphs.

10.4.1.2. *Direct proof*

The most direct technique for showing that a valid constraint induces a facet consists of showing that there exist dim(P) affinely independent solutions satisfying the constraint with equality. If P is full-dimensional, we then obtain $ax \leq \alpha$ defines a facet of P. If P is not full-dimensional, as for the first method, we must also show that $ax \leq \alpha$ is not an equation in $Ax \leq b$ by determining a solution of P that satisfies $ax \leq \alpha$ with strict inequality. We will also illustrate this method on the stable set polytope.

A (simple) graph is said to be *complete* if an edge exists between each pair of vertices. If $G = (V, E)$ is a graph, a subset K of vertices of V is called a *clique* of G if it induces a maximal complete subgraph, that is a complete subgraph which is not strictly contained in any complete subgraph. If $S \subseteq V$ is a stable set, it is clear that S cannot intersect a clique in more than one vertex. This implies that the following constraints

$$x(K) \leq 1, \quad \forall\, K \text{ clique of } G \qquad [10.21]$$

are valid for the polytope $P(G)$.

PROPOSITION 10.18.– *Inequalities [10.21] define facets of* $P(G)$.

Proof. Since $P(G)$ is full-dimensional, it is enough to show that there exist $n = |V|$ stable sets of G whose incidence vectors satisfy [10.21] with equality and are affinely independent. Since K is a clique, for every vertex v in $V \setminus K$, a vertex $v' \in K$ exists such that $vv' \notin E$. Let us consider the sets

– $S_v = \{v\}$ for every $v \in K$,

– $S_v = \{v, v'\}$ for every $v \in V \setminus K$.

It is clear that these sets are stable sets of G. Furthermore, their incidence vectors x^{S_v}, $v \in V$ satisfy [10.21] with equality and are linearly independent. Since 0 does not belong to the affine hull of the incidence vectors x^{S_v}, $v \in V$, from comment 10.1, these points are also affinely independent. ∎

Observe that if uv is an edge such that $\{u, v\}$ is contained in a clique K, then inequality [10.18] corresponding to uv is dominated by [10.21].

10.4.1.3. *Proof by maximality*

A final method for showing that a constraint $ax \leq \alpha$, valid for P, defines a facet, consists of proving that the face $F = \{x \in P \;:\; ax = \alpha\}$ induced by $ax \leq \alpha$ is not strictly contained in a facet of P. In other words, if $F \subseteq \{x \in P \;:\; bx = \beta\}$, where

Consider a vertex $w \in V \setminus W$. Since w is not adjacent to w_0, the set $S'' = \{w, w_0\}$ is then a stable set of G. Since $ax^{S''} = ax^{\{w_o\}} = \alpha$, and thus $bx^{S''} = bx^{\{w_o\}} = \beta$, it follows that $0 = bx^{S''} - bx^{\{w_o\}} = b(w)$. Since w is arbitrary in $V \setminus W$, we have:

$$b(w) = 0 \quad \text{for every } w \in V \setminus W \qquad [10.25]$$

From [10.23]–[10.25], it follows that $b = \rho a$. Furthermore, for every vertex v of G there is a stable set S of G which contains v such that $ax^S = \alpha$. The face defined by $ax \leq \alpha$ is different from a trivial face $\{x \in P(G) : x(v) = 0\}$. This implies that the face induced by $bx \leq \beta$ is not contained in a trivial face. From proposition 10.16, it follows that $b(v) \geq 0$ for every $v \in V$. Since $bx \leq \beta$ defines a facet of $P(G)$, there must exist at least one vertex $v \in V$ such that $b(v) > 0$. Consequently $\rho > 0$. ∎

10.4.2. *Integrality techniques*

Let $P \subseteq \mathbb{R}^n$ be the solutions polyhedron of a combinatorial optimization problem, and let $Ax \leq b$ be a system of valid inequalities for P. In what follows, we discuss techniques which enable us to show that $Ax \leq b$ completely describes P. For this, we assume that every integer solution of $Ax \leq b$ is a solution of the problem.

10.4.2.1. *Extreme points integrality proof*

A first technique for showing that $Ax \leq b$ describes P consists of proving that the extreme points of the polyhedron $\hat{P} = \{x \in \mathbb{R}^n : Ax \leq b\}$ are all integers. This would imply that $\hat{P} \subseteq P$. Since $P \subseteq \hat{P}$, we therefore have $P = \hat{P}$. To demonstrate this technique, we consider the 2-edge connected subgraph problem.

A graph G is said to be k-edge connected, for fixed $k > 0$, if between each pair of vertices of G, there exist at least k edge-disjoint paths. Given a graph $G = (V, E)$ and a weight function $\omega : E \rightarrow \mathbb{R}$ that associates the weight $\omega(e)$ with each edge $e \in E$, *the k-edge connected subgraph problem* is to determine a k-edge connected subgraph (V, T) of G containing all the vertices of V and such that $\sum_{e \in T} \omega(e)$ is minimum. This problem has applications in the design of reliable telecommunications networks [GRÖ 95, KER 05].

Let $G = (V, E)$ be a graph. If $W \subseteq V$, the set of edges having one extremity in W and the other in $V \setminus W$ is called a *cut*, and denoted by $\delta(W)$. If $W = \{v\}$, we write $\delta(v)$ instead of $\delta(\{v\})$. Theorem 10.10 (Menger's theorem), establishes a relationship between edge-disjoint paths and cuts in a graph.

THEOREM 10.10.– (Menger [MEN 27]). *In a graph G, there are k edge-disjoint chains between two vertices s and t if and only if every cut of G that separates s and t contains at least k edges.*

$e \in E \setminus \{e_0\}$. It is clear that $\bar{x}'$ is an extreme point of $Q(G - e_0)$, where $G - e_0$ is the graph obtained from G by removing e_0. Since $\bar{x}'$ is fractional and $Q(G - e_0) =$ TECP$(G - e_0)$, we have a contradiction. ∎

LEMMA 10.2.– *G is 3-edge connected.*

Proof. Assume that G is not 3-edge connected. Since G is 2-edge connected, then G contains a cut consisting of exactly two edges, say e_1 and e_2. Consequently, $x(e_1) = x(e_2) = 1$. Let $G^* = (V^*, E^*)$ be the graph obtained from G by contracting e_1. Let $x^* \in \mathbb{R}^{|E|-1}$ be the restriction of $\bar{x}$ on E^*. Then x^* is a solution of $Q(G^*)$. Furthermore, x^* is an extreme point of $Q(G^*)$. Indeed, if this is not the case, there must then be two solutions y', y'' of $Q(G^*)$ such that $x^* = \frac{1}{2}(y'+y'')$, hence $y'(e_2) = y''(e_2) = 1$. Consider the solutions $y^{*'}$, $y^{*''} \in \mathbb{R}^m$ given by:

$$y^{*'}(e) = \begin{cases} y'(e) & \forall\, e \in E^* \\ 1 & \text{if } e = e_1 \end{cases}$$

and:

$$y^{*''}(e) = \begin{cases} y''(e) & \forall\, e \in E^* \\ 1 & \text{if } e = e_1 \end{cases}$$

It is clear that $y^{*'}$, $y^{*''}$ are solutions of $Q(G)$. We also have $\bar{x} = \frac{1}{2}(y^{*'} + y^{*''})$, a contradiction. Consequently, x^* is an extreme point of $P(G^*)$. Since G^* is series-parallel, $|E^*| < |E|$, and x^* is fractional, this contradicts the minimality of $|E|$. ∎

By lemma 10.1, $\bar{x}(e) > 0$ for every $e \in E$. Since $\bar{x}$ is an extreme point of $Q(G)$, from corollary 10.3, there must be a set of cuts $\{\delta(W_i),\ i = 1, \ldots, t\}$ and a subset of edges $E_1 \subseteq E$ such that $\bar{x}$ is the unique solution of the system:

$$\begin{cases} x(e) = 1, & \forall\, e \in E_1 \\ x(\delta(W_i)) = 2, & i = 1, \ldots, t \end{cases} \qquad [10.29]$$

where $|E_1| + t = |E|$.

LEMMA 10.3.– *Each variable $x(e)$ has a non-zero coefficient in at least two equations of [10.29].*

Proof. It is clear that $x(e)$ must have a non-zero coefficient in at least one equation of system [10.29]. Otherwise every point $\bar{x}'$ such that $\bar{x}'(f) = \bar{x}(f)$ if $f \in E \setminus \{e\}$

$Ax \leq b$, we have $P = \{x : Ax \leq b\}$. The method can be presented as follows. We consider an inequality $ax \leq \alpha$ that defines a facet F of P. By using the structure of the extreme points of P (that is the solutions of the underlying combinatorial problem), we establish certain properties related to a, which imply that $ax \leq \alpha$ is a constraint of the system $Ax \leq \alpha$. We will illustrate this method on the matching polytope.

Given a graph $G = (V, E)$, a subset of pairwise non-adjacent edges is called a *matching*. If each edge of G has a certain weight, *the matching problem* in G is to determine a matching whose total weight is maximum. Edmonds [EDM 65] has shown that this problem can be solved in polynomial time. He also produced a linear system that completely describes the associated polytope.

If $G = (V, E)$ is a graph, the *matching polytope* of G, denoted by $P^c(G)$, is the convex hull of the incidence vectors of the matchings of G. It is not difficult to see that if F is a matching of G, then its incidence vector x^F satisfies the following constraints:

$$x(e) \geq 0 \quad \forall\, e \in E \qquad [10.30]$$

$$x(\delta(v)) \leq 1 \quad \forall\, v \in V \qquad [10.31]$$

$$x(E(S)) \leq \frac{|S|-1}{2} \quad \forall\, S \subseteq V,\ |S| \geq 3 \text{ and odd} \qquad [10.32]$$

Theorem 10.12, established by Edmonds [EDM 65], has been proven by Lovász by applying the technique described above [LOV 79].

THEOREM 10.12.– *For every graph $G = (V, E)$, the matching polytope $P^c(G)$ is given by inequalities [10.30]–[10.32].*

Proof. First of all, we can easily verify that $P^c(G)$ is full-dimensional. Indeed, the sets $\{e\}$, $e \in E$ with the empty set form a family of $|E| + 1$ matchings whose incidence vectors are affinely independent. Consequently, from corollary 10.2, two constraints define the same facet of $P^c(G)$ if and only if one is a positive multiple of the other.

Let $ax \leq \alpha$ be a constraint that defines a facet of $P^c(G)$, and let $\mathcal{C}_a$ be the set of matchings of G whose incidence vectors satisfy $ax \leq \alpha$ exactly. Assume that $ax \leq \alpha$ is different from constraints [10.30] and [10.31]. We will show that $ax \leq \alpha$ is necessarily of the type [10.32].

Since $ax \leq \alpha$ is different from inequalities [10.30], then $a(e) \geq 0$ for every $e \in E$. Indeed, if $a(e) < 0$ for a certain edge e, then every matching in $\mathcal{C}_a$ does not contain e, and, consequently, the face defined by $ax \leq \alpha$ is contained in the face linear program max $\{\omega x \ : \ Ax \leq b\}$ has an integer optimal solution. Indeed, by theorem 10.5, this implies that the extreme points of $\{x \in \mathbb{R}^n : Ax \leq b\}$ are all integers. This technique has been introduced by Edmonds [EDM 65] to show the integrality of system [10.30]–[10.32] for the matchings of a graph. Other techniques, based on projection, can also be used to show the integrality of a polyhedron [SCH 03].

10.5. Integer polyhedra and min–max relations

As has been highlighted in section 10.4, the principal motivation for proving the integrality of a polyhedron is to establish a combinatorial min–max relation between the optimal solutions of the underlying dual problems. This dual relations in combinatorial optimization has been the subject of extensive studies, which have led to the introduction of new concepts such as totally dual integral systems, and blocking and antiblocking polyhedra. In this section, we introduce these concepts and discuss some applications.

10.5.1. *Duality and combinatorial optimization*

Let:

$$\max \{\omega x \ : \ Ax \leq b, \ x \geq 0\} \qquad [10.34]$$

be a linear program (said to be primal) and:

$$\min \{b^T y \ : \ A^T y \geq \omega^T, \ y \geq 0\} \qquad [10.35]$$

be its dual. From linear programming duality, if one of problems [10.34] or [10.35] has an optimal solution, then the other also has one, and the two optimal solutions have the same value. In other words, if [10.34] has an optimal solution $\bar{x}$, then its dual [10.35] has an optimal solution $\bar{y}$ such that $\omega\bar{x} = b^T\bar{y}$. Thus, if two optimal solutions are known for the primal problem and the dual problem, then we obtain a min–max relation between the optimal solutions of the two problems. This can have interesting applications when the two dual problems have a combinatorial interpretation. Indeed, if the system $Ax \leq b$ describes the solutions polyhedron of a combinatorial optimization problem $\mathcal{P}_1$ and $A^T y \geq \omega^T$ describes that of a combinatorial optimization problem $\mathcal{P}_2$, and if [10.34] has an optimal solution, then we obtain a relation of the form:

$$\max \{\omega(F) \ : \ F \in \mathcal{F}_1\} = \min \{b(F) \ : \ F \in \mathcal{F}_2\} \qquad [10.36]$$

where A is the vertex-edge incidence matrix of G (let us remember that $\mathbf{1}$ refers to the vector whose components are all equal to 1). Since A is TU, by theorem 10.13, $P^*(G)$ is integral. Consequently, every extreme point of $P^*(G)$ represents a matching of G, and therefore $P^*(G) \subseteq P^c(G)$. It then follows that $P^*(G) = P^c(G)$.

Let us now consider the maximum cardinality matching problem. Since $P^*(G) = P^c(G)$, this problem is equivalent to the linear program:

$$\max \{\mathbf{1}^T x \; : \; Ax \leq \mathbf{1}, \; x \geq 0\} \qquad [10.37]$$

The dual of [10.37] can be written as:

$$\min \{\mathbf{1}^T y \; : \; A^T y \geq \mathbf{1}, \; y \geq 0\} \qquad [10.38]$$

Note that the variables in [10.38] correspond to the vertices of G and that the constraints correspond to the edges. Since the matrix A is TU, and A^T is also TU, an optimal solution $\bar{y}$ of [10.38] can be supposed to be integer. Therefore $\bar{y}$ is a 0–1 vector, and we can see that in this case $\bar{y}$ represents a set of vertices that covers all the edges of the graph. Such a set is called a vertex cover. From [10.36], we then obtain theorem 10.15.

THEOREM 10.15.– (König [KÖN 31]). *In a bipartite graph, the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.*

If $G = (V, E)$ is a directed graph, the vertex-arc incidence matrix of G is the matrix D whose rows correspond to the vertices of G, and columns to the arcs of G such that for a vertex i and an arc e, the entry $D_{i,e}$ is 1 if i is the initial vertex of e, -1 if i is the terminal vertex of e and 0 if not. This matrix is TU, and, as a consequence, we can obtain the famous maximum-flow minimum-cut theorem [FOR 56, FOR 62].

Unfortunately not all the integer matrices (in $-1, +1, 0$) are TU. Integral polyhedra exist for which the matrix of the corresponding system is not TU.

If $\{x \in \mathbb{R}^n \; : \; Ax \leq 0, \; x \geq 0\}$ is an integral polyhedron where A is not TU, although the linear program:

$$\max\{\omega x \; : \; Ax \leq b, \; x \geq 0\} \qquad [10.39]$$

still has an integer optimal solution, its dual may not have one (even if ω is integer). Consequently, no combinatorial min–max relationship can be obtained. A natural question that can therefore be asked is: under what conditions can the dual of [10.39] have an integer optimal solution each time that [10.39] has one for an integer ω? Edmonds and Giles [EDM 77] have studied this question, and have introduced the concept of a totally dual integral system, which we discuss in what follows.

Note that this theorem generalizes König's theorem 10.15. Extensions of the notion of TDI have also been studied. For more details see Schrijver [SCH 03].

10.5.4. *Blocking and antiblocking polyhedra*

Another aspect of the duality that has been studied in combinatorial optimization is that of blocking and antiblocking polyhedra. These concepts, introduced by Fulkerson [FUL 71, FUL 72], have been the source of several important developments in combinatorial optimization.

10.5.4.1. *Blocking polyhedra*

Let A be a positive $m \times n$ matrix and $w \in \mathbb{R}^m_+$. Consider the linear program:

$$\max\{1^T x \ : \ A^T x \leq w,\ x \geq 0\} \qquad [10.40]$$

Problem [10.40] is called a *packing problem*. Its dual can be written as:

$$\min\{w^T y \ : \ Ay \geq 1,\ y \geq 0\} \qquad [10.41]$$

Let $\mathcal{B}$ be the polyhedron given by $\{y \in \mathbb{R}^n_+ \ : \ Ay \geq 1\}$. Note that $\mathcal{B}$ is bounded and is full-dimensional. Moreover, the system $Ay \geq 1$ may contain redundant constraints. If $Ay \geq 1$ does not contain any redundant inequalities, then the matrix A is said to be *proper*. A matrix is also said to be proper if it does not contain any rows (that is $\mathcal{B} = \mathbb{R}^N_+$ or $\mathcal{B} = \emptyset$). If A is a proper 0–1 matrix, then the rows of A can be considered as incidence vectors of m non-comparable pairwise subsets of a set of n elements. In this case the family of subsets, represented by the rows of A, is said to be a *clutter*.

Let:

$$\hat{\mathcal{B}} = \{z \in \mathbb{R}^n_+ \ : \ z^T x \geq 1,\ \forall x \in \mathcal{B}\} \qquad [10.42]$$

The set $\hat{\mathcal{B}}$ is called a *blocking* of $\mathcal{B}$. Note that if $\mathcal{B} = \emptyset$, then $\hat{\mathcal{B}} = \mathbb{R}^n_+$, and if $\mathcal{B} = \mathbb{R}^n_+$, then $\hat{\mathcal{B}} = \emptyset$. Consequently, $\hat{\mathcal{B}}$ can be seen as the dual of $\mathcal{B}$ and *vice versa*. Fulkerson [FUL 71] established the following relationships between $\mathcal{B}$ and its blocking $\hat{\mathcal{B}}$.

THEOREM 10.20.– (Fulkerson [FUL 71]). *Let A be a proper matrix whose rows are $a^1, \ldots, a^m$. Let $b^1, \ldots, b^r$ be the extreme points of $\mathcal{B}$, and let B be the $q \times n$ matrix whose rows are $b^1, \ldots, b^q$. Then*

Let $G = (V, E)$ be a graph and s and t be two vertices of V. Let A be the matrix whose rows are the incidence vectors of the minimal paths between s and t. Let B be the matrix whose rows are the incidence vectors of the minimal cuts separating s and t. Note that the two matrices A and B are 0–1 matrices. Let $w \in \mathbb{R}^n_+$, where $n = |E|$. By considering w as a capacity vector associated with the edges of G, the problem [10.40] corresponding to A and w is nothing but the maximum flow problem between s and t in G with respect to the capacity vector w. From Fulkerson's max-flow min-cut theorem, we know that the maximum value of a flow between s and t is equal to the minimal capacity of a cut separating s and t, where the capacity of a cut is the sum of the capacities of the edges of the cut. Consequently, the min–max relation [10.43] is satisfied for A and B, which implies that A, B is a blocking pair.

If $w = \mathbb{1}^T$, the programs [10.40] and [10.41] for A can be written as:

$$\max \{\mathbb{1}^T x \ : \ A^T x \leq \mathbb{1},\ x \geq 0\} \qquad [10.44]$$

and:

$$\min \{\mathbb{1}^T y \ : \ Ay \geq \mathbb{1},\ y \geq 0\}. \qquad [10.45]$$

Note that the variables in [10.44] correspond to the minimal paths of G between s and t, and those in [10.45] correspond to the edges of G. Observe that the packing problem [10.44] has an integer optimal solution. Such a solution represents a packing of paths between s and t, that is a set of pairwise disjoint paths between s and t.

Since A and B form a blocking pair, from theorem 10.20, the extreme points of the polyhedron $\{y \in \mathbb{R}^n_+ : Ay \geq \mathbb{1}\}$ are precisely the minimal cuts separating s and t. From [10.43], we thus obtain the following min–max relation, which is nothing but theorem 10.10, Menger's theorem.

THEOREM 10.22.– *The minimum number of edges in a cut separating s and t is equal to the maximum number of pairwise disjoint paths between s and t.*

In a similar way, by using the fact that the min–max relationship is also satisfied for B and A, we obtain theorem 10.23.

THEOREM 10.23.– *The minimum number of edges in a chain between s and t is equal to the maximum number of pairwise disjoint cuts separating s and t.*

Other examples of blocking pairs and matrices are given in [FUL 71, SCH 03].

THEOREM 10.25.– (Fulkerson [FUL 71]). *The min–max relation [10.48] is satisfied for two matrices A and B (in this order) if and only if A and B form an antiblocking pair of matrices.*

If relation [10.48] is satisfied for A and B, from theorem 10.25, it is also satisfied for B and A.

To illustrate this concept, let us consider the matching polyhedron. We have seen in section 10.4 that if $G = (V, E)$ is a graph, the matching polytope of G, $P^c(G)$, is given by inequalities [10.30]–[10.32]. $P^c(G)$ can also be written in the form $P^c(G) = \{x \in \mathbb{R}^E \; : \; Bx \leq 1, \; x \geq 0\}$. If we denote by A the matrix whose rows are the incidence vectors of the matchings in G, the min–max relation [10.48] is nothing but that established by duality. Consequently, A and B form an antiblocking pair of matrices. If $w = 1$, the min–max relation for A and B implies theorem 10.19 (Berge's theorem).

10.6. Cutting-plane method

Given a combinatorial optimization problem, it is generally difficult to characterize the associated polyhedron by a system of linear inequalities. Moreover, if the problem is **NP**-complete, there is very little hope of obtaining such a description. Furthermore, even if it is characterized, the system describing the polyhedron can contain a very large (even exponential) number of inequalities, and therefore cannot be totally used to solve the problem as a linear program. However, by using a *cutting-plane method*, a partial description of the polyhedron can be sufficient to solve the problem optimally. We discuss this method below.

Let us consider a combinatorial optimization problem in the form:

$$\max \{\omega x \; : A \, x \leq b, \; x \text{ integer}\} \qquad [10.49]$$

where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Let P be the convex hull of the solutions of [10.49]. From proposition 10.2 and theorem 10.4, the problem [10.49] is equivalent to the program $\max\{\omega x \; : \; x \in P\}$. If the inequalities of the system $Ax \leq b$ are sufficient to describe the polyhedron P, then every extreme point of the polyhedron $\{x \in \mathbb{R}^n \; : \; Ax \leq b\}$ is integral, and, consequently, the problem [10.49] is equivalent to its linear relaxation:

$$\max \{\omega x \; : \; Ax \leq b\} \qquad [10.50]$$

Unfortunately, this is not always the case, and the polyhedron $\{x \in \mathbb{R}^n \; : \; Ax \leq b\}$ can indeed contain fractional extreme points. Consequently, an optimal solution obtained as a linear combination of constraints of the system $Ax \leq b$. Since $yA^j - \lfloor yA^j \rfloor \geq 0$ and $x_j \geq 0$ for $j = 1, \ldots, n$, the constraint:

$$\sum_{j=1}^{n} \lfloor yA^j \rfloor x_j \leq yb \qquad [10.53]$$

is also satisfied by the solutions of [10.49]. Since the left-hand side of [10.53] is integer, it follows that the inequality:

$$\sum_{j=1}^{n} \lfloor yA^j \rfloor x_j \leq \lfloor yb \rfloor \qquad [10.54]$$

is valid for P. Inequalities of type [10.54] are called *Chvátal–Gomory inequalities*, and the above procedure is known as the *Chvátal–Gomory method.*

As is stated in theorem 10.26, every valid constraint for P can be obtained using the Chvátal–Gomory method.

THEOREM 10.26.– (Schrijver [SCH 80]). *Let $ax \leq \alpha$ be a valid inequality for $P \neq \emptyset$ with $(a, \alpha) \in \mathbb{Z}^{n+1}$. Then $ax \leq \alpha$ is a Chvátal–Gomory inequality.*

A valid inequality for P, equivalent to (or dominated by) a positive linear combination of $Ax \leq b$ is said to be *of Chvátal rank* 0 with respect to $Ax \leq b$. An inequality $ax \leq \alpha$ valid for P is said to be of *Chvátal rank k* with respect to $Ax \leq b$ if $ax \leq \alpha$ is not of Chvátal rank less than or equal to $k - 1$, and if it is equivalent to (or dominated by) a positive linear combination of inequalities, each one can be obtained using the Chvátal–Gomory method from inequalities of Chvátal rank less than or equal to $k-1$. In other words, an inequality valid for P is of Chvátal rank k if k applications of the Chvátal–Gomory method are necessary to obtain this inequality. Thus the constraints of Chvátal rank 1 are those which are not of rank 0 but which are either equivalent to or dominated by a positive linear combination of constraints of $Ax \leq b$ and inequalities obtained using the Chvátal–Gomory procedure from constraints of $Ax \leq b$. Note that the constraints [10.53] are of Chvátal rank 1 with respect to $Ax \leq b$. The valid inequalities that contribute to the resolution of a combinatorial problem, in the context of a cutting-plane method, are generally of Chvátal rank ≤ 1.

Theorem 10.26 implies that every constraint that is valid for the polyhedron P is of finite Chvátal rank with respect to $Ax \leq b$. The maximum Chvátal rank with respect to $Ax \leq b$ of a valid constraint (facet) for P is called *the Chvátal rank* of the polyhedron $\{x \in \mathbb{R}^n \ : \ Ax \leq b\}$. Note that the rank of $\{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ is 0 if and only if $P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$, that is if $\{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ is integral.

Suppose that P is the matchings polytope in a graph $G = (V, E)$, and let $Ax \leq b$ be the system given by inequalities [10.30] and [10.31]. It is not hard to see that inequalities [10.32] are of Chvátal rank 1 with respect to $Ax \leq b$. Since constraints [10.30]–[10.32] completely describe P, it follows that the polyhedron $Ax \leq b$ is of rank 1.

10.6.3. *Branch-and-cut algorithms*

If the cutting-plane algorithm does not allow us to provide an optimal solution of the problem, it is necessary to use a branch-and-bound technique to establish one. This technique allows us to construct a resolution tree where each vertex of the tree corresponds to a subproblem, the initial problem being the one associated with the root. This technique is based on two essential procedures:

– *Branching*: this procedure is simple; it just allows us to divide the problem associated with a given vertex of the tree into two disjoint subproblems by fixing one of the variables x_i to 1 for one of the problems and to 0 for the other. An optimal solution of the problem (corresponding to this vertex of the tree) will thus be optimal for one of the subproblems.

– *Bounding*: the aim of this procedure is to establish an upper bound (lower bound in the case of minimization) for the optimal value of the problem associated with a vertex of the tree.

To solve problem [10.49], we can start by solving a linear relaxation of the problem using a cuts algorithm. If an optimal solution is not found in this phase (called the *cutting phase*), we choose a fractional variable x_i, and we apply the branching procedure. In this way we create two subproblems (two vertices of the solution tree, which we link to the root). We establish an upper bound for each subproblem by solving the relaxed programs. If for one of the subproblems the optimal solution is integer, we stop its exploration. Otherwise, we choose one of the vertices, say S, and we divide the associated problem into two subproblems. We then create two new vertices, say S_1 and S_2, which we link to the *father* vertex S. By repeating this procedure, we construct a tree where the vertices correspond to the subproblems created. If the optimal solution for one of the problems is feasible for the initial problem or not as good as a feasible solution already found, then we stop developing the corresponding vertex. This one is declared *sterile*. At each step, we choose a pending vertex of the tree that is not sterile. We divide the corresponding problem into two subproblems and calculate a bound for each of the subproblems created. The algorithm stops when all the pending vertices of the tree are sterile. In this case, the best feasible solution found is optimal.

To calculate a bound for each vertex of the tree, we can simply solve the linear program obtained from the program of the father vertex by adding either the equation $x_i = 0$, or the equation $x_i = 1$. However, this bound can be weak, and can lead to a very slow resolution process, especially when the problem is large. However, if we add violated constraints to this relaxation, we can obtain better bounds and further accelerate the resolution of the program. *A branch-and-cut algorithm* is a branch-and-bound technique in which we apply the cutting-plane algorithm to calculate the bound of each subproblem. This method, introduced by Padberg and Rinaldi [PAD 91] where $\hat{x} \notin P$, the problem consists of finding a hyperplane that separates $\hat{x}$ and P (see Figure 10.3). If P is given by a system of inequalities $Ax \leq b$, then we talk of a separation problem associated with the system $Ax \leq b$.

In a branch-and-cut algorithm, we solve a sequence of separation problems on each vertex of the branching tree, each problem allowing generation of one (or several) violated constraints. The complexity of the algorithm will therefore naturally depend on that of the separation problem of the different classes of inequalities used in the algorithm. Using the ellipsoid method, introduced by Khachian [KHA 79] for linear programming, Grötschel *et al.* [GRÖ 81] have shown that if we can solve the separation problem in polynomial time for a polyhedron P, then we can solve the optimization problem for P, $\max\{\omega x \ : \ x \in P\}$ in polynomial time. The converse of this result is true. If we can optimize in polynomial time, we can also separate in polynomial time. Consequently, the complexity of a cutting-plane algorithm on a polyhedron $\{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ does not depend on the number of constraints in the system $Ax \leq b$ (even if it is exponential), but rather on the complexity of the separation problem that is associated with it. So, to solve an optimization problem of the form $\max\{\omega x \ : \ Ax \leq b\}$, using a cutting-plane algorithm, we do not need to know the system $Ax \leq b$ explicitly. It is sufficient just to be able to verify if a solution $\hat{x}$ satisfies $A\hat{x} \leq b$, and, if not, to determine *one* constraint among $Ax \leq b$ that is violated by $\hat{x}$.

To illustrate this concept, let us again consider the stable set problem in a graph $G = (V, E)$. The formulation of the problem, given by inequalities [10.18] and [10.19], contains a polynomial number of constraints and, consequently, the separation problem corresponding to these constraints can be solved in polynomial time. Let us further consider constraints the [10.51] given below, which are also valid for the stable set polytope.

$$x(V(C)) \leq \frac{|V(C)| - 1}{2}, \quad \forall\, C \text{ odd cycle of } G \qquad [10.55]$$

Constraints [10.55] are called *odd cycle inequalities*. As will be shown below, the branching problem for these constraints can also be solved in polynomial time.

THEOREM 10.27.– (Grötschel *et al.* [GRÖ 88]). *Inequalities [10.55] can be separated in polynomial time.*

Proof. Let $\hat{x} \in \mathbb{R}^V$. Since inequalities [10.18] and [10.19] can be separated in polynomial time, to solve the separation problem for constraints [10.55], we can assume that $\hat{x} \geq 0$ and that $\hat{x}$ satisfies inequalities [10.18]. If $e = ij \in E$, let

10.7.1. *Spin glass models and the maximum cut problem*

A *spin glass* is a system obtained by a small dilution (1%) of a magnetic material (iron) in a non-magnetic material (gold). Physicists' interest in this material comes from observing a peak in the curve of what is called the magnetic *susceptibility* as a function of temperature. Such a peak is generally an indication of a transition phase, a change of state of the system, hence the search for models that are likely to explain this phenomenon.

In a spin glass, the magnetic atoms are laid out randomly in space. Between two atoms i, j, there exists an interaction energy:

$$H_{ij} = -J(R)S_iS_j$$

where S_i (S_j) is the spin of the atom i (j), and $J(R)$ is a function that depends on the distance R between the two atoms. To model these systems, physicists have constructed a simplified model: they assume that the spins are situated at the nodes of a regular mesh (instead of being randomly distributed) and are defined by unidimensional vectors (instead of being tridimensional) S_i, taking the values $+1$ and -1. These meshes are generally square or cubic. They further assume that the interactions between the spins only take place between the closest neighbors, and that their energies (J_{ij}) are random variables taking positive or negative values. The interactions then correspond to the links of the mesh.

The energy of the system corresponds to a configuration S of spins (that is an assignment of $+1$ and -1 to the spins), given by:

$$H(S) = -\sum_{ij \in L} J_{ij}S_iS_j \qquad [10.57]$$

where L is the set of the links and J_{ij} the interaction between the spins i and j. The problem that physicists study is to determine a configuration S that minimizes the energy [10.57] of the system. Such a configuration is called *the fundamental state of the system* and the problem is called the *fundamental state problem*. Physicists traditionally use Monte Carlo type heuristics to determine approximate solutions for this problem, even in the case where the mesh is square (planar). As is shown in the following, this problem can be reduced to the maximum cut problem.

We can associate with a spin glass system, a graph $G = (V, E)$ where the vertices correspond to the spins, and two vertices are linked by an edge if there is a link between the spins corresponding to the vertices. We associate the weight $\omega_{ij} = -J_{ij}$

$$0 \leq x(e) \leq 1 \quad \forall\, e \in E \qquad [10.60]$$

It is not hard to see that every integer solution of the system above represents a cut of G. Consequently , these constraints induce an integer formulation of the maximum cut problem. Constraints [10.59] are called *cycle inequalities.* Given a cycle C, a *chord* of C is an edge whose two extremities are in C and are not consecutive when running through C. Theorem 10.28 gives the necessary and sufficient conditions for constraints [10.59] and [10.60] to define facets of $P_c(G)$.

THEOREM 10.28.–

1) An inequality [10.59] defines a facet of $P_c(G)$ if and only if C does not have a chord.

2) An inequality $x(e) \geq 0$ ($x(e) \leq 1$) defines a facet of $P_c(G)$ if and only if e does not belong to a triangle.

Since constraints [10.59] and [10.60] formulate the maximum cut problem as an integer program, and, from theorem 10.28, can define facets, it would be useful to have a polynomial separation algorithm for these constraints. This would allow us to use them efficiently in the context of a cutting-plane method for the problem. It is clear that constraints [10.60] can be separated in polynomial time. In what follows, we show that constraints [10.59] can also be separated in polynomial time.

By changing the variables from $x(e)$ to $1-x(e)$, constraints [10.59] can be written as:

$$\sum_{x \in C \setminus F} x(e) + \sum_{e \in F} (1 - x(e)) \geq 1, \; \forall\, C \text{ cycle of } G,\; F \subseteq C,\; |F| \text{ odd} \qquad [10.61]$$

If $\hat{x} \in \mathbb{R}^n_+$, the separation problem of constraints [10.59] with respect to $\hat{x}$ reduces to checking whether for every C, by associating a weight $1 - \hat{x}(e)$ with an odd number of edges of C and a weight $\hat{x}(e)$ with the other edges of C, the total weight of C is greater than or equal to 1. To solve this problem, we will consider an auxiliary graph.

Let $G' = (V', E')$ be the graph obtained from G in the following way. For every vertex i of G, we consider two vertices i' and i'' in G'. For every edge ij of G, we consider the edges $i'j'$ and $i''j''$ with a weight $x(ij)$ and the edges $i'j''$ and $i''j'$ with a weight $1 - x(ij)$. As we will see, the separation problem of constraints [10.61] reduces to determining a shortest path in G' between two vertices i' and i''. Let us denote by E_{ij} the set of edges $\{i'j', i'j'', i''j', i''j''\}$ for $ij \in E$. Observe that every path in G' between two vertices i' and i'', which uses at most one edge from each set E_{ij}, corresponds to a cycle in G going through the vertex i. Denote by V_1' (resp. V_2') the set of vertices i' (resp. i'') for i in V. Note that an edge e in G' has a weight

Gerards [GER 85] has shown that the separation problem of constraints [10.62] can be reduced to a polynomial sequence of shortest path problems and can therefore be solved in polynomial time.

THEOREM 10.31.– *Let $K_p = (W, T)$ be a complete subgraph of G of order p. So the inequality:*

$$x(T) \leq \lfloor \frac{p}{2} \rfloor \lceil \frac{p}{2} \rceil. \qquad [10.63]$$

is valid for $P_c(G)$. Furthermore, it defines a facet of $P_c(G)$ if and only if p is odd.

Inequalities [10.63] can be separated in polynomial time if p is fixed.

Branch-and-cut algorithms based on these classes of facets (and on other families of valid constraints) have been developed to solve planar and non-planar instances of the spin glass fundamental state problem [BAR 88, JÜN 98, LIE 03, SIM 95]. This approach has proved to be the most effective for this problem.

10.8. The survivable network design problem

With the introduction of optical technology, the telecommunications field has seen substantial evolution in recent years. Indeed, fiber-optics offer large transmission capacity, and thus allow the transfer of great quantities of information. Therefore, current networks tend to have a sparse topology (almost a tree). However, the failure of one or more links (or nodes) of a telecommunications network can have catastrophic consequences if the network is not in a position to be able to provide rerouting paths. So, designing a sufficiently survivable network, that is one which can continue to work in the event of a failure, has today become one of the objectives of telecommunications operators.

Survivability is generally expressed in terms of connectivity in the network. We ask that between each pair of nodes, depending on their importance in the network, a minimum number of disjoint paths exist, in such a way that in the case of failure, there is always at least one path that allows traffic to flow between the nodes. If the design of each link in the network carries a certain cost, the problem that is then posed is of designing a network whose topology satisfies the survivability conditions and is of minimum cost.

This problem has attracted much attention in recent years. Several methods of solving the problem have been developed, in particular polyhedral techniques. These have been effective for optimally solving instances of great size [GRÖ 92a, GRÖ 92b, KER 04]. In what follows, we discuss these techniques for certain variants of the problem. First of all we give a formulation of the problem in terms of graphs.

According to Menger's theorem 10.10, the ESNDP is equivalent to the following integer program:

$$\min \sum_{e \in E} c(e)x(e)$$

$$0 \leq x(e) \leq 1, \quad \forall\, e \in E \qquad [10.64]$$

$$x(\delta(W)) \geq con(W) \quad \forall\, W \subseteq V, \emptyset \neq W \neq V \qquad [10.65]$$

$$x(e) \in \{0,1\} \quad \forall\, e \in E \qquad [10.66]$$

Constraints [10.65] are called *cut inequalities.*

Let $G = (V, E)$ be a graph and $r = (r(v),\ v \in V)$ be a vector of connectivity types. Let:

$$\text{ESNDP}(G) = conv\{x \in \mathbb{R}^E \ : \ x \text{ satisfies } [10.64] - [10.66]\}$$

be the convex hull of the solutions of the ESNDP problem. ESNDP(G) is called *edge-survivable subgraph polytope.*

Given a graph $G = (V, E)$ and a vector $r = (r(v),\ v \in V)$, we say that an edge $e \in E$ is *essential* if the graph $G - e$ is not edge-survivable. The essential edges must therefore belong to every solution of the problem. If E^* denotes the set of essential edges in G, then we have the following result.

PROPOSITION 10.20.– (Grötschel, Monma [GRÖ 90]). *The dimension of ESNDP(G) is equal to* $|E| - |E^*|$.

10.8.2. *Valid inequalities and separation*

To lighten the presentation, we consider the *weak survivability* case, that is $r(v) \in \{0, 1, 2\}$ for each vertex $v \in V$. Most of the results established in this case can easily be extended to the general case. Furthermore, given that the survivability conditions in this case induce a topology that has shown itself to be sufficiently effective in practice, this variant has been intensively investigated.

A first class of valid inequalities for the polytope ESNDP(G) is that given by the cut inequalities [10.65]. The separation problem for these constraints is equivalent to the minimum cut problem with positive weights on the edges, and can therefore be solved in polynomial time. Given a solution $\hat{x} \in \mathbb{R}^E$, we associate with each edge e a capacity $\hat{x}(e)$, and we calculate the maximum flow between each pair of vertices. From the max-flow min-cut theorem, if for a pair s, t, the flow between s and t is strictly less than $\min\{r(s), r(t)\}$, then the constraint [10.65] induced by the minimum cut between s and t is violated by $\hat{x}$. This separation can be implemented in $O(n^4)$ using the Gomory–Hu tree [GOM 61].

where $(V_1, \ldots, V_p)$ is a partition of V, can be reduced to the minimization of a submodular function, and can therefore be solved in polynomial time (a function $f : 2^S \longrightarrow \mathbb{R}$, where S is a finite set, is said to be *submodular* if $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for all subsets A and B of S). Recently, Barahona and Kerivin [BAR 04] have given a polynomial combinatorial algorithm to separate constraints [10.69] in this case. As a consequence of the complexities of the separation problems of [10.67] and [10.69], the separastion problem of constraints [10.68] is polynomial if $r \in \{1, 2\}^V$.

10.8.2.3. *F-partition inequalities*

Let us now assume that the connectivity types are all equal to 2, that is $r(v) = 2$ for every $v \in V$. A class of valid inequalities for the ESNDP(G) in this case has been introduced by Mahjoub [MAH 94] as follows. Consider a partition $(V_0, V_1, \ldots, V_p)$ of V and let $F \subseteq \delta(V_0)$ be of odd cardinality. By adding up the following valid inequalities:

$$x(\delta(V_i) \geq 2, \quad \forall\, i = 1, \ldots, p$$

$$-x(e) \geq -1, \quad \forall\, e \in F$$

$$x(e) \geq 0, \quad \forall\, e \in \delta(V_1) \setminus F$$

we obtain:

$$2x(\Delta) \geq 2p - |F|$$

where $\Delta = \delta(V_0, \ldots, V_p) \setminus F$. By dividing by 2 and rounding up the second member to the next integer, we obtain:

$$x(\Delta) \geq p - \lfloor \frac{|F|}{2} \rfloor. \qquad [10.70]$$

Inequalities [10.70] are called *F-partition inequalities*. Note that these inequalities are of Chvátal rank 1 with respect to the system given by the trivial constraints and the cut constraints. Let us also note that if $|F|$ is even, inequality [10.70] can be obtained from the trivial inequalities and the cut inequalities.

Inequalities [10.70] are a special case of a more general class of valid inequalities given by Grötschel *et al.* [GRÖ 92b] for the ESNDP(G). Kerivin *et al.* [KER 04] have considered a subclass of F-partition constraints called *odd wheel inequalities* and have given sufficient conditions for these to define facets. They have also extended these inequalities for the case where the connectivity types are 1 or 2 for each vertex of the graph.

The separation problem for the F-partition constraints is still an open problem. However, if F is fixed, as has been shown by Baïou *et al.* [BAÏ 00], the problem can be solved in polynomial time.

For separating the F-partition inequalities [10.70] and [10.72], Kerivin *et al.* propose two heuristics [KER 04]. The first is based on the concept of critical extreme points introduced by Fonlupt and Mahjoub [FON 99]. The procedure consists of applying some graph reduction operations, and of looking for odd cycles formed by edges whose value is fractional. If such a cycle is found, then an F-partition constraint violated by $\bar{x}$ is detected. The second heuristic allows us to transform cuts, with as many edges $e \in E$ with $\bar{x}(e) = 1$ as possible, into F-partitions. This is done by calculating a Gomory–Hu tree with respect to the weights $(1 - \bar{x}(e),\ e \in E)$. If $\delta(W)$ is a cut of the Gomory–Hu tree, the F-partition constraint is therefore generated by considering the partition given by W and the vertices in $V \setminus W$, and by choosing a set of edges $F \subseteq \delta(W)$ (the same process can be applied for the partition induced by $V \setminus W$ and the vertices in W).

The numerical results presented in [KER 04], concerning uniform instances where $r(v) = 2$ for every $v \in V$ having up to 417 vertices and instances with $r \in \{1, 2\}^V$ having up to 101 vertices, show that these different classes of inequalities are useful for solving the problem optimally. In particular, in the case where $r(v) = 2$ for every $v \in V$, the F-partition inequalities seem to play a decisive role in establishing the optimal solution. They allow us to improve the bound at the root of the branching tree, and to thus decrease its size. When $r \in \{1, 2\}^V$, the partition constraints [10.68] seem to be the most important deciding factor in solving the problem. Like the F-partitions constraints for the uniform case, these constraints allow us to significantly reduce the bound at the root, and to considerably reduce the size of the branching tree.

10.9. Conclusion

In this chapter we have discussed polyhedral approaches in combinatorial optimization. We have closely examined polyhedron descriptions in terms of facets and in terms of extreme points and their implementations in the context of a branch-and-cut technique. As has been highlighted, these approaches have been shown to be powerful in exactly solving difficult combinatorial optimization problems. As examples, we have presented applications to spin glass in statistical physics and to survivable network design problems, for which these approaches are particularly effective.

For certain problems, it is possible that a branch-and-cut algorithm does not give an optimal solution even after an enormous computation time. It is then useful in such cases to have an approximate solution with a relatively very small error. For this, it is useful to be able to compute, in each iteration of the algorithm, a feasible solution of the problem. This gives (in the maximization case) a lower bound on the optimal value. This bound with that given by the linear relaxation allows us to calculate the relative error of the current solution. One of the most commonly used techniques for calculating feasible solutions is that known as *random rounding*. The method consists of transforming a fractional solution into a feasible integer solution by randomly

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