DESIGN OF SURVIVABLE NETWORKS: POLYHEDRAL TECHNIQUES

A. Ridha Mahjoub
LAMSADE, Université Paris-Dauphine, Paris, France

Contents

1. **Polyhedral techniques**

    1.1. Polyhedral Approach
    1.2. Separation and Optimization
    1.3. Cutting plane method
    1.4. Branch&Cut

2. **Design of Survivable Networks**

1.1. Polyhedral Approach

A Combinatorial Optimization (C.O.) problem is a problem of the form

$$\mathcal{P}=\max\{c(F)=\sum_{e\in F} c(e),\ F\in \mathcal{F}\}$$

where $\mathcal{F}$ is the set of solutions of $\mathcal{P}$,
$\mathcal{F} \subset 2^E$ for a set ground set E and $c(F)$ is the weignt of F.

With $F \in \mathcal{F}$, we associate a {0,1} vector $x^F \in \mathbb{R}^E$, called the incidence vector of F given by

$$x^F_i = \begin{cases} 1 & \text{if } i \in F \\ 0 & \text{if } i \in E\setminus F \end{cases}$$

A C.O. problem can be formulated as a 0-1 program.

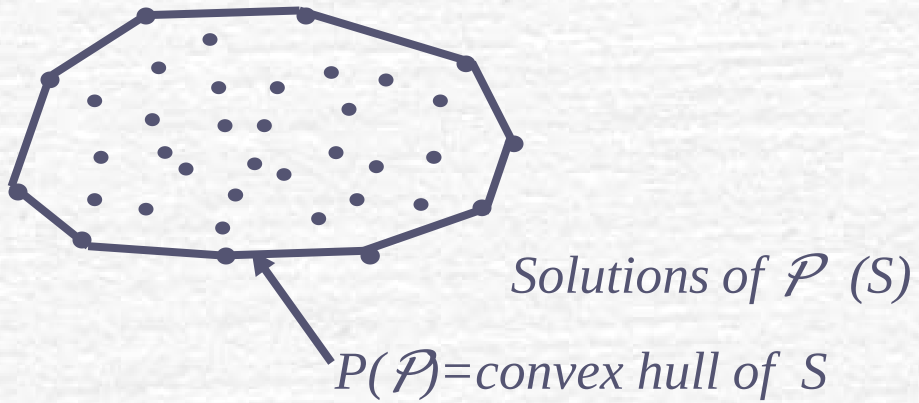
Idea : Reducing the problem to a linear program.

$$\text{Max} \sum c_j x_j$$

Subject to:

$$\sum a_{ij} x_j \leq b_i \,, \; i = 1,\ldots,m$$

$$x_i \in \{0,1\}, \; i = 1,\ldots,n$$



0-1 Program

A C.O. problem can be formulated as a 0-1 program.

Idea : Reducing the problem to a linear program.

Max $\sum c_j x_j$

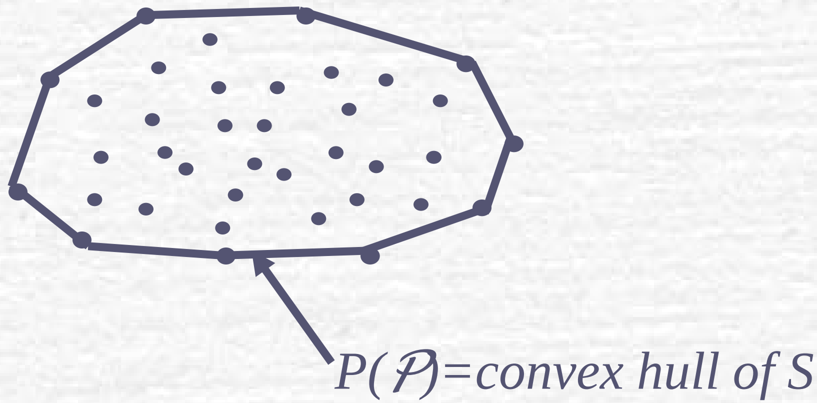
Subject to:

$\sum a_{ij} x_j \leq b_i$, $i=1,\ldots,m$

New Constraints

$x_i \geq 0$, $i=1,\ldots,n$

Linear Program



$\mathcal{P} \Leftrightarrow \max\{cx, x \in P(\mathcal{P})\}$

Polyhedral Approach:

Let $\mathcal{P}$ be a C.O. on a ground set E, $|E|=n$.

1. Represent the solutions of $\mathcal{P}$ as 0-1 vectors.
2. Consider these vectors as points of R^n, and define the convex hull $P(\mathcal{P})$ of these points.
3. Characterize $P(\mathcal{P})$ by a linear inequality system.
4. Apply linear programming for solving the problem.

This approach has been initiated by Edmonds in 1965 for the Matching Problem.

Step **3.** is the most difficult.

- If the problem is polynomial, generally it is possible to characterize the associated polytope!

- If the problem is NP-complete, there is a very little hope to get such a description.

Question: How to solve the problem when it is NP-complete.

A further difficulty:

The number of (necessary) constraints may be exponentail.

The Traveling Salesman Problem

For 120 cities,
The number of (necessary) contraints is $\geq 10^{179}$
($\cong 10^{100}$ times the number of atoms in the globe)
(number of variables: 7140.)

To solve the TSP on 120 cities,
(Grötschel 1977), used only 96 contraints among the 10^{179} known constraints..

1.2. Separation and Optimization

With a linear system

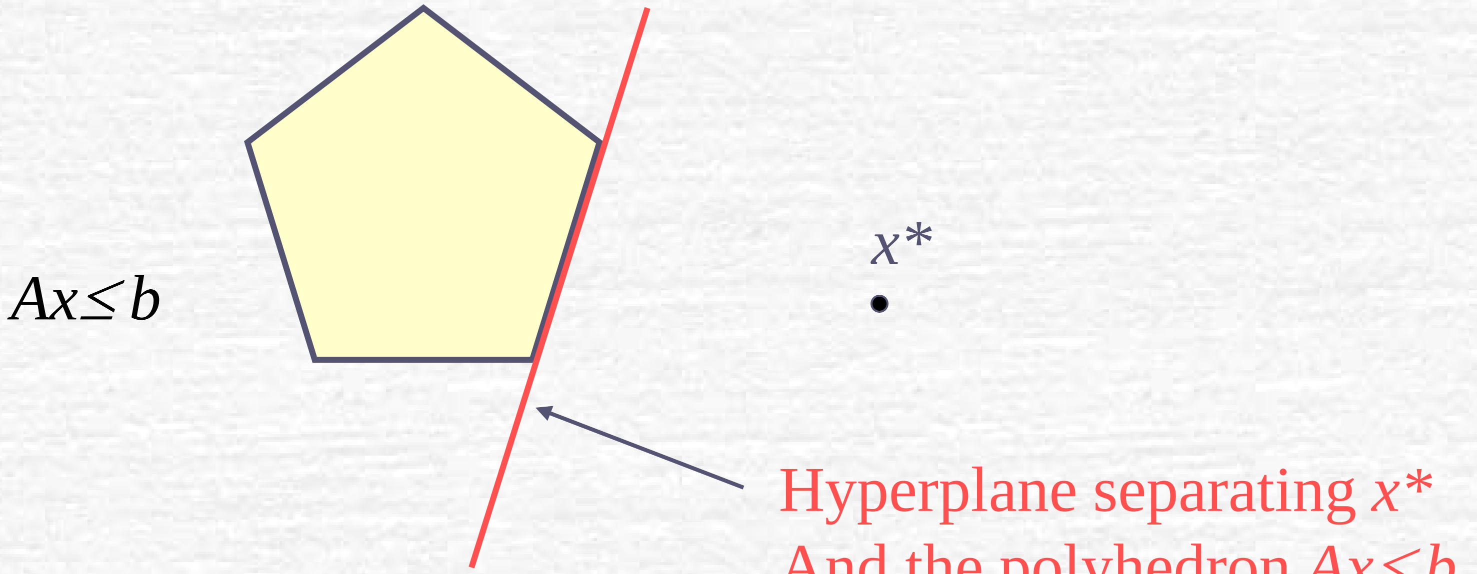
$$Ax \leq b$$

we associate the following problem:

Given a solution x^*, verify whether x^* satisfies $Ax \leq b$,
and if not, determine a constraint of $Ax \leq b$ which is violated by x^*.

This problem is called the separation problem associated with $Ax \leq b$.

If x^* does not verify system $Ax \leq b$, then there is a hyperplane that separates x^* and the polyhedron $Ax \leq b$.



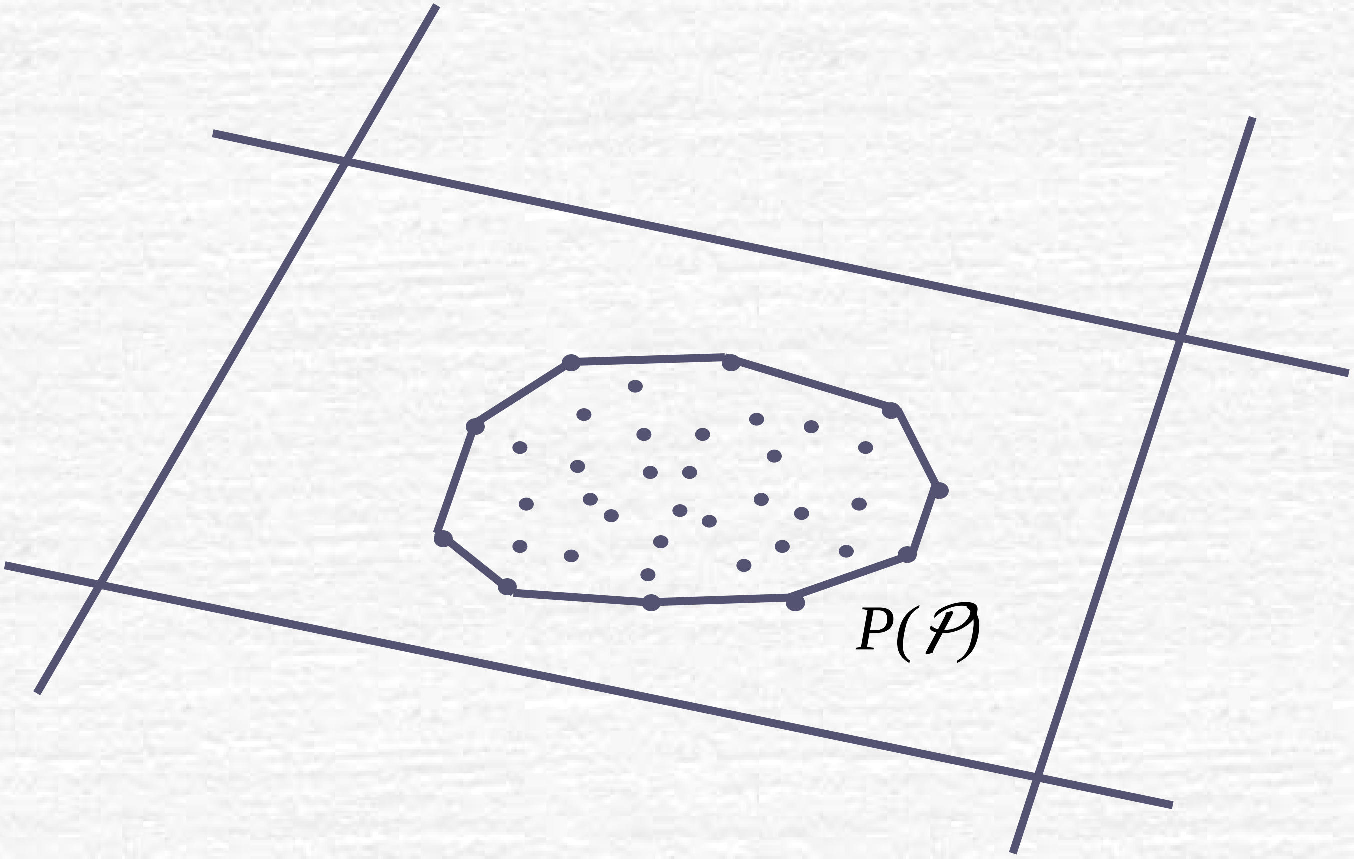
Theorem: (Grötschel, Lovász, Schrijver, 1981)

Given a linear program

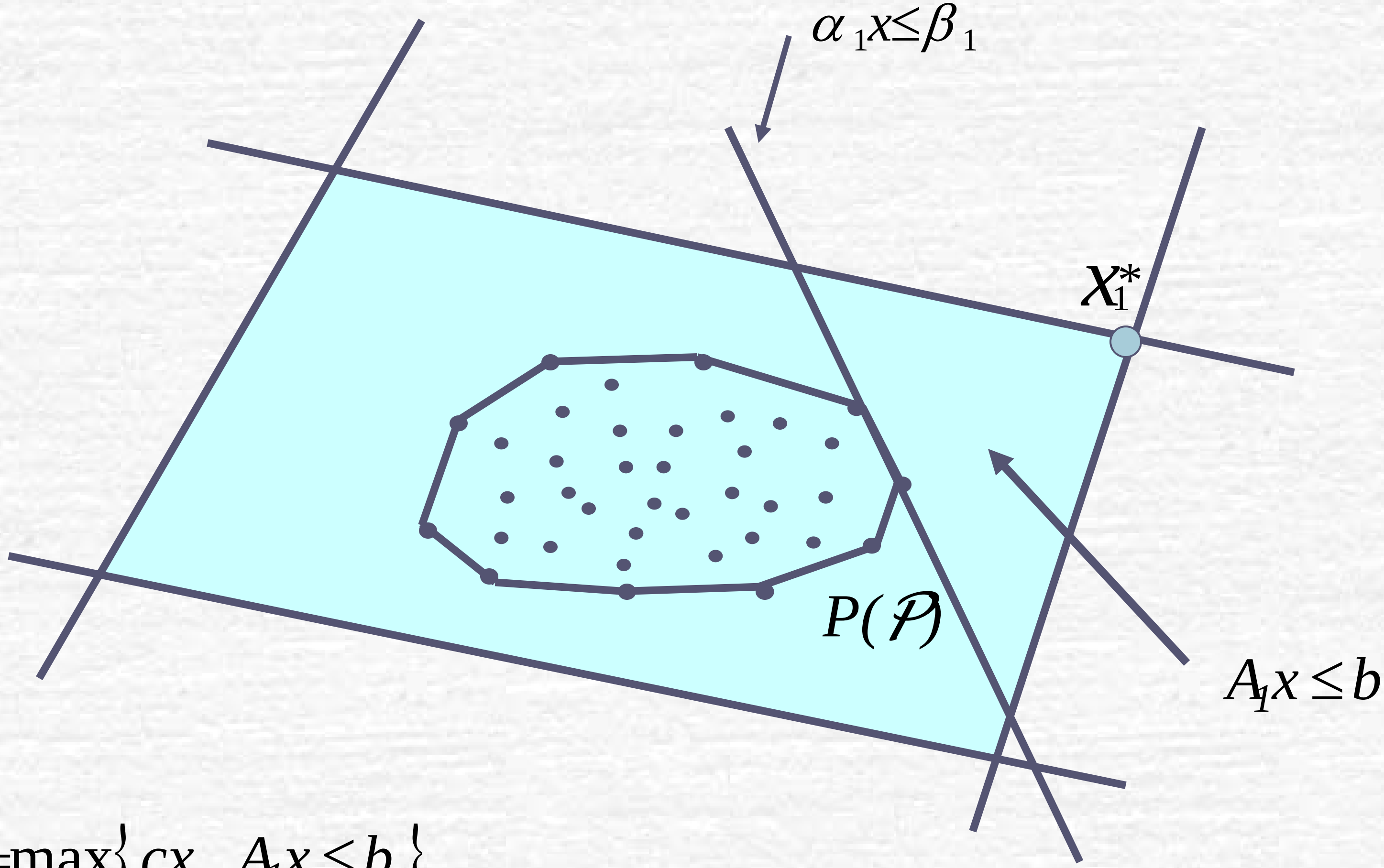
$$P = \max\{cx, Ax \leq b\},$$

there is a polynomial time algorithm for P if and only if there is a polynomial time algorithm for the separation problem associated with $Ax \leq b$.

1.3. Cutting plane method

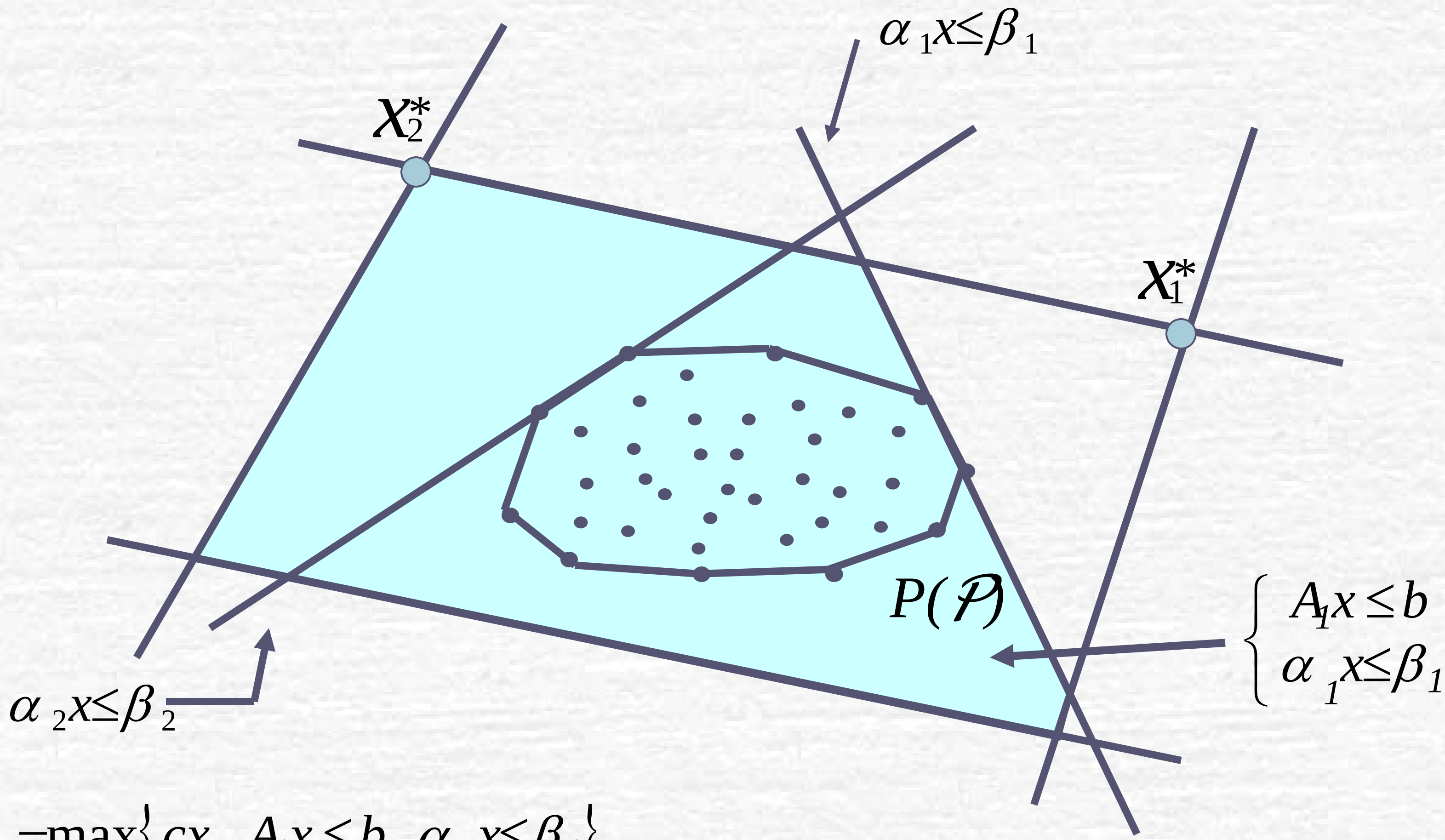


1.3. Cutting plane method



$$P_1 = \max\{ cx ,\ A_1 x \leq b \}$$

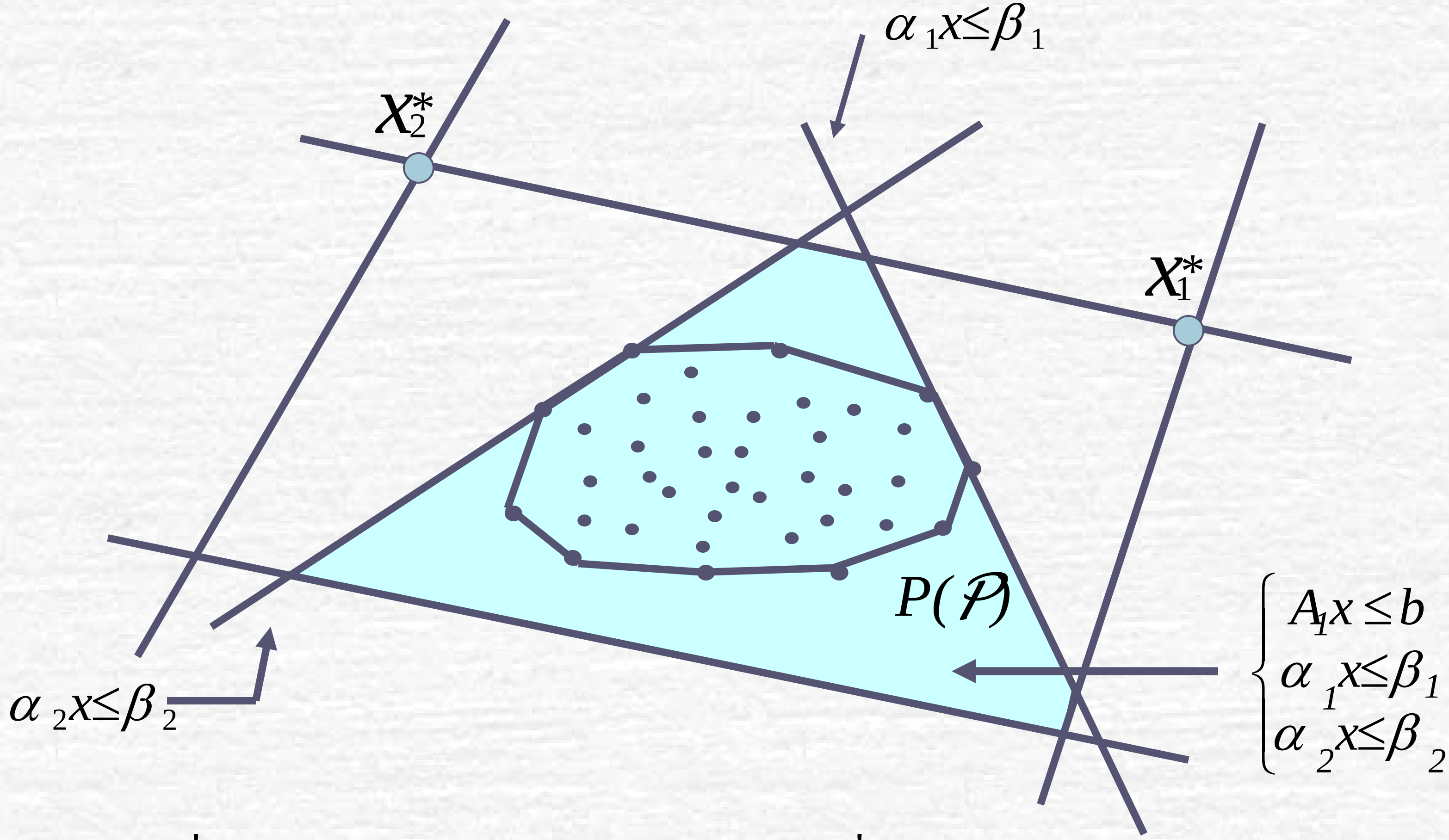
1.3. Cutting plane method



$$P_2 = \max\{cx,\ A_1 x \leq b,\ \alpha_1 x \leq \beta_1\}$$

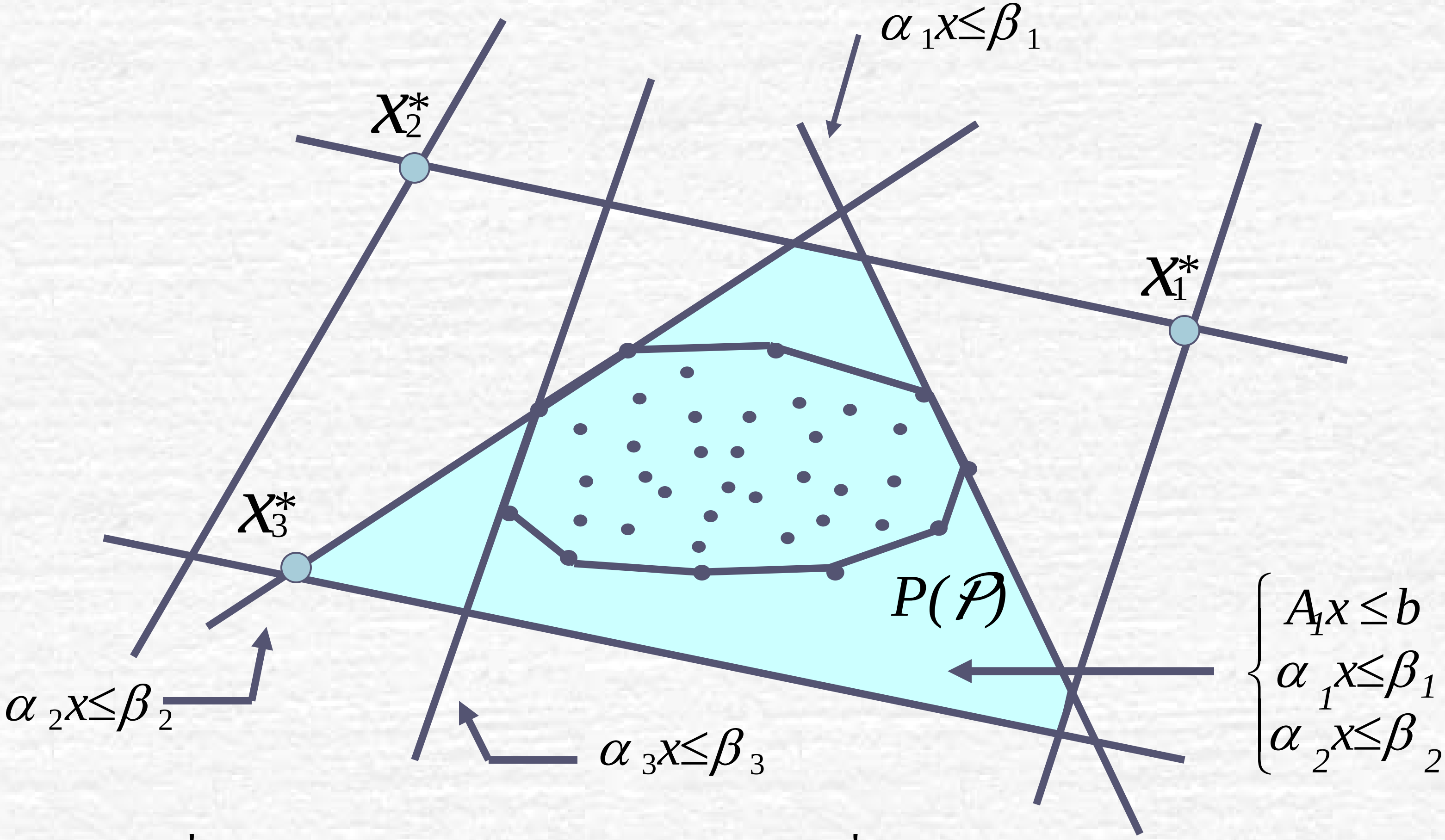
1. Polyhedral Techniques

1.3. Cutting plane method

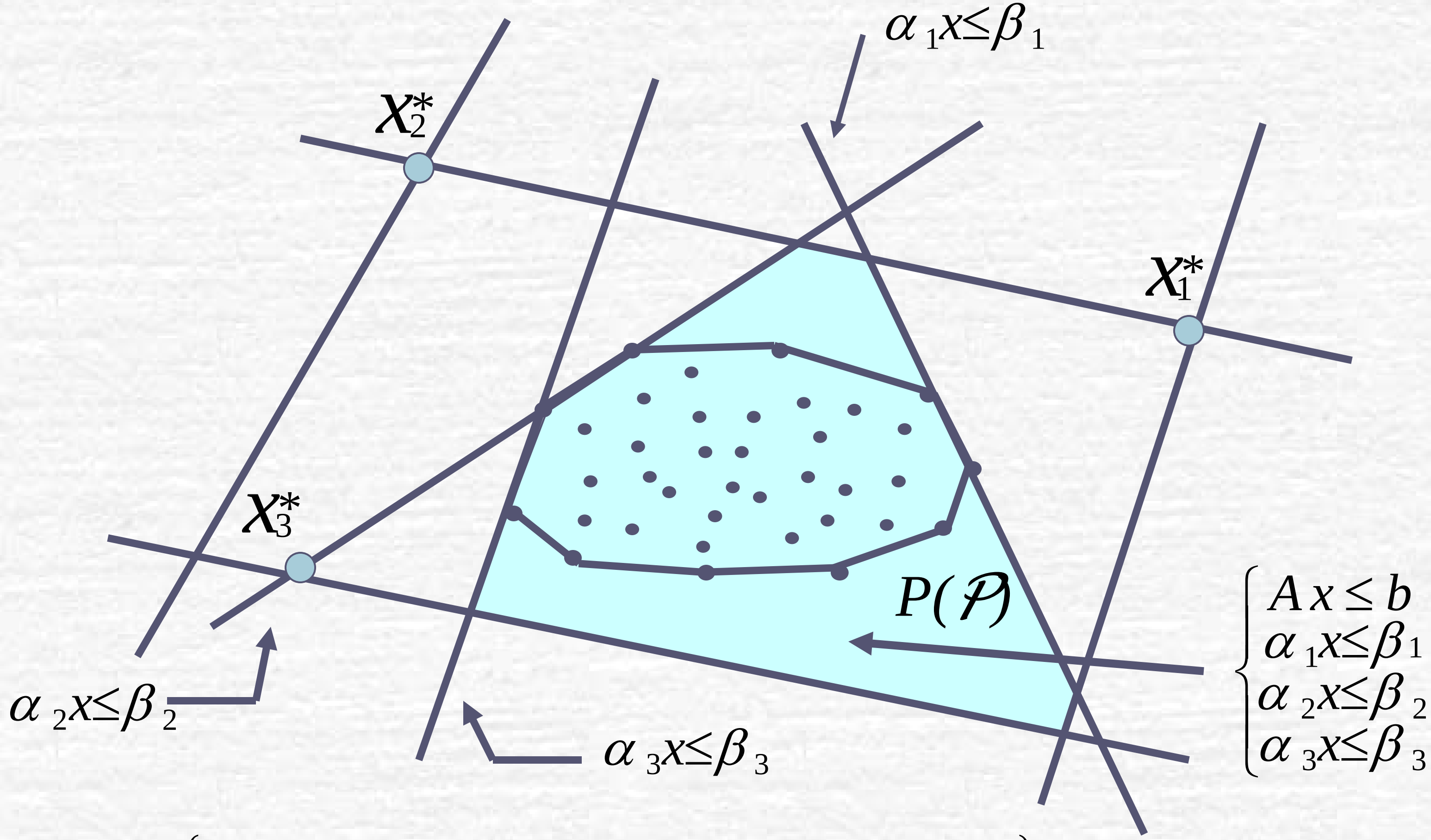


$$P_3 = \max\{cx,\ A_1 x \leq b,\ \alpha_1 x \leq \beta_1,\ \alpha_2 x \leq \beta_2\}$$

1.3. Cutting plane method



$$P_3 = \max\{cx,\ A_1 x \leq b,\ \alpha_1 x \leq \beta_1,\ \alpha_2 x \leq \beta_2\}$$



$$P_4 = \max\{\, cx,\ Ax \leq b,\ \alpha_1 x \leq \beta_1,\ \alpha_2 x \leq \beta_2,\ \alpha_3 x \leq \beta_3 \}$$

1.4. Branch&Cut Method

- A Branch&Bound based method.

- On each node of the tree we solve a linear relaxation of the problem by the cutting plane method.

1) If an optimal solution in not still found, select a (pending) node of the tree and a fractional varaiable x_i. Consider two sub-problems by fixing x_i to 1 and x_i to 0 (branching phase).

2) Solve each sub-problem by generating new violated constraints (cutting phase).

Go to **1)**.

Remarks:

-The polyhedral approach (Branch&Cut) is powerful for solving NP-hard C.O. problems. It also permits to *prove* polynomiality.

- Generally, it is difficult to find polynomial time separation algorithms. Then separation heuristics could be efficient in this case.

-If there is a huge number of variables, one can combine a Branch&Cut algorithm with a column generation method (Branch-and-Cut-and-Price).

frequency assignment
vehicule routing

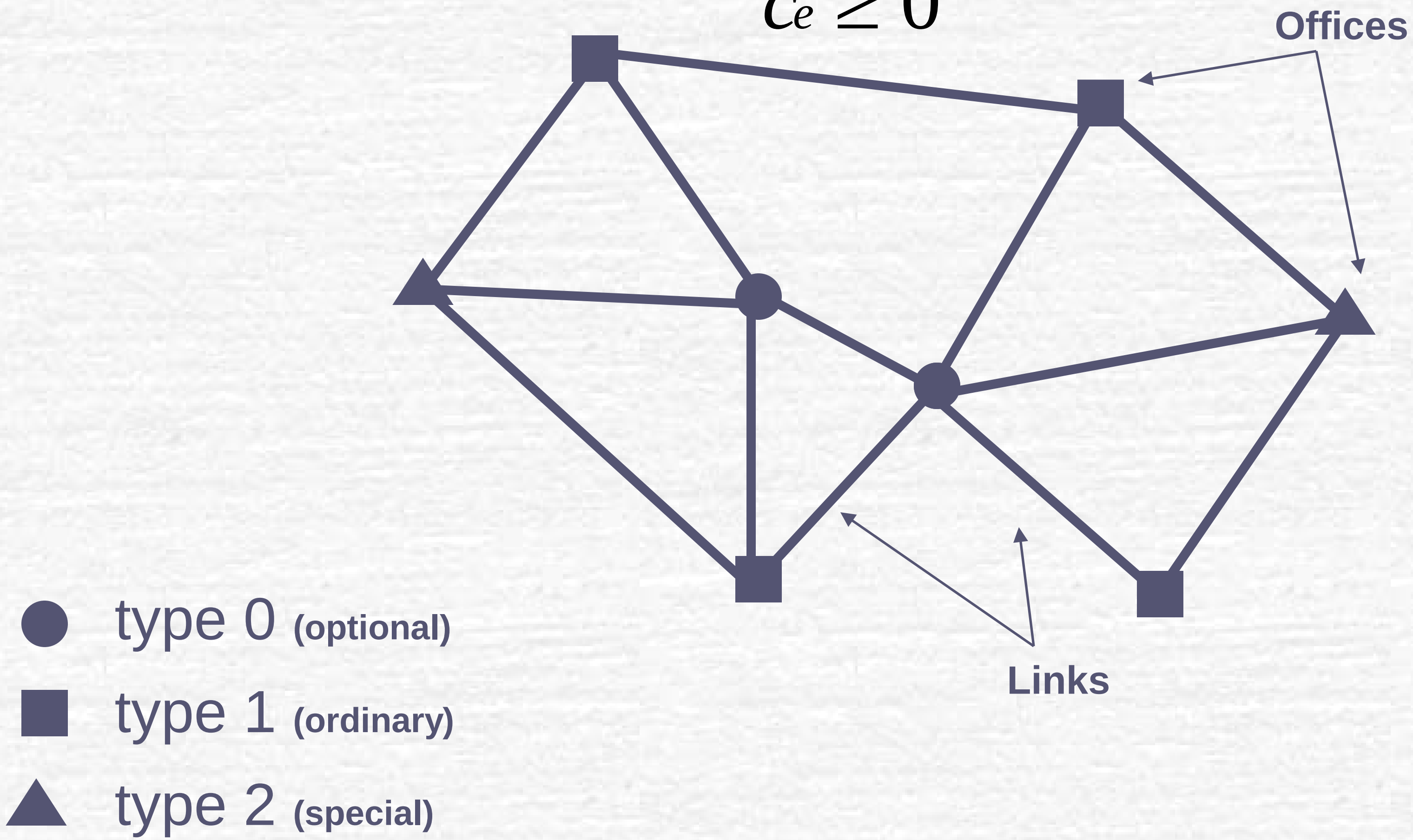
Contents

1. **Polyhedral techniques**

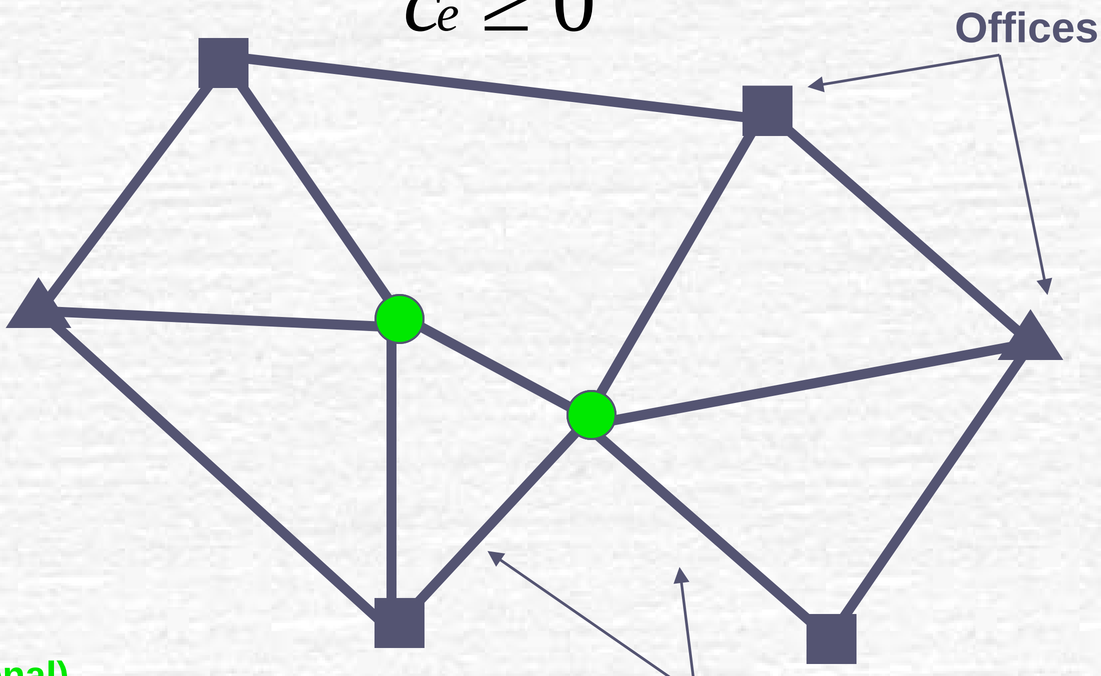
   1.1. Introduction
   1.2. Polyhedral Approach
   1.3. Separation and Optimization
   1.4. Cutting plane method
   1.5. Branch&Cut

2. **Design of Survivable Networks**

$$c_e \geq 0$$

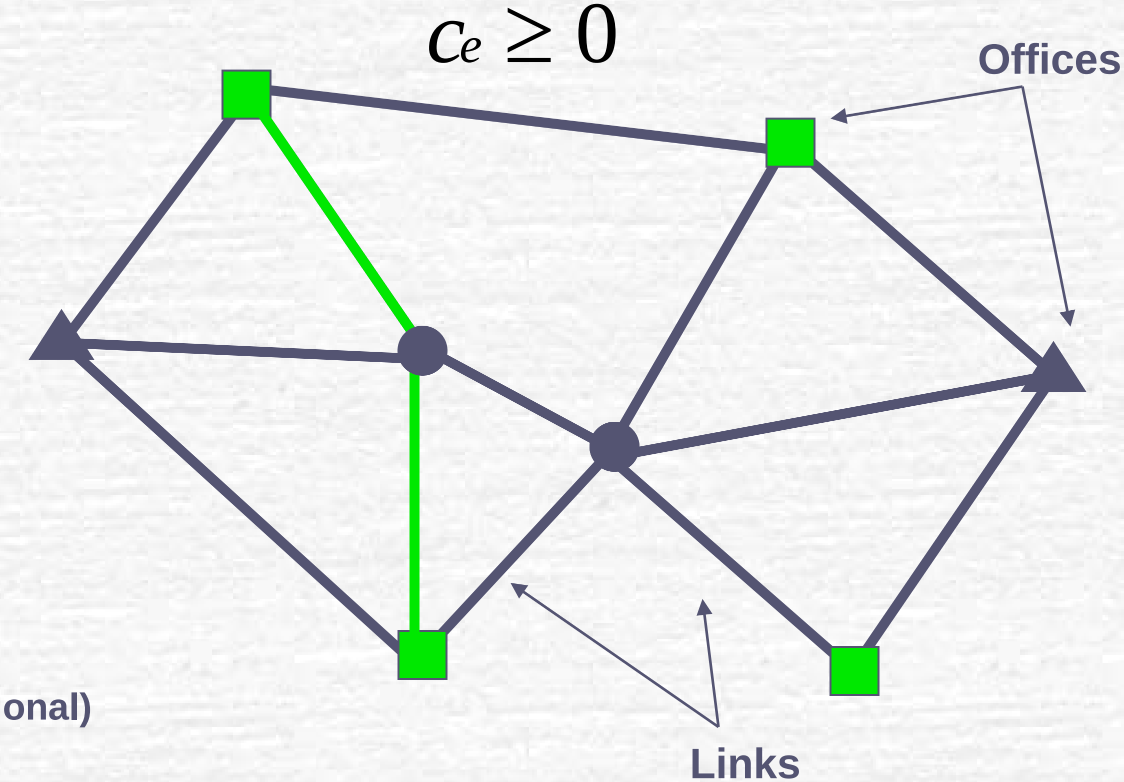

Offices
Links
type 0 (optional)
type 1 (ordinary)
type 2 (special)

$c_e \geq 0$



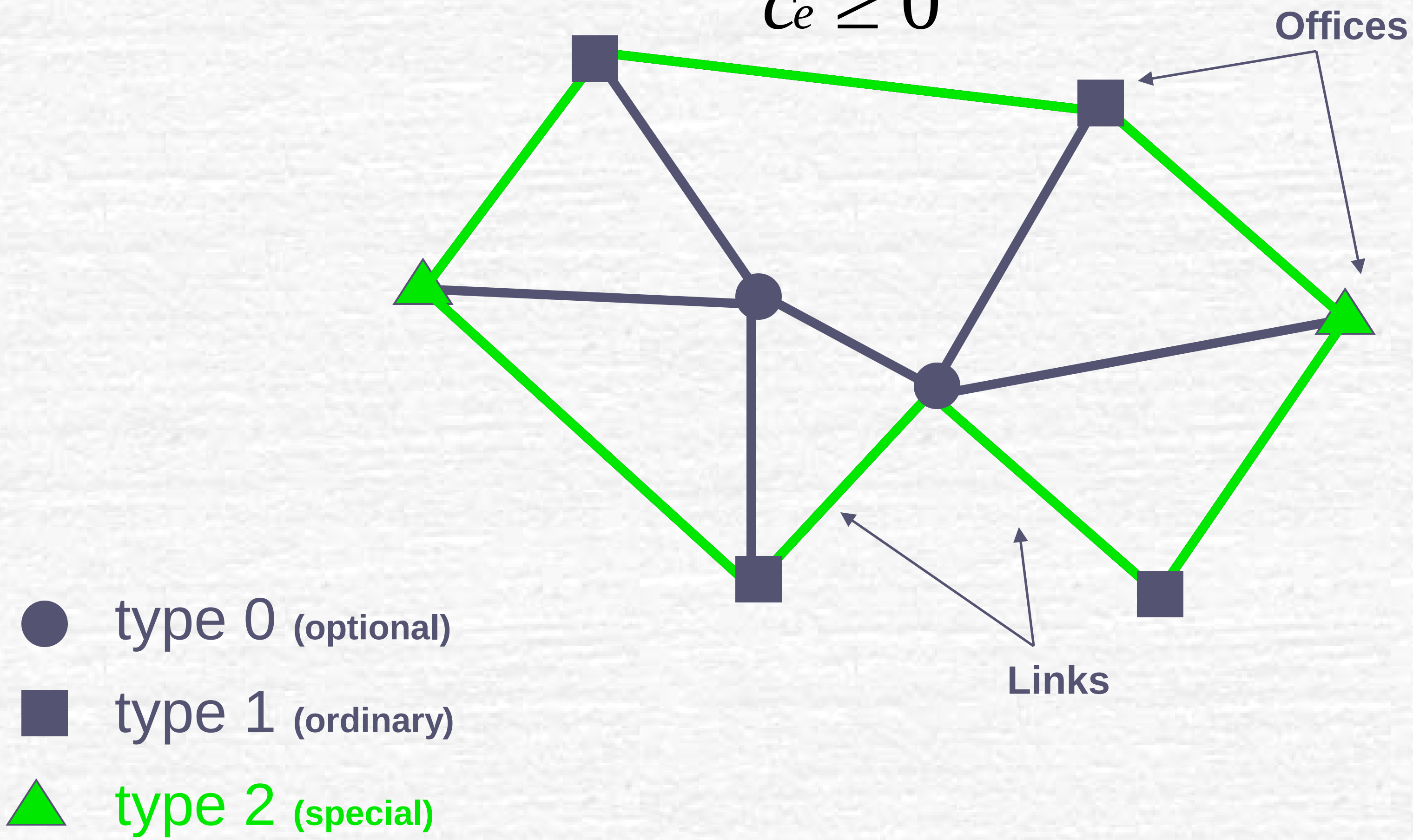
- type 0 **(optional)**
- type 1 **(ordinary)**
- type 2 **(special)**

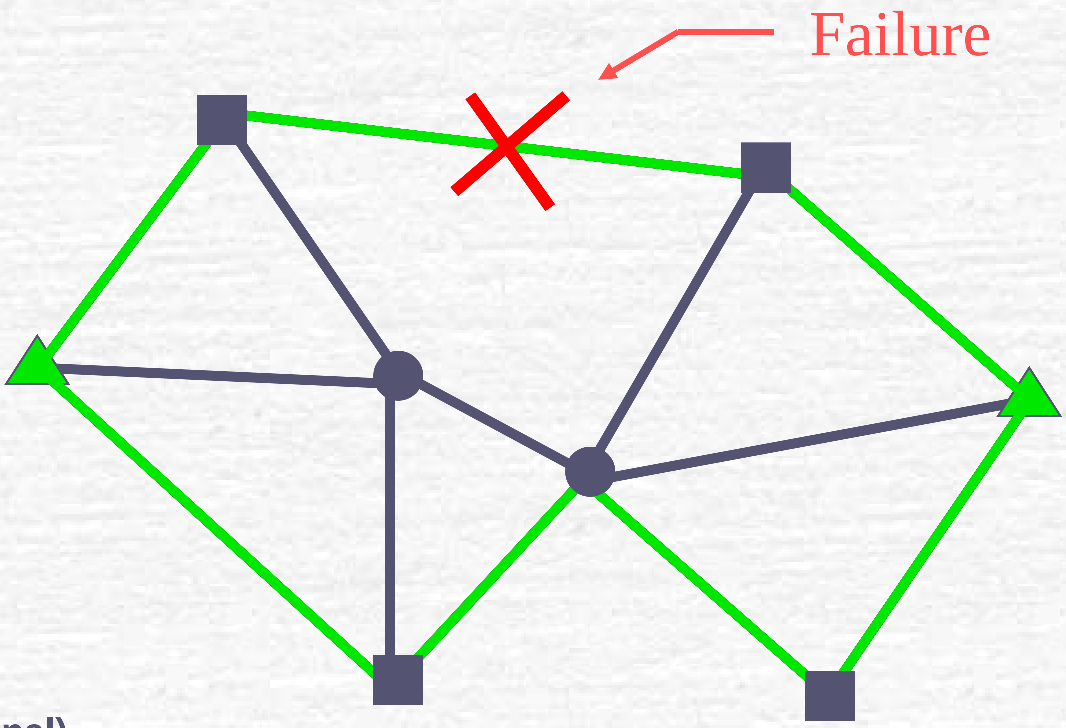
2. Network survivability



- type 0 (optional)
- type 1 (ordinary)
- type 2 (special)

$$c_e \geq 0$$


Offices
Links
type 0 (optional)
type 1 (ordinary)
type 2 (special)



- type 0 **(optional)**
- type 1 **(ordinary)**
- type 2 **(special)**

Survivability

The ability to restore network service in the event of a catastrophic failure.

Goal

Satisfy some connectivity requirements in the network.

Motivation

Design of optical communication networks.

Contents

2. Design of Survivable Networks

2.1. A general model
2.2. Heuristics
2.3. Polyhedral results
2.4. Separation
2.5. Critical extreme points
2.6. A Branch&Cut algorithm
2.7. Survivability with bounded length
2.8. Capacitated survivable networks

2.1. A General model

A first model

Given an undirected graph $G=(V,E)$ with
a weight on each edge,
a nonnegative integer matrix $R=(r_{ij})$ of connectivity requirements,

find a minimum weight subgraph of G such that between every pair of nodes i, j of V, there are at least r_{ij} edge (node)-disjoint paths

Frish (1967), Steiglitz (1969)
Chou and Frank (1970)
Winter (1985), Grötschel, Monma (1990)

A practical model: node types

Let $G=(V,E)$ be a graph. If s is a node of G, we associate with s a connectivity type $r(s) \in N$.

If s,t are two nodes, let

$$r(s,t)=min(r(s),r(t))$$

G is said to be survivable if for every pair of nodes s,t, there are at least $r(s,t)$ edge (node)-disjoint paths between s and t.

(Grötschel, Monma, Stoer (1992))

The Survivable Network Design Problem (SNDP)

Given weights on the edges of G, find a minimum weight survivable subgraph of G.

The SNDP is also known as the generalized Steiner tree problem and the multiterminal synthesis problem.

Special cases:

- *r(v)*=1 for every *v*: the minimum spanning tree problem.

- *r(v)*=1 for two nodes *s*,*t* and 0 elsewhere: the shortest path problem between *s* and *t*.

- *r(v)*∈{0,1} for every *v*: the Steiner tree problem.

- *r(v)*=*k* for every *v* (*k* fixed): the *k*-edge (*k*-node) connected subgraph problem .

The SNDP is NP-hard in general.

Polynomially solvable cases

c(e)=c for all *e* (uniform costs): (Chou and Franck (1970))

Given a set of nodes V, construct a minimum weight graph on V satisfying the (edge) connectivity requirements (parallel edges are allowed).

c(e)=0/1 for all *e*: The augmentation problem (edge case)

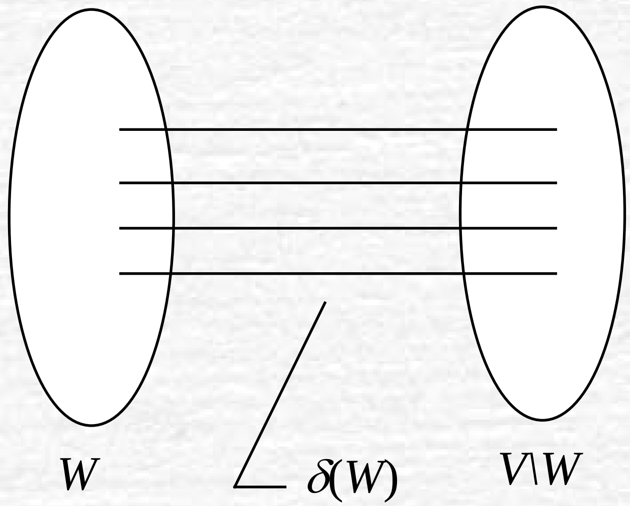
(parallel edges are allowed): (Franck) (1992)).

Polynomial time algorithms have also been devised for

special classes of graphs (like series-parallel graphs)

special types of node connectivities

Formulation of the SNDP (edge case)



If $W \subset V$, $\emptyset \neq W \neq V$, let
$r(W) = \max\{r(s) \mid s \in W\}$
$con(W) = \min\{r(W), r(V\backslash W)\}$

$r(W)$ is the connectivity type of W.

$\delta(W)$ is called a *cut* of G.

$$\sum_{e \in \delta(W)} x(e) = x(\delta(W)) \geq con(W)$$

cut inequalities

The (edge) SNDP is equivalent to the following integer program

$$\min \sum_{e \in E} c(e)x(e)$$

Subject to

$$x(\delta(W)) \geq con(W) \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

$$0 \leq x(e) \leq 1 \qquad \text{for all } e \in E,$$

$$x(e) \in \{0,1\} \qquad \text{for all } e \in E.$$

Follows from Menger's theorem (1927).

$$\min \sum_{e\in E} c(e)x(e)$$

Subject to

$x(\delta(W)) \geq con(W)$ for all $W \subset V$, $\emptyset \neq W \neq V$

$0 \leq x(e) \leq 1$ for all $e \in E$,

The linear relaxation can be solved in polynomial time (by the ellipsoid method).

2.2. Heuristics

Steiglitz, Weiner and Kleitman (1969): (general case):

Local search heuristic

Monma & Shallcross (1989): ($r(v) \in \{1,2\}$ for all v):

based on heuristics for the traveling salesman problem

Ko & Monma (1989): ($r(v)=k$ for all v) (The k-edge (node) connected subgraph problem): extension of Monma & Shallcross heuristic.

Heuristics with worst case garantee

A function f: $2^V \longrightarrow Z_+$ is called *proper* (Goemans & Williamson (1995)) if it satisfies the following

- $f(\emptyset)=0$,
- $f(S)=f(V \setminus S)$ for all $S \subseteq V$
- If $A \cap B=\emptyset$, then $f(A \cup B) \leq \max \{f(A), f(B)\}$

The connectivity function $f(S)=con(S)$ is proper.

SNDP with arbitrary proper connectivity function

Without multiple copies of edges

$$\min \sum_{e\in E} c(e)x(e)$$

$x(\delta(W)) \geq f(W)$ for all $W \subset V$, $\emptyset \neq W \neq V$,

$0 \leq x(e) \leq 1$ for all $e \in E$,
$x(e) \in \{0,1\}$ for all $e \in E$.

Primal-Dual polynomial $2f_{max}$-approximation algorithm, where $f_{max}=\max\{f(S), S \subset V\}$.

Williamson, Goemans, Mihail, Vazirani (1995)

Generalizes a factor 2 when $f(W)= 0$ or 1 (Goemans & Williamson) (1995)

$2\mathcal{H}(f_{max})$-approximation algorithm where $\mathcal{H}(f_{max}) = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{f_{\max}}$ is the harmonic function.

Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson (1996)

Factor 2 approximation algorithm when the function f is weakly supermodular

Jain (2001)

$f(V)=0$, and for every $A,B \subseteq V$ at least one of the following holds:
- $f(A)+f(B) \leq f(A \cap B)+ f(A \cup B)$ or
- $f(A)+f(B) \leq f(A \setminus B)+ f(B \setminus A)$.

If a function is proper then it is weakly supermodular.

Case when copies of edges are allowed

Jain's algorithm also works when multiple copies of an edge are allowed: $\Rightarrow$ factor 2 approximation algorithm

min{$2r_{max}$,$2p$}-approximation algorithm
Goemans and Bertsimas (1993)

where p is the number of distinct connectivity requirement values and $r_{max} = \max\{r(u), v \in V\}$.

2 log r_{max}-approximation algorithm
Agrawal, Klein and Ravi (1995)

$2\mathcal{H}(f_{max})$- approximation algorithm
Goemans & Williamson (1992)

2.3. Polyhedral Results

Let SNDP(G) be the convex hull of the solutions of SNDP, *i.e.*

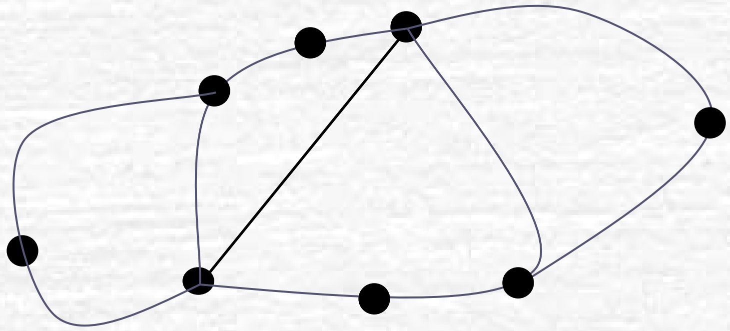
$$\text{SNDP}(G) = conv\{x \in R^E \mid x \text{ is a (an integer) solution of SNDP}\}.$$

SNDP(G) is called the survivable network design polyhedron.

2.3.1. Restricted graphs

A graph is said to be series-parallel if it can be constructed from an edge by iterative application of the following operations:

1) *Addition of parallel edges*
2) *Subdivision of edges*



Theorem: (Kerivin & M. (2002))
If G is series-parallel and r(v) is even for every v, then SNDP(G) is given by the trivial and the cut inequalities.

Generalizes Cornuéjols, Fonlupt and Naddef (1995), Baïou & M. (1996), Didi-Biha & M. (1999).

Corollary:
If G is series-parallel and r(v) is even for every v, then SNDP can be solved in polynomial time.

k-Connectivity with k odd

Let $(V_1,\ldots,V_p)$ be a partition of V. Chopra (1994) showed that

$$x(\delta(V_1,\ldots,V_p)) \geq \lceil k/2 \rceil p-1 \qquad (1)$$

is valid for the SNDP(G) when G is outerplanar (a subclass of series-parallel graphs), k is odd and an edge can be used more than once. Here $\delta(V_1,\ldots,V_p)$ is the set of edges between the V_i's.

Theorem: Chopra (1994)

If G is outerplanar, k odd and multiple eges are allowed, then the k-edge connected polyhedron is given by inequalities (1) and $x(e) \geq 0$ *for all e.*

Generalized by Didi Biha & M. (1996) to series-parallel graphs (with and without possibility of multiple copies of edges) .

General graphs

Low connectivity case: $r(v) \in \{0,1,2\}$

2.4.2. Valid inequalities:

Trivial inequalities:

$$0 \leq x(e) \leq 0 \quad \text{for all } e \in E$$

Cut inequalities:

$$x(\delta(W)) \geq con(W) \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

Partition inequalities:

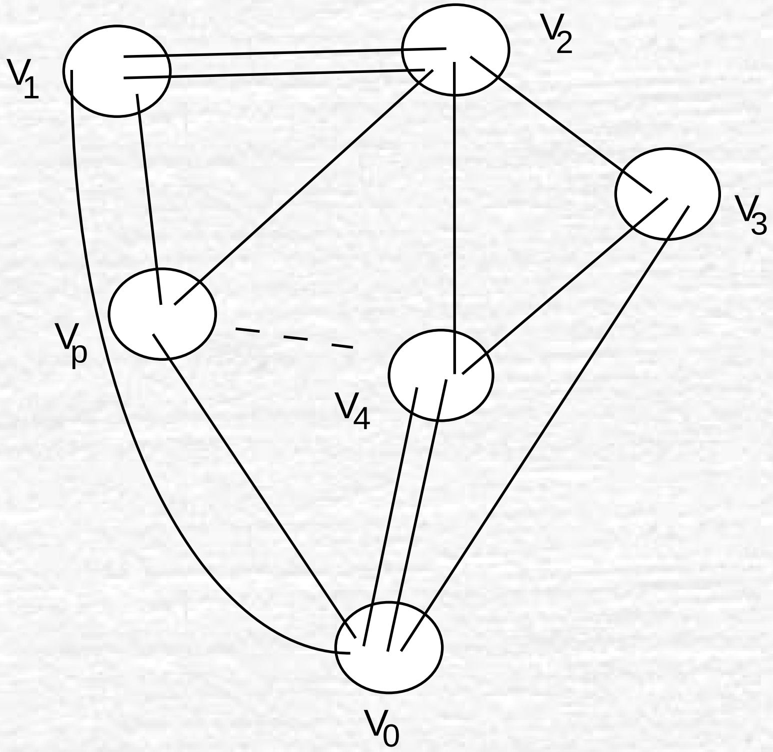
Let $V_1,\dots,V_p$, $p \geq 2$, be a partition of V such that $con(V_i) \geq 1$ for all V_i. Then the following inequality is valid for SNDP(G).

$$x(\delta(V_1,\dots,V_p)) \geq \begin{cases} p-1, & \text{if } con(V_i)=1 \text{ for all } V_i \\ p, & \text{if not,} \end{cases}$$

(Grötschel, Monma and Stoer (1992))

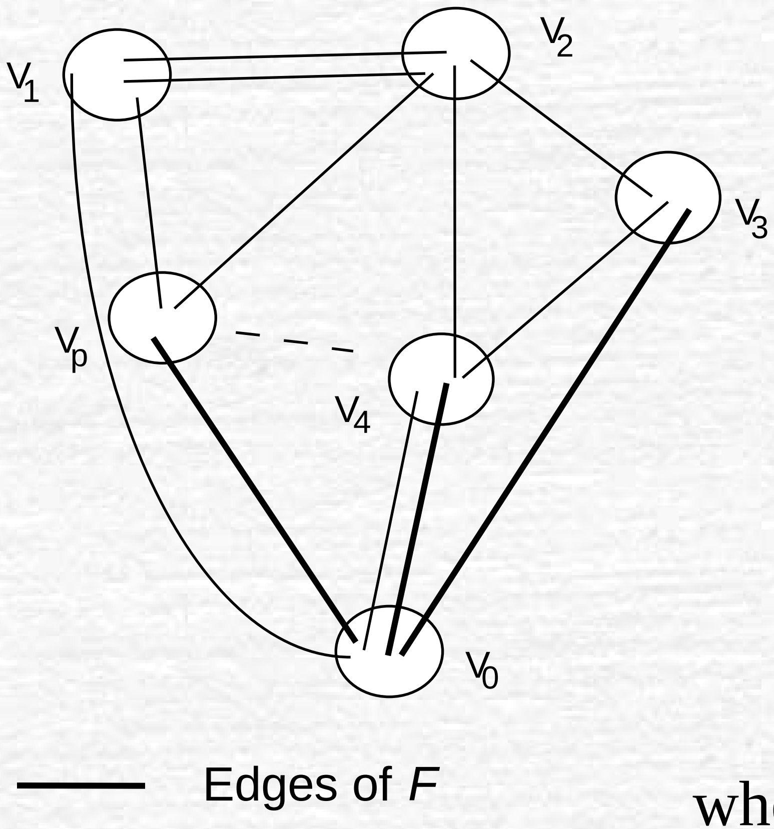
F-partition inequalities:

Let $V_0, V_1, \ldots, V_p$ be a partition of V such that $con(V_i)=2$ for all V_i



F-partition inequalities:

Let $V_0,V_1,\ldots,V_p$ be a partition of V such that $con(V_i)=2$ for all V_i



Let F be a set of edges of $\delta(V_0)$ and $|F|$ id odd.

$$x(\delta(V_i))\geq 2, \qquad i=1,\ldots,p$$

$$-x(e)\geq -1, \qquad e\in F$$

$$x(e)\geq 0, \qquad e\in \delta(V_0)\setminus F$$

$$\Rightarrow \qquad 2x(\Delta)\geq 2p-|F|,$$

where $\Delta=\delta(V_0,V_1,\ldots,V_p)\setminus F$

Then

$$x(\Delta) \geq p - \frac{|F| - 1}{2}$$

is valid for the SNDP(G).

These inequalities are called *F-partition inequalities.* (M. (1994))

Further valid inequalities related to the traveling salesman polytope have been given by Boyd & Hao (1994) for the 2-edge connected subgraph polytope. And general valid inequalities for the SNDP have been introduced by Grötschel, Monma and Stoer (1992) (generalizing the *F*-partition inequalities).

2.4. Separation

Consider the constraints

$$x(\delta(V_1,\ldots,V_p)) \geq p\text{-}1.$$

called *multicut inequalities*.

These arise as valid inequalities in many connectivity problems.

The separation problem for these inequalities reduce to $|E|$ min cut problems Cunningham (1985) .

It can also be reduced to $|V|$ min cut problems Barahona (1992).

Both algorithms provide the *most violated* inequality if there is any.

F-partition inequalities

($r(v)$ =2 for all node v)

Theorem. (Barahona, Baïou & M.) *If F is fixed, then the separation of F-partition inequalities can be solved in polynomial time.*

Let $G'=(V',E')$ be the graph obtained by deleting the edges of F. Hence the F-partition inequalities can be written as

$$x(\delta(V_0,\dots,V_p)) \geq p-(|F|-1)/2$$

where $(V_0,\dots,V_p)$ is a partition of V' such that for each edge $uv \in F$, $|\{u,v\} \cap (V_0)|=1$.

There are $2^{|F|}$ possibilities for assigning these nodes.

For each possibility we contract the nodes that must be in V_0 and solve the separation problem for the inequalities.

$$x(\delta(V_0,\ldots,V_p)) \geq p-(|F|-1)/2$$

where $|F|$ is fixed. These are partition inequalities, and hence the separation can be done in polynomial time.

Partition inequalities

These inequalities can be written as

$$x(\delta(V_1,\ldots,V_p)) \geq p\text{-}1, \quad \text{if } con(V_i)=1 \text{ for all } V_i$$
$$\geq p, \quad \text{if not,}$$

For any partition $(V_1,\ldots,V_p)$ of V.

If $r(v) \in \{0,1,2\}$, the separation problem is NP-hard (Grötcshel, Monma, Stoer (1992)).

Theorem: (Kerivin, M. (2002)) *The separation of the partition inequalities when $r(v) \in \{1,2\}$ for all v can be done in polynomial time.*

The separation reduces to minimizing a submodular function. (A function f: 2^V ----> R is said to be *submodular* if

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B), \text{ for all } A, B \subset V.$$

Recently Barahona and Kerivin (2004) showed that the problem reduces to $O(|V|^4)$ minimum cut problems.

2.5. Critical extreme points of the 2-edge connected subgraph polytope

(Fonlupt & M. (1999))

We suppose $r(v)=2$ for all v.

Consider the linear relaxation of the problem:

$$\min \sum_{e\in E} c(e)x(e)$$

$$x(\delta(W)) \geq 2 \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

$$0 \leq x(e) \leq 1 \qquad \text{for all } e \in E.$$

2.5. Critical extreme points
of the 2-edge connected subgraph polytope

(Fonlupt & M. (1999))

We suppose $r(v)=2$ for all v.

Consider the linear relaxation of the problem:

$$\min \sum_{e \in E} c(e)x(e)$$

$P(G)$

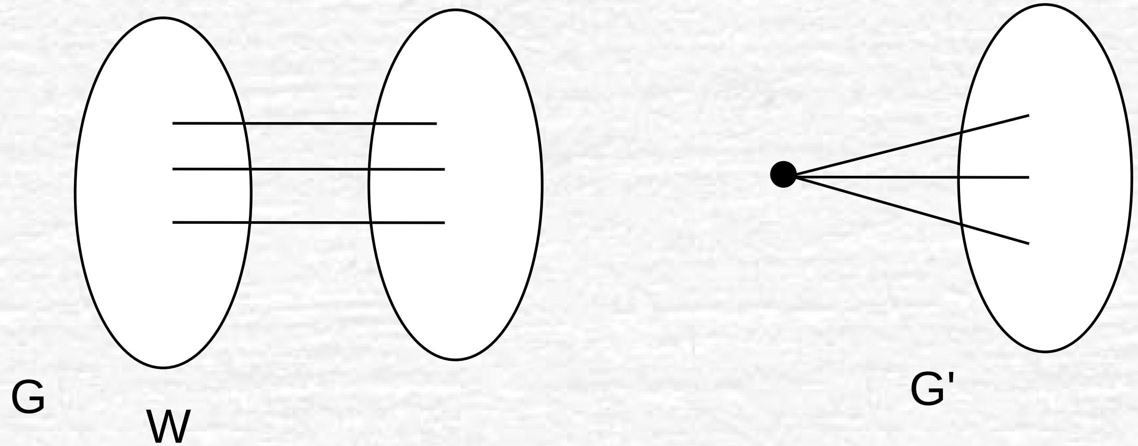
$x(\delta(W)) \geq 2$	for all $W \subset V$, $\emptyset \neq W \neq V$
$0 \leq x(e) \leq 1$	for all $e \in E$.

Reduction Operations

Let x be a fractional extreme point of $P(G)$.

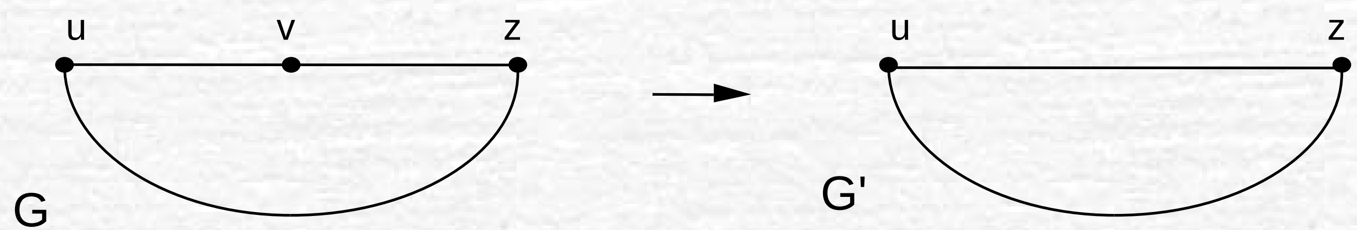
O_1: delete edge e such that $x(e)=0$,

O_2: contract a node set W such that the subgraph induced by W, $G(W)$ is 2-edge connected and $x(e)=1$ for every $e \in E(W)$.



$G(W)$ is 2-edge connected and $x(e)=1$ for every $e \in E(W)$.

O_3: contract an edge having one of its endnodes of degree 2.



Lemma: *Let x be an extreme point of P(G) and x' and G' obtained from x and G by applications of operations* O_1, O_2, O_3. *Then x' is an extreme point of P(G'). Moreover if x violates a cut, a partition or an F-partition inequality, then x' so does.*

Domination

Let x and y be fractional two extreme points of $P(G)$. Let $F_x=\{e\in E \mid x(e) \text{ is fractional}\}$ and $F_y=\{e\in E \mid y(e) \text{ is fractional}\}$.

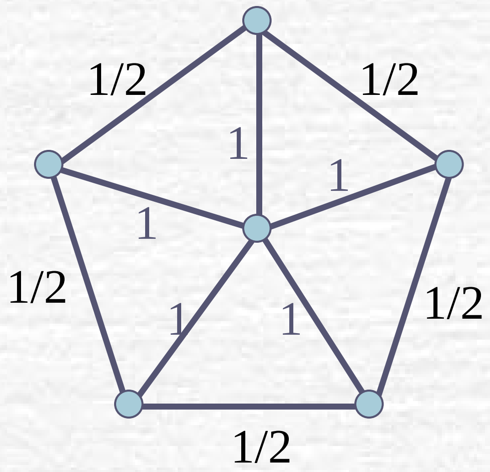
We say that x dominates y if $F_y \subset F_x$.

Question:

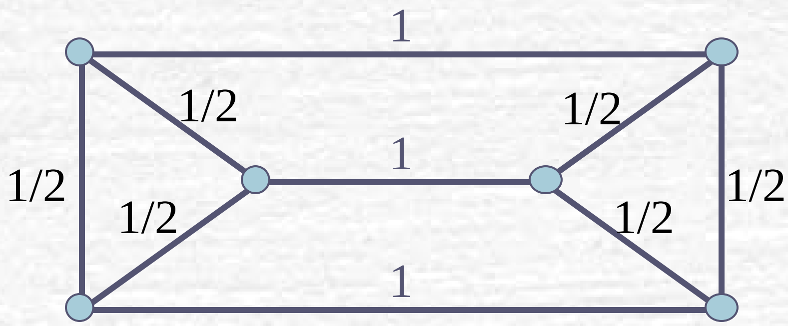
Characterise the minimal fractional extreme points.

<u>Definition :</u> A fractional extreme point *x* of *P(G)* is said to be *critical* if:
1) none of the operations O_1, O_2, O_3 can be applied for it,
2) it does not dominate any fractional extreme point of *P(G)*.

Example:



Critical



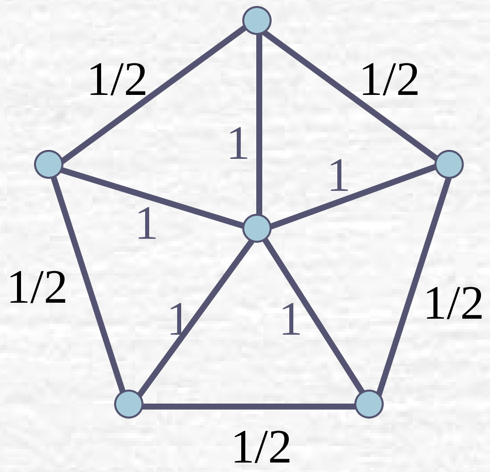
Non-critical

Definition : A fractional extreme point *x* of *P(G)* is said to be *critical* if:

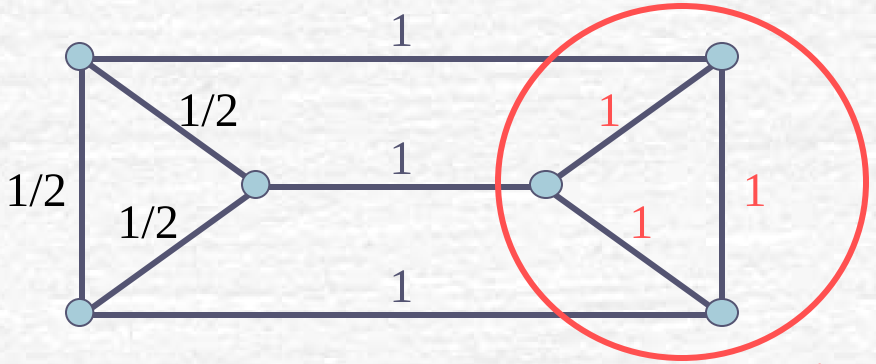
1) none of the operations O_1, O_2, O_3 can be applied for it,

2) it does not domine any fractional extreme point of *P(G)*.

Example:



Critical

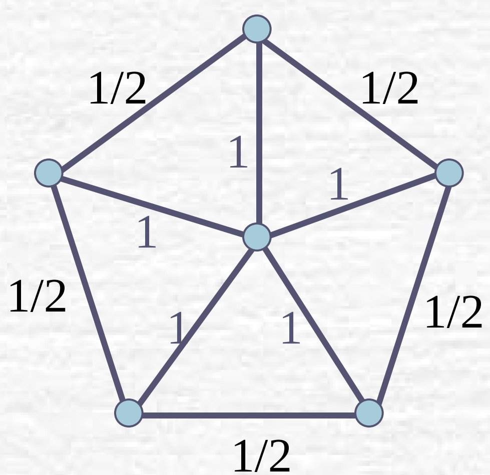


Non-critical

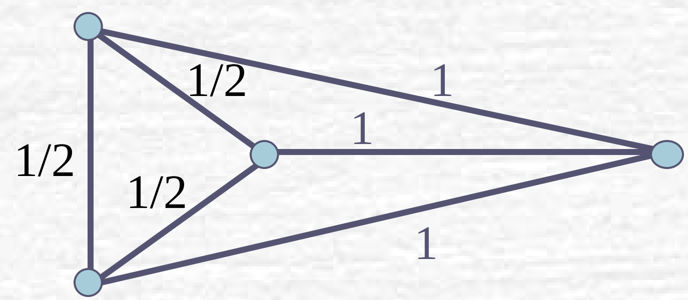
Definition : A fractional extreme point *x* of *P(G)* is said to be *critical* if:

1) none of the operations O_1, O_2, O_3 can be applied for it,
2) it does not domine any fractional extreme point of *P(G)*.

Example:



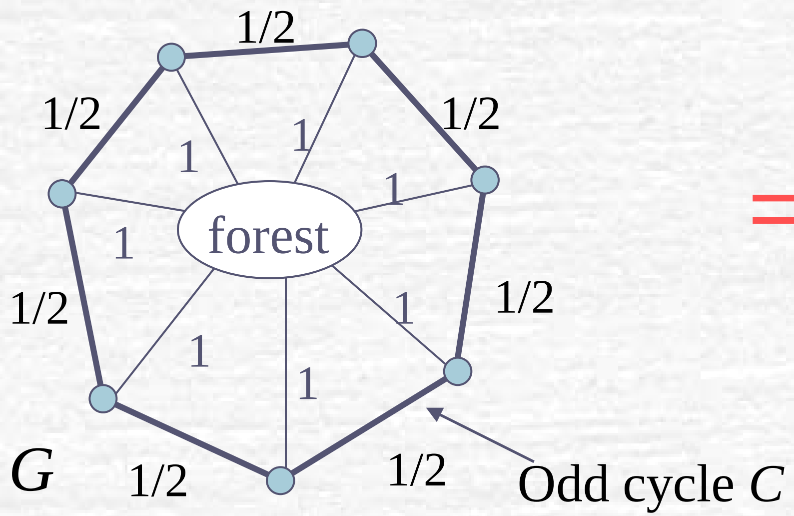
Critical



Extreme point

Critical

Theorem: *An extreme point of P(G) is critical if and only if G and x are of the following form:*



$$\Rightarrow \quad \sum_{e \in C} x(e) \geq \frac{|C|+1}{2}$$

is valid and defines a facet

(it is an *F*-partition inequality)

<u>Theorem:</u> *If x is a critical extreme point of P(G), then x can be separated (in polynomial time) by an F-partition inequality.*

The concept of critical extreme points has been extended (with respect to appropriate reduction operations) to 2-node connected graphs and (1,2)-survivable networks (Kerivin, M., Nocq (2001)), And to *k*-edge connected graphs (Didi Biha & M. (2004)).

2.6. Branch&Cut algorithm

(Kerivin, Nocq, M. (2001))

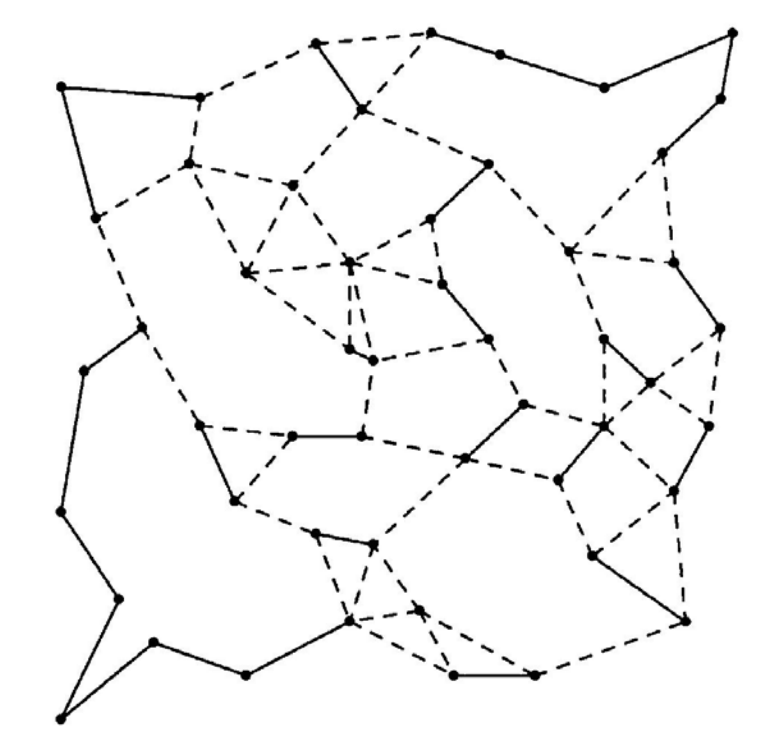
$r(v) \in \{1,2\}$ for all v

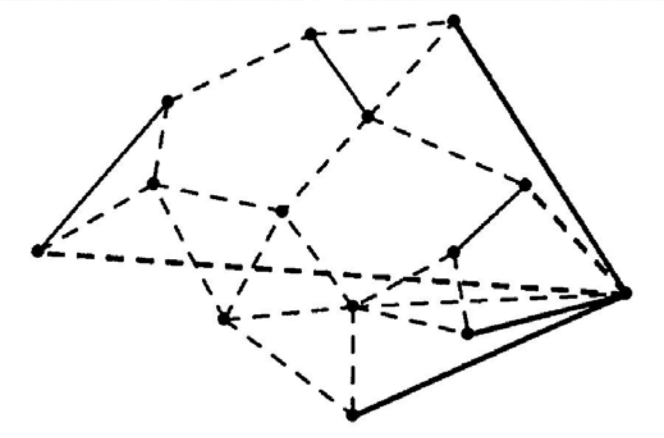
Used constraints:

trivial inequalities
cut inequalities
F-partition inequalities
partition inequalities

If x is a fractional extreme point (critical or not), we apply the reduction operations. Let G' and x' be the graph and the solution thus obtained.

If a cut, a partition or an F-partition constraint is violated by x' for G', then it can be lifted to a constraint of the same type violated by x for G.





F

G' 14 nodes

$x(\delta(V_1,\ldots,V_p)\backslash F)\geq 11$

G 51 nodes

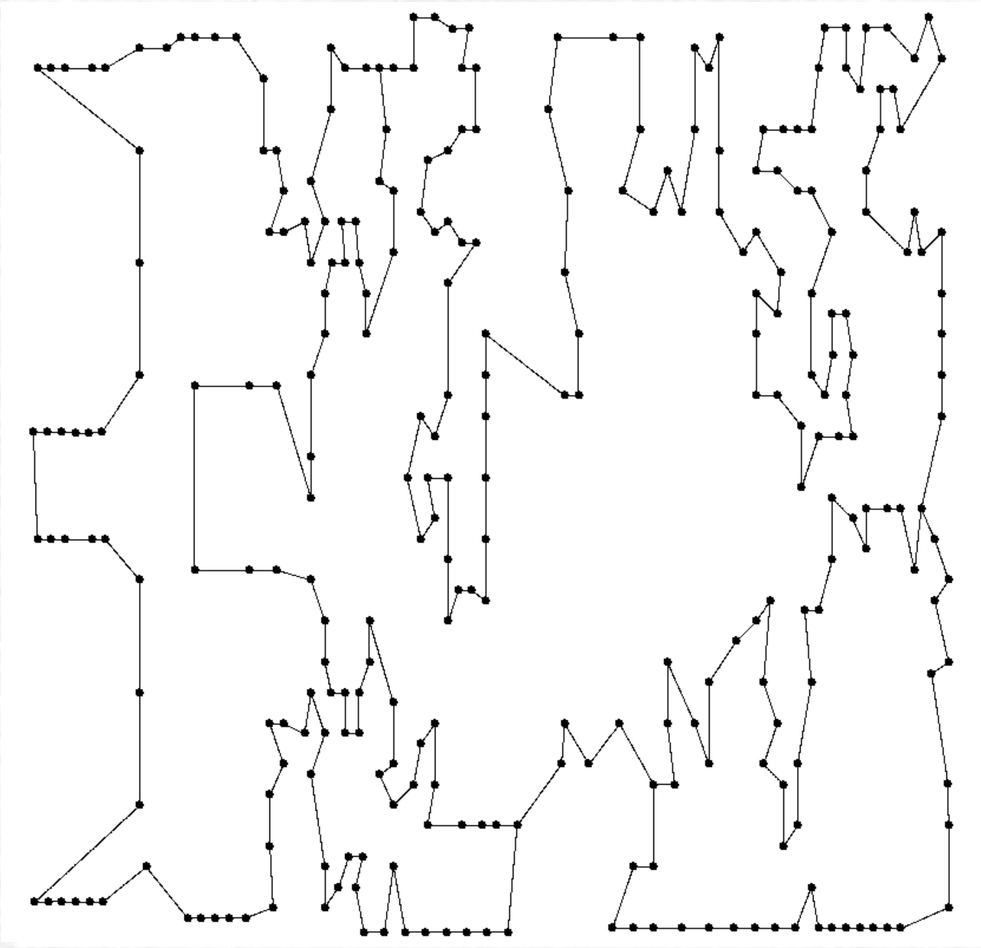
This contraint cuts the extreme point of *G'* and that of *G*.

#nodes 299
(type 2)

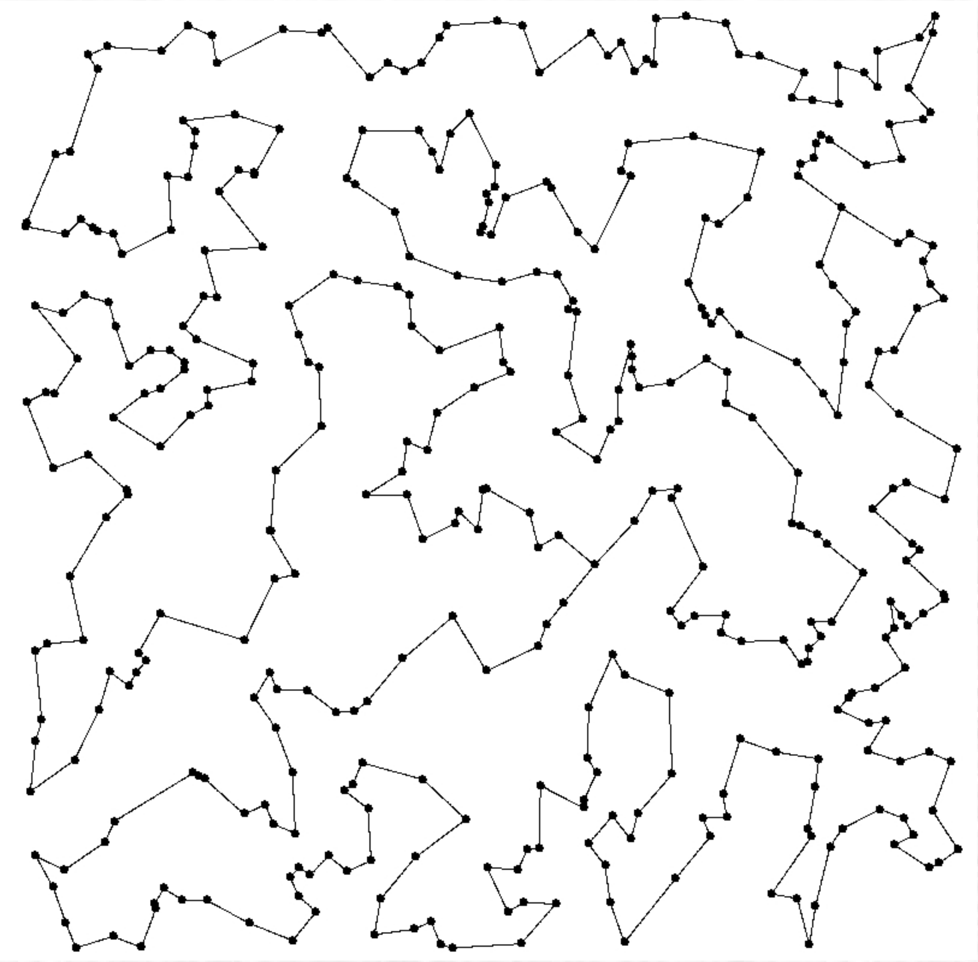
#variables 44551

#constraints 357

CPU Time 142 sec



#nodes (type 2) 2-node connected	**400**
#variables	**79400**
#constraints	**1369**
CPU Time	**152 min**



#nodes 48

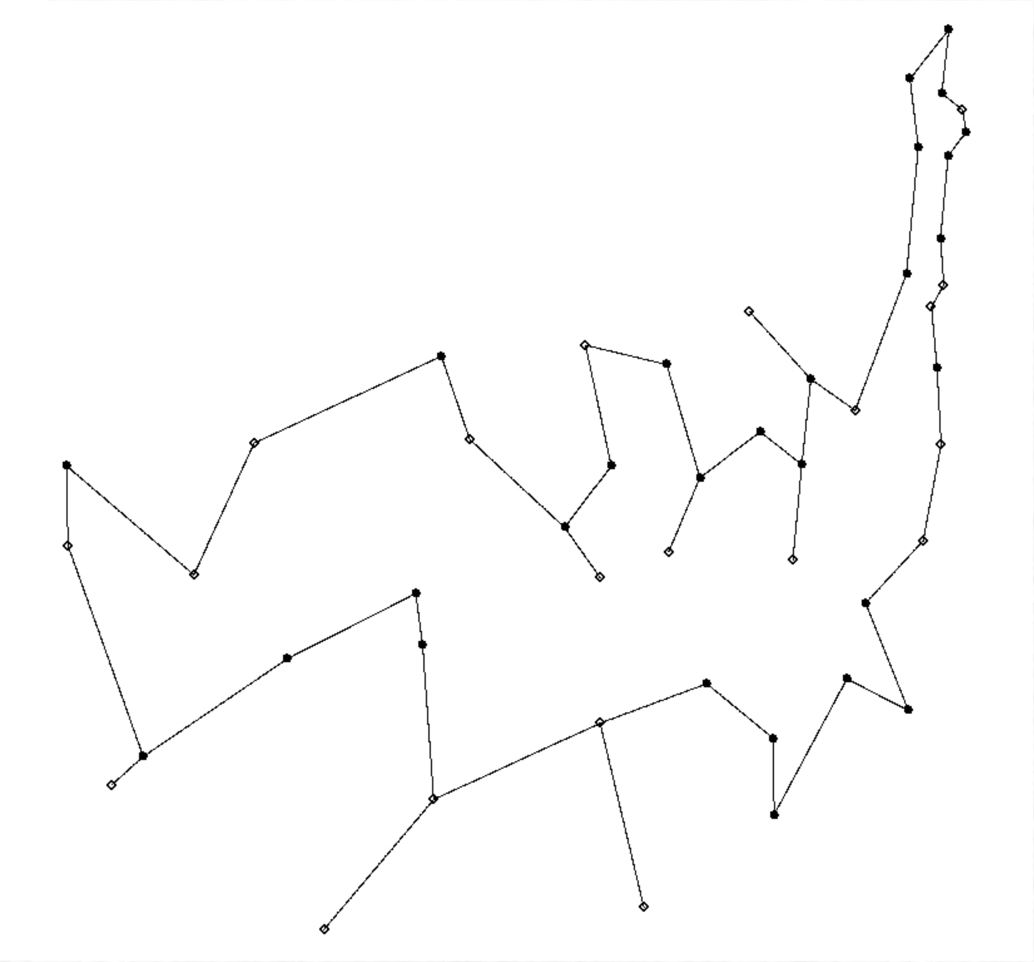
#type 1 20

#type 2 28

#variables 1 128

#constraints 428

CpuTime 202 sec



2.7. Survivable networks with length constraints

Motivation: to have effective routing cost

Local rerouting:

Each edge must belong to a bounded cycle (ring).
SONET/SDH networks

End-to-end rerouting:

the paths between the terminals should not exceed a certain length (a certain number of hops) (hop-constrained paths).
ATM networks, INTERNET

2.7.1. Bounded rings

2-node connected graphs

Fortz, Labbé, Maffioli (1999)
Fortz, Labbé (2002)

Valid inequalities
Separation algorithms
Lower bounds on the optimal value
Cutting plane algorithms

2-edge connected graphs

Fortz, M., McCormick, Pesneau (2003)

2.7.2. Hop-constrained paths

The minimum hop constrained spanning tree problem

Determine a minimum spanning tree such that the number of links between a root node and any node in the tree does not exceed a bound L.

(NP-hard (even for *L*=2))

Multicommodity flow formulation
Lagrangean relaxations

Gouveia (1998)
Gouveia & Requejo (2001)
Gouveia & Magnanti (2000)

The minimum hop-constrained path problem

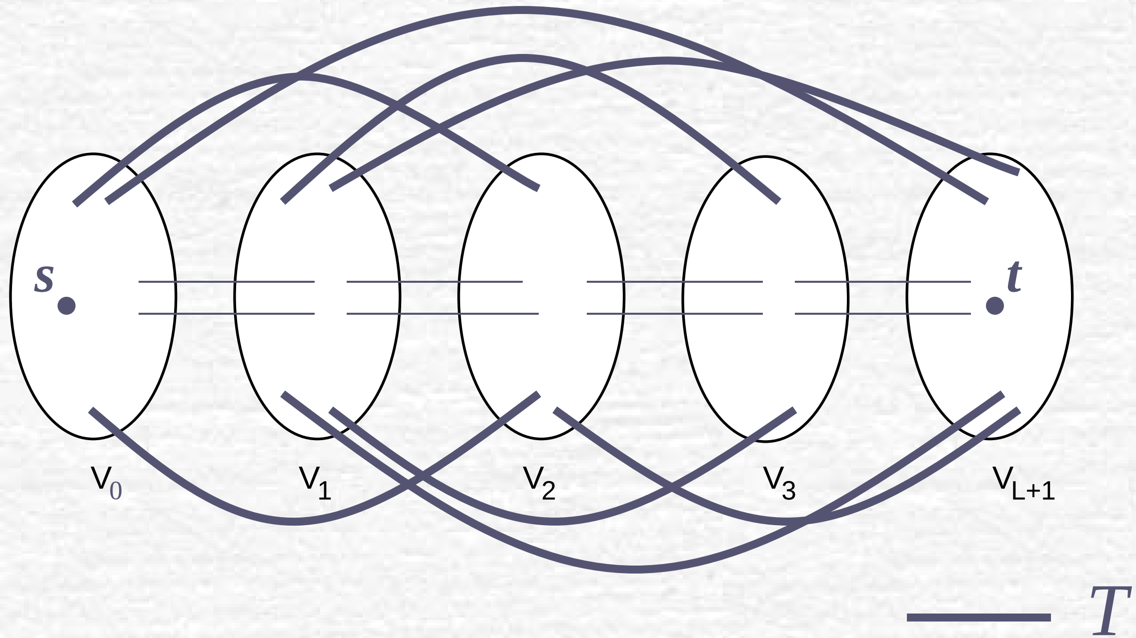
Determine a minimum path between two given nodes s and t, of length no more than L (L fixed).

Dahl & Gouveia (2001)

Formulation in the natural space of variables
Valid inequalities
Description of the associated polytope when L=2,3.

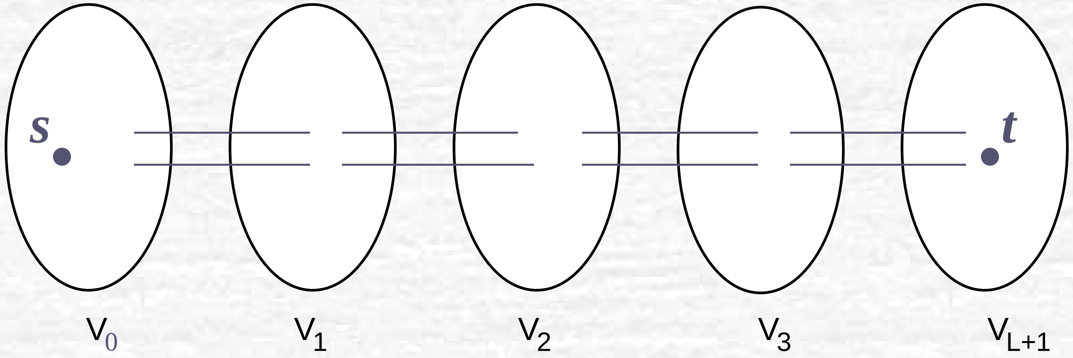
The *L*-star inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



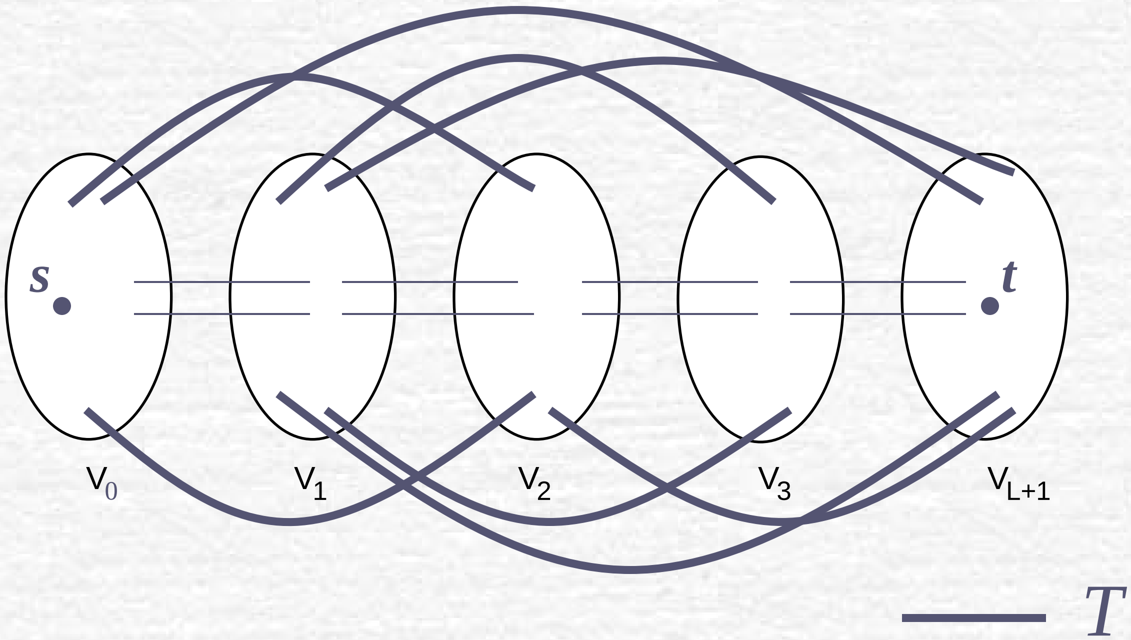
The *L*-star inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



The *L*-star inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



$\boldsymbol{x(T) \geq 1}$

(*L*-star inequalities)

<u>Theorem:</u> (Dahl (1999))

The L-star inequalities together with the cut inequalities (separating s and t) and the trivial inequalities completely describe the L-path polyhedron when $L \leq 3$.

If at least K paths are required between s and t, *then*

$$x(T) \geq K$$

is valid for the corresponding polytope.

The separation problem for the L-star inequalities can be solved in polynomial time, if $L \leq 3$.

Fortz, M., McCormick, Pesneau (2003)

The hop-constrained network design problem (HCNDP):

Given a graph with weights on the edges, a set of terminal-pairs (origines-destinations), two intgers K, L, find a minimum weight subgraph such that between each pair of terminals there are at least K paths of length no more than L.

***K*=1, *L*=2** (Dahl, Johannessen (2000))

Formulation of the problem
Valid inequalities
Greedy approximation algorithms
Cutting plane algorithm

***K*=2, *L*=3, and only one pair of terminals**

Huygens, M., Pesneau (2003)

Formulation of the problem
Complete description of the associated polytope by the trivial, the cut and the *L*-star inequalities

$\Rightarrow$ a polynomial time cutting plane algorithm for the problem (when *K*=2, *L*=3)

No formulation (using the design variables) is known for the problem when $K = 2$ and $L = 4$.

Length constrained 2-connected graphs

Ben Ameur (2000)

Classes of length constrained 2-connected graph
Lower bounds on the number of edges
Valid inequalities for the 2-connected polytope with length constraints

2.8. Capacitated Survivable Networks

Given

- a graph $G=(V,E)$ (the supply graph) ,
- a set of demands $\{d_{uv}\}$ between pairs of origines-destinations (u,v),
- a set $\{C_e^t, t=1,\ldots,T_e\}$ of discrete capacities, specified for each edge e
- a cost K_e^t for each capacity C_e^t,
- for every demand d_{uv}, a parameter $0<\rho_{uv}<1$ representing the fraction of demand d_{uv} that must be satisfied if an edge (or a node) fails.

The problem: Stoer & Dahl (1994)

Which capacities to install on the edges such that for every single edge (or node) failure, at least the fraction ρ_{uv} of d_{uv} can be routed for every demand d_{uv}, and the total cost is minimum.

Stoer & Dahl (1994, 1998) proposed

- a mixed integer programming formulation,
- valid and facet defining inequalities (some of the inequalities obtained by exploiting the knapsack structure of some subsystems) and the 2-connected topology,
- a cutting plane algorithm.

A more general model with path length and routing constraints

Alevras, Grötschel, Wessäly (1997, 1998)

Mixed integer programming formulation
Cutting plane algorithms

Cut subsystem

Let C be a cut. Usually inequalities of the following form are valid.

Demand inequality $$\sum_{e \in C} x(e) \geq D \qquad (1)$$

Survivability inequalities $$\sum_{f \in C \setminus \{e\}} x(f) \geq L, \qquad e \in C \qquad (2)$$

$$x(e) \in Z_+ , \qquad e \in C \qquad (3)$$

Let $P_n(D,L)$ ($|C|=n$) be the polyhedron given by (1)-(3).

$P_n(D,L)$ (and some extensions) have been studied by

Bienstock and Muratore (1997)

Muratore (1998)

- Structural properties of the extreme points of $P_n(D,L)$
- Description of valid and facet defining inequalities
- Development of cutting plane algorithm for solving capacitated SNDP.

Magnanti & Wang (1997) studied the same polyhedron but without constraint (1) (capacity constraint) and with different right hand sides for the survivability inequalities.

Conclusion

- The Survivable network design problems are difficult to solve (even special cases).

- The problems with length constraints remain the most complicated SNDP . A better knowledge of their facial structure would be usefull to establish efficient cutting plane techniques.

- The capacitated SNDP needs more investigation, from both the algorithmic and polyhedral points of view.

-Develop usefull cutting plane and column generation techniques for the very general model with length constraints, capacity assignment and routing….?