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On the dominating set polytope

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Abstract

In this paper, we study the dominating set polytope in the class of graphs that decompose by one-node cutsets where the pieces are cycles. We describe some classes of facets and procedures to construct facets of the polytope in that class of graphs, and establish some structural properties. As a consequence we obtain a complete description of the polytope by a system of inequalities when the graph is a cycle. We also show that the separation problem related to that system can be solved in polynomial time. This yields a polynomial time cutting plane algorithm for the minimum weight dominating set problem in that case. We further discuss the applications for the class of cactus graphs.

1. Introduction

Given a graph $G = (V, E)$, a node subset $D \subseteq V$ of G is called a *dominating set* if every node of $V \setminus D$ is adjacent to at least one-node of D. Given a weight vector $w \in \mathbb{R}^V$ associated with the nodes of G, the *minimum weight dominating set problem* (MWDSP for short) consists of finding a dominating set D of G such that $\sum_{u \in D} w(u)$ is minimum. This problem is a well known intractable problem.

The MWDSP arises in many applications [6,7,13], in particular, in those involving the strategic placement of men or pieces on the nodes of a network. The recent book by Haynes

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et al. [13] illustrates many interesting examples, including sets of representatives, school bus routing, (r, d)-configurations, radio stations … etc.

The MWDSP has been extensively investigated from an algorithmic point of view [5–7, 9–11]. It is NP-hard in general. The cardinality version has been shown to be polynomially solvable in several classes of graphs such as cactus graphs [14] and the class of series-parallel graphs [15]. However, to the best of our knowledge, no polynomial time algorithms are known for the MWDSP in these graphs. A complete survey of the algorithmic complexity of the DSP can be found in [5,13].

If $G = (V, E)$ is a graph and $S \subseteq V$ a node subset of G, then the 0-1 vector $x^S \in \mathbb{R}^V$ with $x^S(u) = 1$ if $u \in S$ and $x^S(u) = 0$ otherwise, is called the *incidence vector* of S.

The convex hull of the incidence vectors of all dominating sets of G, denoted by $P_D(G)$, is called the *dominating set polytope* of G. Thus the MWDSP is equivalent to the linear program $\min\{w^{\mathrm{T}}x : x \in P_D(G)\}$.

Let $G = (V, E)$ be a graph. If $u \in V$ is a node of G, the *neighborhood* of u in G, denoted by $N(u)$, is the node set consisting of u together with the nodes which are adjacent to u. If $u \in V$, we let $N^*(u) = N(u) \setminus \{u\}$. If $S \subseteq V$ and $b : V \longrightarrow \mathbb{R}$, $b(S)$ will denote $\sum_{u \in S} b(u)$. The MWDSP is equivalent to the following integer program

$$\min\ w^{\mathrm{T}}x$$
$$x(u) \geq 0 \quad \text{for all } u \in V, \tag{1.1}$$
$$x(u) \leq 1 \quad \text{for all } u \in V, \tag{1.2}$$
$$x(N(u)) \geq 1 \quad \text{for all } u \in V, \tag{1.3}$$
$$x(v) \text{ integer} \quad \text{for all } v \in V.$$

Inequalities (1.1) and (1.2) are called *trivial inequalities* and inequalities (1.3) are called *neighborhood inequalities*. In [3] the following has been shown.

Theorem 1.1. *Let C be a chordless cycle on n-nodes. Then the inequality*

$$x(C) \geq \left\lceil \frac{|C|}{3} \right\rceil \tag{1.4}$$

is valid for $P_D(C)$. Moreover it defines a facet of $P_D(C)$ if and only if either $|C| = 3$ or $|C| \geq 4$ and $|C|$ is not a multiple of 3.

Inequalities (1.4) are called *cycle inequalities*.

If $G = (V, E)$ is a graph and $u \in V$ is a node that is not adjacent to any node of $V \setminus \{u\}$, then u is said to be *isolated*. It is not hard to see that if G does not contain isolated nodes, then $P_D(G)$ will be full dimensional. In the rest of the paper we consider the graphs that do not have isolated nodes.

In contrast to many NP-hard combinatorial optimization problems, the polyhedral aspect of the MWDSP has not received much attention. To the best of our knowledge, the polytope $P_D(G)$ has been characterized only in the class of threshold graphs [16], and the class of strongly chordal graphs [9] within the framework of totally balanced matrices. Our aim, in this paper, is indeed to study the polytope $P_D(G)$ in a further class of graphs, namely the class of cactus graphs. In particular, we give a complete description of $P_D(G)$ on a cycle which was an open question for many years.

Given a graph $G = (V, E)$ and two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G, G is called *k-sum* of G_1 and G_2, where k is a positive integer, if $V = V_1 \cup V_2$, $|V_1 \cap V_2| = k$, and the subgraph $(V_1 \cap V_2, E_1 \cap E_2)$ is complete. The set $V_1 \cap V_2$ is called a *k-node cutset*. When $k = 1$ we will write one-node cutset and one-sum for 1-node cutset and 1-sum.

In [3] (see also [2,8]), the polytope $P_D(G)$ is studied in the graphs that decompose by one-node cutsets. It is shown that if G decomposes into G_1 and G_2, then the dominating set polytope of G can be described from two linear systems related to G_1 and G_2.

A *cactus* is a graph that can be decomposed by one-node cutsets into cycles and edges. Let Γ (Γ') be the class of graphs G that may be obtained by means of one-sums from a chordless cycle C (an edge e) and a family of 5-cycles so that all the articulation nodes belong to C (e) (see the end of this section). As it has been pointed out in [3], to give a complete description of $P_D(G)$ (a polynomial time algorithm) when G is a cactus, one has to know such a description (such an algorithm) for the classes Γ and Γ'.

If G is a graph of Γ', then it is not hard to see that $P_D(G)$ is given by inequalities (1.1)–(1.4). However, if G is a graph of Γ, $P_D(G)$ may have further types of facets. In this paper we describe some classes of facets and procedures to construct facets of the polytope $P_D(G)$ in that class of graphs and give some structural properties. As a consequence, we obtain a complete description of the polytope of $P_D(G)$ when G is a cycle. This yields a polynomial time cutting plane algorithm for solving the MWDSP in a cycle. To the best of our knowledge, this is the first polynomial time algorithm for the MWDSP in these graphs. From [3], this is a first and essential step for devising an efficient algorithm for the MWDSP in the class of cactus graphs. We finally discuss some applications for the class of cactus graphs.

The paper is organized as follows. In the next section, we describe some facets and procedures to construct facets of the polytope $P_D(G)$ when G is a graph of Γ, and give some structural properties. In Section 3 we characterize $P_D(G)$, by a system of inequalities, when G is a cycle. In Section 4 we study the separation problem for this system, and show that the minimum weight dominating set problem can be solved in this case in polynomial time by a cutting plane algorithm. In Section 5 we give some concluding remarks.

In the rest of this section we give some notations.

We consider finite, undirected and loopless graphs. We denote a graph by $G = (V, E)$ where V is the *node set* and E the *edge set*. If $G = (V, E)$ is a graph and $e \in E$ is an edge whose endnodes are u and v, then we write $e = (u, v)$. A path P (cycle C) in G whose sequence of nodes is $v_1, \ldots, v_k$ will be denoted by $P = \{v_1, \ldots, v_k\}$ (resp. $C = \{v_1, \ldots, v_k\}$).

If $G = (V, E)$ is a graph of Γ, then we denote by $C = \{1, 2, \ldots, n\}$ and $T_1, \ldots, T_r$, respectively, the cycle and, say, the r 5-cycles from which G can be obtained by means of one-sums. We let $T_j = \{v_j, w_1^j, w_2^j, w_3^j, w_4^j\}$ where $v_j \in C$ and w_1^j, w_4^j are the nodes of T_j that are adjacent to v_j. Given two-nodes u, v of C, we denote by $C(u, v)$ the path $(u, u+1, \ldots, u+t)$ of C between u and v, where t is such that $u + t = v$ (the integers are modulo n).

2. Facets and structural properties

In this section we identify various facets of the polytope $P_D(G)$, and describe some structural properties.

2.1. Facets

In what follows we introduce a class of facets of $P_D(G)$ when G is a cycle. It is a special case of a more general class of facets defining inequalities of the set covering polytope given in [17]. We also give a procedure that permits us to construct facets from facets. The results are given without proof, for the proof see [4].

Theorem 2.1. *Let $C = \{1, \ldots, n\}$ be a cycle. Let $W = \{v_1, \ldots, v_p\}$ be a subset of $p \geq 3$-nodes of C where p is odd and $v_1 < v_2 < \cdots < v_p$. Suppose that $|C(v_i + 1, v_{i+1} - 1)| = 3k_i$, $k_i \geq 1$, for $i = 1, \ldots, p \pmod{p}$. Then the constraint*

$$2 \sum_{v \in W} x(v) + \sum_{v \in C \setminus W} x(v) \geq \sum_{i=1}^{p} k_i + \left\lceil \frac{p}{2} \right\rceil$$

defines a facet of $P_D(C)$.

The following theorem describes the relation between the facets of $P_D(G)$ and those of $P_D(\bar{G})$ where $\bar{G}$ is the one-sum of G and a 5-cycle. Here G is arbitrary. It describes a procedure of construction of facets for $P_D(\bar{G})$. This may be very useful for characterizing the dominating set polytope in cactus graphs by means of one-sums.

Theorem 2.2. *Let $G = (V, E)$ be a graph and $a^{\mathrm{T}} x \geq \alpha$ be a facet defining inequality of $P_D(G)$. Let $u \in V$ and $\delta = \min\{a(v),\ v \in N_G(u)\}$. Let $\bar{G}$ be the graph that is the one-sum of G and a 5-cycle $C = \{u,\ w_1, w_2, w_3, w_4\}$ where $\{u\} = C \cap V$. (Recall that u is adjacent to w_1 and w_4.)*

(i) *If $a^{\mathrm{T}} x \geq \alpha$ is valid for $P_D(\bar{G})$ then $a^{\mathrm{T}} x \geq \alpha$ defines a facet of $P_D(\bar{G})$.*

(ii) *If $a^{\mathrm{T}} x \geq \alpha$ is not valid for $P_D(\bar{G})$ and there exists a dominating set A^* of G such that $a^{\mathrm{T}} x^{A^*} = \alpha$, $A^* \cap N_G(u) = \{\bar{u}\}$, $u \neq \bar{u}$ and $(A^* \setminus \{\bar{u}\}) \cup \{w_1, w_3\} \in D(\bar{G})$, where $\bar{u}$ is a node of $N_G(u)$ such that $a(\bar{u}) = \delta$, then*

$$a^{\mathrm{T}} x + \delta(x(w_1) + x(w_4)) \geq \alpha$$

defines a facet of $P_D(\bar{G})$.

The next theorem, which is also given without proof, describes the converse operation of Theorem 2.2. For the proof see [4].

Theorem 2.3. *Let $\bar{G} = (\bar{V}, \bar{E})$ be a graph which is the one-sum of $G = (V, E)$ and a cycle $C = \{u, w_1, w_2, w_3, w_4\}$, where $\{u\} = C \cap V$. Let $a'^{\mathrm{T}} x \geq \alpha'$ be a facet defining inequality of $P_D(\bar{G})$ such that $a'(\bar{u}) = a'(w_1) = a'(w_4) > 0$ where $\bar{u}$ is a node of $N_G(u)$ such that $a'(\bar{u}) = \min\{a'(v) : v \in N_G(u)\}$. Let*

$$a(v) = a'(v) \quad \text{for all } v \in V,$$
$$\alpha = \alpha'.$$

Then $a^{\mathrm{T}} x \geq \alpha$ defines a facet of $P_D(G)$.

2.2. Structural properties

In what follows we shall give some structural properties of the facet defining inequalities of $P_D(G)$ when G is a cycle. These properties will establish some relations between the coefficients of the facet defining inequalities different from (1.1)–(1.4). These will be useful in the next section for characterizing the polytope $P_D(G)$ in this case. In fact, using these properties, we shall show that any facet defining inequality of $P_D(G)$, when G is a cycle, different from (1.1)–(1.4), is necessarily of type (2.1).

Let $C = \{1, \ldots, n\}$ be a cycle. Let $a^{\mathrm{T}} x \geq \alpha$ be a constraint that defines a facet of $P_D(C)$, different from constraints (1.1)–(1.4). As it is shown in [3], we have that $a(j) \geq 0$ for all $j \in C$. Let $D(C) = \{W \subseteq V : W$ is a dominating set of $C\}$, and $\Delta_a = \{A \in D(C) : a^{\mathrm{T}} x^A = \alpha\}$.

Since $P_D(C)$ is full dimensional, and, hence, there exists a unique (up to multiplication by a positive scalar) linear system that defines $P_D(C)$, the only constraints valid for $P_D(C)$ and satisfied with equality by all the members of Δ_a are positive multiples of $a^{\mathrm{T}}x = \alpha$. We have the following lemmas. The first one is a direct consequence of the fact that $a^{\mathrm{T}}x \geq \alpha$ is different from inequalities (1.1)–(1.4).

Lemma 2.1. (i) *For every node $v \in V$, there is a node set $A \in \Delta_a$ ($A' \in \Delta_a$) such that $v \in A$ ($v \notin A'$).*
(ii) *For every node $v \in V$, there is a node set $\tilde{A} \in \Delta_a$ such that $|\tilde{A} \cap N(v)| \geq 2$.*

The following lemmas are given without proof, for the proof see [4].

Lemma 2.2. *If $a(v) \leq a(v+1)$, $a(v+2)$ (resp. $a(v) \leq a(v-1)$, $a(v-2)$) for some $v \in C$, then either $a(v) = a(v+1)$ or $a(v) = a(v+2)$ (resp. $a(v) = a(v-1)$ or $a(v) = a(v-2)$).*

Lemma 2.3. *For every $v \in C$, at least one of the following statements holds:*

(a) $a(v) \leq a(v+1)$,
(b) $a(v+1) \leq a(v+2)$.

Lemma 2.4. *For every $v \in C$ we have*

$$\min\{a(v), a(v+1)\} \leq \min\{a(v-1), a(v+2)\}.$$

Lemma 2.5. *If $a(v) > a(v+1)$ for some $v \in C$, then $a(v-2) = a(v-1) = a(v+1) = a(v+2)$.*

From Lemmas 2.4 and 2.5, we obtain the following.

Remark 2.1. One of the following statements holds.

(i) $a(u) = a(v)$ for all $u, v \in C$.
(ii) There exist p-nodes $u_1, \ldots, u_p$ of C such that $a(u_i) > a_0$ for $i = 1, \ldots, p$ and $a(v) = a_0$ for all $v \in C \setminus \{u_1, \ldots, u_p\}$ where $a_0 = \min\{a(v), v \in C\}$. Moreover $a_0 > 0$ and $C(u_i + 1, u_{i+1} - 1) \neq \emptyset$ for $i = 1, \ldots, p$ (modulo p).

Lemma 2.6. *Let u and v, $u < v$, be two-nodes of C such that*

(i) $a(u-1) = a(w)$, *for all $w \in C(u+1, v-1)$,*
(ii) $a(u-1) < a(u), a(v)$.

Then $|C(u+1, v-1)| = 3t$ for some $t \geq 1$.

Proof. First note that by Remark 2.1, it follows that $C(u+1, v-1) \neq \emptyset$ and $a(v) > a(v+1)$. Consider a node set A of Δ_a. We have the following claims.

Claim 1. If $|C(u+1, v-1)| = 3t' + 2$, $t' \geq 0$, then $u \in A$ if and only if $v \in A$.

Proof of Claim 1. Indeed, if A contains u, then A must contain the nodes $u+3s$, $s = 1, \ldots, t'+1$. Suppose that this is not the case, and that A contains for instance $u + 3s_0 - 1$ for some $s_0 \in \{1, \ldots, t' + 1\}$. (The proof is similar if A contains a node $u + 3s_0 - 2$ for some $s_0 \in \{1, \ldots, t' + 1\}$.) We may suppose, without loss of generality, that $u + 3s_0 - 1$ is the first node of A not of type $u + 3s$ when going from u to v. Thus $u + 3, \ldots, u + 3(s_0 - 1) \in A$. Now consider the set $A' = (A \setminus \{u, u+3, \ldots, u+3(s_0-1)\}) \cup \{u-1, u+2, \ldots, u+3s_0-4\}$. It is easy to see that A is a dominating set of G. Since $a(u-1) = a(w)$ for all $w \in C(u+1, v-1)$ and $a(u-1) < a(u)$, it follows that $a^{\mathrm{T}} x^{A'} < \alpha$, which is impossible. Consequently, $u + 3s \in A$, for $s = 1, \ldots, t' + 1$, and hence $v \in A$.

If $v \in A$, then by symmetry we have $u \in A$.

Claim 2. If $|C(u+1,\ v-1)| = 3t' + 1,\ t' \geq 0$ then

$$|A \cap C(u, v)| = t' + 1. \tag{2.1}$$

Proof of Claim 2. First suppose that $u \in A$. Then, as in Claim 1, it can be shown that A contains the nodes $u + 3s$, $s = 1, \ldots, t'$. Hence (2.1) holds. If $u \notin A$, then by symmetry, we may suppose that $v \notin A$. It is not hard to see in this case that exactly $t' + 1$-nodes of $C(u+1,\ v-1)$ are needed to dominate this path. In consequence, (2.1) holds.

Now suppose that $C(u+1, v-1) \neq 3t$. If $C(u+1, v-1) = 3t' + 2$ (resp. $C(u+1, v-1) = 3t' + 1$) for some $t' \geq 0$, then it follows by Claim 1 (resp. Claim 2) that the incidence vector of every set A of Δ_a satisfies the equation

$$x(u) - x(v) = 0, \tag{2.2}$$

$$\left(\text{resp. } \sum_{j \in C(u,v)} x(j) = t' + 1\right). \tag{2.3}$$

Since $a \geq 0$, the equation $a^{\mathrm{T}} x = \alpha$ cannot be a positive multiple of (2.2). Also as $a(u) > a(w)$ for all $w \in C(u+1, v-1)$ and $C(u+1, v-1) \neq \emptyset$, $a^{\mathrm{T}} x = \alpha$ cannot be a positive multiple of (2.3). This contradicts the fact that $a^{\mathrm{T}} x \geq \alpha$ defines a facet. $\square$

3. The polytope $P_D(G)$ on a cycle

We have the following.

Theorem 3.1. *If $C = \{1, \ldots, n\}$ is a cycle, then $P_D(C)$ is defined by inequalities* (1.1)–(1.4) *and* (2.1).

Proof. Let $a^{\mathrm{T}} x \geq \alpha$ be a constraint that defines a facet of $P_D(C)$ different from inequalities (1.1)–(1.3). We will show that it is either of type (1.4) or of type (2.1).

If $a(u) = a(v)$ for all $u,\ v \in C$, then $a^{\mathrm{T}} x \geq \alpha$ is of type (1.4) $\left(\text{i.e., } \sum_{j \in C} x(j) \geq \lceil \frac{n}{3} \rceil\right)$. Now suppose that there exist p-nodes $v_1, \ldots, v_p$ of C such that $a(v_i) > a_0$ for $i = 1, \ldots, p$ where $a_0 = \min\{a(v) : v \in C\}$. By Remark 2.1, we have that $a(u) = a_0$ for all $u \in C \setminus \{v_1, \ldots, v_p\}$. Further, by Lemma 2.6, we have that $|C(v_i + 1, v_{i+1} - 1)| = 3k_i$, $k_i \geq 1$, for $i = 1, \ldots, p$ (mod p).

In what follows we are going to show that $a(v_i) = 2a_0$ for $i = 1, \ldots, p$. By Lemma 2.1(i) there is a dominating set $A \in \Delta_a$ that contains v_i. Let $A' = (A \setminus \{v_i\}) \cup \{v_i - 1, v_i + 1\}$. Obviously, $A' \in D(C)$. Therefore,

$$a(v_i) \leq a(v_i - 1) + a(v_i + 1). \tag{3.1}$$

Now let v_i, $i \in \{1, \ldots, p\}$ and $A \in \Delta_a$. We have the following claims.

Claim 1. If $v_{i-1} \in A$ (resp. $v_{i+1} \in A$), then

(i) $|A \cap C(v_{i-1} + 1, v_{i+1} - 1)| = k_{i-1} + k_i$,

(ii) $v_{i+1} \notin A$ (resp. $v_{i-1} \notin A$).

Proof of Claim 1.

(i) Suppose that $v_{i-1} \in A$, the statement when $v_{i+1} \in A$ can be obtained by symmetry. Then A must contain the nodes $v_{i-1} + 3s,\ s = 1, \ldots, k_{i-1} + k_i$. Indeed, suppose that A contains for instance a node $v_{i-1} + 3s_0 - 2$ for some $s_0 \in \{1, \ldots, k_{i-1} + k_i\}$. We may assume that $v_{i-1} + 3s_0 - 2$ is the first node of the path $C(v_{i-1} + 1, v_{i+1} - 1)$ not of the form $v_{i-1} + 3s$ that belongs to A. In consequence, we have that the nodes $v_{i-1}, v_{i-1} + 3, \ldots, v_{i-1} + 3(s_0 - 1)$ all belong to A. Let $A' = (A \setminus \{v_{i-1}, v_{i-1} + 3, \ldots, v_{i-1} + 3(s_0 - 1)\}) \cup \{v_{i-1} - 1, v_{i-1} + 2, \ldots, v_{i-1} + 3s_0 - 4\}$. Observe that the nodes $v_{i-1}+3, \ldots, v_{i-1}+3(s_0-1), v_{i-1}+2, \ldots, v_{i-1}+3s_0-4$ are all different from v_i. It is not hard to see that A' is a dominating set of C. As $a(u) = a_0$ for all $u \in C \setminus \{v_1, \ldots, v_p\}$ and $a(v_{i-1}) > a_0$, it follows that $a^{\mathrm{T}} x^{A'} < \alpha$, which is impossible.

(ii) By (i), $v_{i-1} + 3s \in A$ for $s = 1, \ldots, k_{i-1} + k_i$. Note that $v_{i-1} + 3(k_{i-1} + k_i) = v_{i+1} - 2$. If $v_{i+1} \in A$, then by considering the dominating set $(A \setminus \{v_{i+1}\}) \cup \{v_{i+1} + 1\}$ we get $a(v_{i+1} + 1) \geq a(v_{i+1})$, a contradiction.

Claim 2. For every node v_i, $i \in \{1, \ldots, p\}$, there is a set $A_i \in \Delta_a$ such that $v_i - 3, v_i - 1, v_i + 1, v_i + 3 \in A_i$.

Proof of Claim 2. By Claim 1, if $v_{i-1} \in A$ $(v_{i+1} \in A)$, then $|A \cap C(v_{i-1}, v_{i+1})| = k_{i-1} + k_i + 1$. Now since $a^{\mathrm{T}} x \geq \alpha$ is facet defining, there must exist a solution $A_i \in \Delta_a$ such that $|A_i \cap C(v_{i-1}, v_{i+1})| > k_{i-1} + k_i + 1$. For otherwise, every set $A \in \Delta_a$ would contain exactly $k_{i-1}+k_i+1$-nodes of $C(v_{i-1}, v_{i+1})$, and in consequence, $a^{\mathrm{T}} x = \alpha$ would be a positive multiple of the equation $\sum_{v \in C(v_{i-1}, v_{i+1})} x(v) = k_{i-1}+k_i+1$. As $C(v_{i+1}+1, v_{i-1}-1) \neq \emptyset$ and $a(u) > 0$ for all $u \in C(v_{i+1} + 1, v_{i-1} - 1)$, this is impossible.

So there is $A_i \in \Delta_a$ that contains at least $k_{i-1}+k_i+2$-nodes of $C(v_{i-1}+1, v_{i+1}-1)$. Note that (by Claim 1) A_i does not contain neither v_{i-1} nor v_{i+1}. Consider the set $A_i' = \{v_{i-1}+1, v_{i-1}+4, \ldots, v_{i-1}+1+3k_{i-1}, v_i+1, v_i+3, \ldots, v_i+3k_i\}$. Note that $|A_i'| = k_{i-1}+k_i+2$. Also note that A_i' dominates all the nodes of $C(v_{i-1}, v_{i+1})$. As $v_{i-1}, v_{i+1} \notin A_i$, $v_{i-1}+1, v_{i+1}-1 \in A_i'$, $a(v_i) > a_0$ and $a(w) = a_0$ for all $w \in C(v_{i-1}+1, v_{i+1}-1) \setminus \{v_i\}$, we may assume that $A_i' \subseteq A_i$. This ends the proof of the claim.

Now for v_i, $i \in \{1, \ldots, p\}$, by Claim 2, there exists a set $A_i \in \Delta_a$ such that $v_i - 3, v_i - 1, v_i + 1, v_i + 3 \in A_i$. Let $\bar{A}_i = (A_i \setminus \{v_i - 1,\ v_i + 1\}) \cup \{v_i\}$. As $\bar{A}_i \in D(C)$, it follows that

$$a(v_i) \geq a(v_i - 1) + a(v_i + 1). \tag{3.2}$$

By (3.1) and (3.2) we have that $a(v_i) = 2a_0$. This implies that $a^{\mathrm{T}} x \geq \alpha$ is of type (2.1). $\square$

4. Separation and algorithmic consequences

The separation problem for a class of inequalities consist of deciding whether a given vector $\overline{x} \in \mathbb{R}^V$ satisfies the inequalities, and if not to find an inequality that is violated by $\overline{x}$. An algorithm that solves this problem is called a *separation algorithm*.

Clearly, the separation problem for inequalities (1.1)–(1.4) can be solved in polynomial time for a cycle. In what follows we shall show that inequalities (2.1) can also be separated in polynomial time.

Theorem 4.1. *Inequalities (2.1) can be separated in polynomial time.*

Proof. As $|C(v_i + 1, v_{i+1} - 1)| = 3k_i$ for $i = 1, \ldots, p \mod(p)$, it is not hard to see that inequality (2.1) can be written as

$$\sum_{v \in W} \left(2x(v) - \frac{1}{2}\right) + \sum_{v \in C \setminus W} \left(x(v) - \frac{1}{3}\right) \geq \frac{1}{2}. \tag{4.1}$$

Let us denote by $U = \{u_1, \ldots, u_n\}$ the node set of C and let $\bar{x} \in \mathbb{R}^n$. Remark that at least one of the nodes of C belongs to W. In consequence, to prove the theorem, it suffices to show that the separation of inequalities (4.1), when a fixed node, say v, is in W, can be done in polynomial time. In what follows we will prove this for $v = u_1$. As it will turn out, the separation problem in this case can be reduced to a shortest path problem in an appropriate directed graph.

Let $H = (U \cup U', F)$ be the bipartite directed graph where $U' = \{u'_1, \ldots, u'_n\}$ is a copy of U with node u'_i corresponding to u_i, and F is the set of arcs defined as follows: first consider in F the arcs (u_1, u'_{3k+2}) for all $k \geq 1$ such that $3k + 2 \leq n - 4$. Then add, in a recursive way

- for any arc (u_i, u'_j) in F, the arcs (u'_j, u_{j+3k+1}) for all $k \geq 1$ such that $j + 3k + 1 \leq n$, and
- for any arc (u'_i, u_j) in F, the arcs (u_j, u'_{j+3k+1}) for all $k \geq 1$ such that $j + 3k \leq n$, where the indices are mod (n). (That is if $j + 3k + 1 = n$, then $u'_{j+3k+1} = u'_1$.)

Note that H is acyclic. Indeed, if there is a circuit in H, then it must go through node u_1. However, no arcs are going into u_1.

With every arc $e = (u_j, u'_{j+3k+1})$ (resp. $e = (u'_j, u_{j+3k+1})$) associate the weight

$$w(e) = 2x(u_j) - \frac{1}{2} + \sum_{t=j+1}^{j+3k} \left(x(u_t) - \frac{1}{3}\right).$$

Separating (4.1) for $u_1 \in W$ is equivalent to calculating a shortest path in H from u_1 to u'_1. In fact, first note that, as graph H is bipartite, any path between u_1 and u'_1 is odd. Moreover, a sequence of nodes $L = (u_1 = u_{i_1}, u'_{i_2}, \ldots, u_{i_p}, u'_1)$ is a path of H from u_1 to u'_1 if and only if $W = \{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$ is a set of nodes of C such that p is odd and $|C(u_{i_j}+1, u_{i_{j+1}}-1)| = 3k_j$ for some integer k_j for $j = 1, \ldots, p$ where the indices are modulo p. In addition, the weight of L, $w(L)$ is equal to the left hand side of the constraint (4.1) induced by W. Thus if $w(L) < \frac{1}{2}$, then one gets a violated inequality. Otherwise, all the inequalities of type (4.1) with $u_1 \in W$ are fulfilled by $\bar{x}$.

As graph H is acyclic, computing a shortest path in H can be done in polynomial time using for instance Bellman algorithm [1]. □

By Theorem 4.1, it follows from [12] that MWDSP can be solved in polynomial time on a cycle using a cutting plane algorithm. Hence we can state the following.

Corollary 4.2. *The MWDSP is polynomially solvable on a cycle.*

5. Final remarks

In [3] it is shown that if a graph $G = (V, E)$ is the one-sum of a graph (W, F) and a 5-cycle $C = \{u, w_1, w_2, w_3, w_4\}$ where $\{u\} = C \cap V$ (and w_1, w_4 are adjacent to u), then any facet defining inequality of $P_D(G)$ whose support intersects $\{w_1, w_2, w_3, w_4\}$ is of the form $\sum_{v \in W} x(v) + x(w_1) + x(w_4) \geq \alpha$.

Consider a graph $G = (V, E)$ that is the one-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $\{u\} = V_1 \cap V_2$, and let $\bar{G}_i = (\bar{V}_i, \bar{E}_i)$ for $i = 1, 2$, be the graph obtained as a one-sum of G_i and a 5-cycle $(u, w_1^i, w_2^i, , w_3^i, w_4^i)$. In [3] it is also shown that if $\sum_{v\in V_1} a^1(v)x(v) + x(w_1^1) + x(w_4^1) \geq \alpha^1$ and $\sum_{v\in V_2} a^2(v)x(v) + x(w_1^2) + x(w_4^2) \geq \alpha^2$ are facet defining inequalities for $P_D(G_1)$ and $P_D(G_2)$, respectively, then the inequality

$$\sum_{v\in V_1\setminus\{u\}} a^1(v)x(v) + \sum_{v\in V_2\setminus\{u\}} a^2(v)x(v) + (a^1(u) + a^2(u) - 1)x(u) \geq \alpha^1 + \alpha^2 - 1$$

is valid and defines a facet for $P_D(G)$.

Now let $p \geq 3$ be an odd integer. Consider a cycle $C = \{1, \ldots, 4p\}$ on $4p$-nodes. Let $v_1, \ldots, v_p$ be the nodes of C such that $|C(v_i + 1, v_{i+1} - 1)| = 3$ for $i = 1, \ldots, p$ (where the indices are modulo p). Let H be the graph obtained by connecting to node v_i, $p - i$ 5-cycles, for $i = 1, \ldots, p - 1$. Let W be the set of nodes of H which are not adjacent to nodes in C. From Theorems 2.1 and 2.2, one can construct a facet of $P_D(H)$ with coefficient 2 for nodes $v_1, \ldots, v_p$, 0 for the nodes in W and 1 for the rest of nodes.

Also consider the graph K which is the one-sum of a cycle $C' = \{1, \ldots, 12\}$ of 12-nodes and a 5-cycle. Let u, u', u'' be the nodes of C such that $|C(u+1, u'-1)| = |C(u'+1, u''-1)| = 3$. Suppose u is the node common to C' and the 5-cycle. Similarly, by Theorems 2.1 and 2.2, one can construct a facet of $P_D(K)$ with coefficient 2 for node u, u', u''; 0 for the nodes of the 5-cycle which are not adjacent to C' and 1 for the rest of the nodes.

Let G be the graph obtained from H and copies of K by recursive application of the one-sum operation with respect to $v_1, \ldots, v_{p-1}$ and u until all the 5-cycles of H disappear. From the composition in [3] given above, G induces a facet of $P_D(G)$ whose coefficients are $p + 2 - i$ for node v_i, for $i = 1, \ldots, p$ and 1 for the other nodes. This leads to the following remark.

Remark 5.1. Given an integer $p > 0$, there exists a graph $G \in \mathcal{C}$ (where $\mathcal{C}$ is the class of cactus graphs) such that $P_D(G)$ has a facet defining inequality with the coefficients $1, 2, \ldots, p$.

Finally let us note that, in the light of the results given above, a complete description of the $P_D(G)$ in the class of cactus graphs should use the procedure given in Theorem 2.2 together with a complete description of $P_D(G)$ in the class Γ. For this class, these results motivate us to give the following conjecture.

Conjecture 5.2. *If G is a graph of Γ, then a constraint different from inequalities (1.1)–(1.3) is facet defining for $P_D(G)$ if and only if it can be obtained from a constraint of $P_D(C)$ by application of the procedure of Theorem 2.2.*

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