# A Note on $K_{i}$-Perfect Graphs 

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#### Abstract

$K_{i}$-perfect graphs are a special instance of $F-G$ perfect graphs, where $F$ and $G$ are fixed graphs with $F$ a partial subgraph of $G$. Given $S$, a collection of $G$-subgraphs of graph $K$, an $F-G$ cover of $S$ is a set of $T$ of $F$-subgraphs of $K$ such that each subgraph in $S$ contains as a subgraph a member of $T$. An $F-G$ packing of $S$ is a subcollection $S^{\prime} \subseteq S$ such that no two subgraphs in $S^{\prime}$ have an $F$-subgraph in common. $K$ is $F-G$ perfect if for all such $S$, the minimum cardinality of an $F-G$ cover of $S$ equals the maximum cardinality of an $F-G$ packing of $S$. Thus $K_{1}$-perfect graphs are precisely $K_{i-1}-K_{i}$ perfect graphs. We develop a hypergraph characterization of $F-G$ perfect graphs that leads to an alternate proof of previous results on $K_{i}$-perfect graphs as well as to a characterization of $F-G$ perfect graphs for other instances of $F$ and $G$.


## 1. INTRODUCTION

The notion of $K_{i}$-perfectness was introduced in [2]. Given a fixed integer $i \geq 2$ and a graph $G$, we let $K_{i}(G)$ denote the set of $K_{i}$ s of $G$ (a $K_{i}$ is a complete graph of order $i$ ). For any $S \subseteq K_{i}(G)$, a $K_{i}$-cover of $S$ is a collection $T$ of $K_{i-1} \mathrm{~s}$ of $G$ such that each $K_{i}$ in $S$ contains at least one $K_{i-1}$ in $T$. A $K_{i}$-packing of $S$ is a collection of $K_{i}$ in $S$ that pairwise do not intersect in a $K_{i-1}$. Clearly, for any $S \subseteq K_{i}(G)$, the minimum cardinality of any $K_{i}$-cover of $S$ is greater than or
equal to the maximum cardinality of any $K_{i}$-packing of $S . G$ is defined to be $K_{i}$-perfect if equality holds for all $S \subseteq K_{i}(G)$. $K_{i}$-perfectness was studied in [2] where it was shown that a graph is $K_{i}$-perfect if and only if it is conformal (i.e., contains no $K_{i+1}$ ) and the $K_{i}$-intersection graph (on vertex set $K_{i}(G)$ with $X, Y$ an edge if and only if $X$ and $Y$ intersect in a $K_{i-1}$ ) has no odd hole (i.e., induced cycle) of length at least 5 .

In this note we extend the notion of $K_{i}$-perfectness to that of $F-G$ perfection where $F$ and $G$ are arbitrary fixed graphs. By studying a hypergraph based on $F$ and $G$ we develop a characterization of a large class of $F-G$ perfect graphs. This characterization subsumes the previously mentioned one for $K_{i}$-perfectness. Later in the note we provide further characterizations for specific $F, G$ combinations as well as comment on further extensions.
$P_{i}$ and $C_{i}$ denote respectively the path and cycle on $i$ vertices.

## 2. $F$ - $G$ PERFECTION

Throughout this section we assume that subgraph refers to partial subgraph unless explicitly stated otherwise. Let $F$ and $G$ be fixed graphs with the property that $G$ contains at least one subgraph isomorphic to $F$. For a graph $K$, we let $F(K)$ (respectively $G(K)$ ) denote all subgraphs of $K$ that are isomorphic to $F$ (respectively $G$ ). For a set $S \subseteq G(K)$ we define an $F-G$ covering of $S$ to be a collection $T \subseteq F(K)$ such that each subgraph in $S$ contains at least one element of $T$. Similarly, an $F-G$ packing of $S$ is a collection $S^{\prime} \subseteq S$ such that no two subgraphs in $S^{\prime}$ share a copy of $F$. For any $S \subseteq G(K)$ we denote by $c(F, G, S)$ and $p(F, G, S)$ the minimum cardinality of an $F-G$ covering of $S$ and the maximum cardinality of an $F-G$ packing of $S$, respectively. As an example, consider the graph $K$ in Figure 1 where $F$ is $P_{3}, G$ is $C_{4}$ and $S=\{\{b, c, f, e\},\{e, g, i, f\},\{d, f, h, g\}\}$. Then $c\left(P_{3}, C_{4}, S\right)=2$ as $\{\{e, f\},\{d, f\}\}$ is a minimum $P_{3}-C_{4}$ cover of $S$. Similarly $p\left(P_{3}, C_{4}, S\right)=2$ as established by $\{\{b, c, f, e\},\{d, f, h, g\}\}$.

Lemma 2.1. For any graph $K$ and set $S \subseteq G(K), c(F, G, S) \geq p(F, G, S)$.
Proof. If $S^{\prime} \subseteq S$ is an $F-G$ packing of $S$ and $T \subseteq F(K)$ is an $F-G$ cover of $S$, then each subgraph of $S^{\prime}$ contains a subgraph in $T$ and no subgraph in $T$ appears in more than one subgraph in $S^{\prime}$.

The relation $c(F, G, S) \geq p(F, G, S)$ and quantification over all subsets of $G(K)$ leads to our new notion of perfection. $K$ is $F-G$ perfect if for all $S \subseteq G(K), c(F, G, S)=p(F, G, S)$. Using this terminology, $K_{i}$-perfection is equivalent to $K_{i-1}-K_{i}$ perfection. The graph $K$ in Figure 1 is not $P_{3}-C_{4}$ perfect since for $S=\{\{b, c, f, e\},\{b, c, i, g\},\{e, f, i, g\}\}$ we have $c\left(P_{3}, C_{4}, S\right)=2$ whereas $p\left(P_{3}, C_{4}, S\right)=1$. It is easy to see that the graph in Figure 2 is $P_{3}-C_{4}$ perfect.


FIGURE 1. $K$.

We now provide a characterization of $F-G$ perfect graphs by studying the underlying hypergraph structure.

## 3. A HYPERGRAPH CHARACTERIZATION

Our hypergraph notation follows, in general, that of Berge [1]. One modification is our use of the term cycle; a cycle of length $n \geq 2$ is an alternating sequence of distinct vertices and hyperedges $x_{0}, E_{0}, x_{1}, E_{1}, \ldots, x_{n-1}, E_{n-1}$ such that (with $\bmod n$ arithmetic) $x_{i} \in E_{i} \cap E_{i-1}$ for $i=0, \ldots, n-1$ and $E_{i} \cap E_{j} \neq \phi$ if and only if $i-j=-1,0$, or 1 . The term linear refers to a hypergraph that has no cycles of length 2 (equivalently if and only if for all distinct hyperedges $E$ and $\left.E^{\prime},\left|E \cap E^{\prime}\right| \leq 1\right)$. The line graph of the hypergraph $H$ is the graph $L(H)$ whose vertices are the hyperedges of $H$ and whose edges are those pairs $\left\{E_{1}, E_{2}\right\}$ for which $E_{1} \cap E_{2} \neq \phi$. As in [1] a hypergraph $H^{\prime}$ is a partial hypergraph of $H$ if the hyperedges of $H^{\prime}$ are in $H$ and every vertex of $H^{\prime}$ is in at least one hyperedge (of $H^{\prime}$ ).

In his classic paper on perfection, Lovász [3] introduced the notion of normal hypergraph. A vertex cover $T$ of hypergraph $H$ is a collection of vertices such


FIGURE 2. $\mathrm{A} P_{3}-C_{4}$ perfect graph.
that every hyperedge contains at least one vertex in $T$; a matching is a collection of hyperedges of $H$ that are pairwise disjoint. Finally, $H$ is normal if and only if for every partial hypergraph $H^{\prime}$ of $H, \tau\left(H^{\prime}\right)$, the minimum size of a vertex cover of $H^{\prime}$ is equal to $\nu\left(H^{\prime}\right)$, the maximum size of a matching in $H^{\prime}$. We now relate normality to $F-G$ perfection of graph $K$ by associating to $K$ a hypergraph $H(F, G, K)$ on vertex set $F(K)$. For each $G^{\prime} \in G(K)$ construct a hyperedge of $H(F, G, K)$ on the vertices corresponding to copies of $F$ that are contained in $G^{\prime}$. The hypergraph $H\left(P_{3}, C_{4}, K\right)$ for the $K$ of Figure 1 is presented in Figure 3.

For $S \subseteq G(K)$ let $H_{s}(F, G, K)$ denote the partial hypergraph of $H(F, G, K)$ with $S$ the set of hyperedges of $H_{s}$. From the definitions we immediately see the one-to-one correspondence between the $F-G$ covers of $S(\subseteq G(K)$ ) and the vertex covers of $H_{s}(F, G, K)$. Similarly, we have a one-to-one correspondence between the $F-G$ packings of $S$ and the matchings of $H_{s}(F, G, K)$. Thus $c(F, G, S)=\tau\left(H_{s}(F, G, K)\right)$ and $p(F, G, S)=\nu\left(H_{s}(F, G, K)\right)$. Since the partial hypergraphs of $H(F, G, K)$ are in a one-to-one correspondence with the subsets $S \subseteq G(K)$, we arrive at the following essential observation.

Proposition 3.1. $K$ is $F-G$ perfect if and only if $H(F, G, K)$ is normal.
As pointed out in [2] this observation does not provide the characterization of $K_{i}$-perfect graphs. Examination of $H\left(K_{i-1}, K_{i}, K\right)$ shows that this hypergraph is always linear since two distinct $K_{i} \mathrm{~s}$ cannot overlap in more than one $K_{i-1}(i \geq 2)$. Linearity also holds for other pairs of $F$ and $G$, for example for $F=P_{i-1}, G=P_{i}$ and $F=P_{i-1}, G=C_{i}(i \geq 3)$. Therefore, we now concentrate on linear hypergraphs and develop necessary and sufficient conditions for them to be normal.

A hypergraph $H$ has the Helly property [1] if for every collection of hyperedges $E_{1}, \ldots, E_{n}$ of $H$ with pairwise nonempty intersection, $\bigcap_{i=1}^{n} E_{i}=\phi$. The next two lemmas follow from the definitions.

Lemma 3.2. A linear hypergraph has the Helly property if and only if it has no cycle of length three.


FIGURE 3. $H\left(P_{3}, C_{4}, K\right)$.

Lemma 3.3. A linear hypergraph $H$ has an odd cycle of length greater than or equal five if and only if $L(H)$ has an odd hole of length greater than or equal five.

In [3] Lovász proved the following:
Theorem 3.4. If hypergraph $H$ has the Helly property and $L(H)$ is perfect then $H$ is normal.

We can now characterize normal linear hypergraphs.
Theorem 3.5. A linear hypergraph $H$ is normal if and only if it does not contain a cycle of odd length.

Proof. If $H$ contains a cycle of odd length, then it is easy to see that it is not normal. Conversely suppose that $H$ has no odd cycle. By Lemma 3.2, $H$ has the Helly property. If $L(H)$ is perfect, Theorem 3.4 would imply that $H$ is normal. We now show that $L(H)$ does not contain a diamond (i.e., a $K_{4}$ with one edge removed). Suppose $e_{1}, e_{2}, e_{3}, e_{4}$ are distinct hyperedges of $H$ such that $e_{i} \cap e_{j}=\phi$ if and only if $i=2$ and $j=4$. Since $H$ has no cycle of length 3 , let $e_{1} \cap e_{2} \cap e_{3}=\{x\}$. Since $x \notin e_{4}, e_{4} \cap e_{3} \neq \phi$ and $e_{1} \cap e_{3}=\{x\}, H$ must contain a cycle of length 3 on the hyperedges $e_{1}, e_{3}$, and $e_{4}$, a contradiction (see Figure 4).

A well known result due to Tucker [4] states that a graph that does not contain an induced diamond is perfect if and only if it does not contain an induced odd cycle of length greater than or equal five. Lemma 3.3 implies that $L(H)$ is perfect and hence we conclude that $H$ is normal.

Combining Proposition 3.1 and Theorem 3.5 we obtain the following characterization of $F-G$ perfection for those graphs $K$ where $H(F, G, K)$ is linear.


FIGURE 4. Structure on $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$.

Theorem 3.6. If $K$ is a graph where no two subgraphs isomorphic to $G$ overlap in more than one copy of $F$, then $K$ is $F-G$ perfect if and only if $H(F, G, K)$ contains no odd cycles.

In particular for $F=K_{i-1}$ and $G=K_{i}(i \geq 2)$ the above theorem implies that $K$ is $K_{i}$-perfect if and only if $H\left(K_{i-1}, K_{i}, K\right)$ does not contain an odd cycle. The conformality of $K$ (i.e., $K$ does not contain a $K_{i+1}$ ) corresponds to $H\left(K_{i+1}, K_{i}, K\right)$ having the Helly property (equivalently no cycle of length three), so the characterization of $K_{i}$-perfection given in [2] follows immediately from Theorem 3.6. We now use Theorem 3.6 to study other particular examples of $F-G$ perfection.

## 4. PARTICULAR EXAMPLES OF F - G PERFECTION

Our first example is $K_{1, i-1}-K_{1, i}$ perfection ( $K_{1, i}$ denotes a star of degree $i$ ). Since any two $K_{1, i}$ s can overlap in at most one $K_{1, i-1}$, Theorem 3.6 applies; a further characterization is given below ( $\Delta(K)$ will denote the maximum degree of a vertex of the graph $K$ ).

Theorem 4.1. $K$ is $K_{1, i-1}-K_{1, i}$ perfect if and only if
(i) $i=1$ and $K$ is bipartite,
(ii) $i=2$ and $K$ is the disjoint union of paths and even cycles,
(iii) $i \geq 3$ and $\Delta(K) \leq i$.

Proof. For $i=1$, the $K_{1,0}-K_{1,1}$ perfect graphs are precisely the $K_{2}$-perfect graphs studied in [2]. There it was shown that a graph is $K_{2}$-perfect if and only if it is bipartite. This is also seen immediately from Theorem 3.6 since $H\left(K_{1,0}, K_{1,1}, K\right) \cong K$. For $i=2$, we see that $K$ is $K_{1,1}-K_{1,2}$ perfect if and only if $H\left(K_{1,1}, K_{1.2}, K\right)(\cong L(K))$ is bipartite, and hence if and only if $K$ is the disjoint union of paths and even cycles.

For $i \geq 3$ first assume that $\Delta(K) \leq i$. Now no two $K_{1, i} s$ of $K$ intersect in a $K_{1, i-1}$ so $H\left(K_{1, i-1}, K_{1, i}, K\right)$ is a matching and hence is normal. Thus $K$ is $K_{1, i-1}-K_{1, i}$ perfect. Conversely, if $\Delta(K)>i$, then $K$ contains a $K_{1, i+1}$ as a subgraph which implies that $H\left(K_{1, i-1}, K_{1, i}, K\right)$ contains a cycle of length 3 (note $i+1 \geq 3$ ) so it is not normal. Then $K$ is not $K_{\mathrm{t}, i-1}-K_{\mathrm{t}, i}$ perfect.

Our next example is for $F=P_{i-1}$ and $G=P_{i}(i \geq 2)$. Since any $P_{i}$ contains exactly two $P_{i-1} \mathrm{~s}, H\left(P_{i-1}, P_{i}, K\right)$ is itself a graph. From Theorem 3.6 we see that a graph $K$ is $P_{i-1}-P_{i}$ perfect if and only if $H\left(P_{i-1}, P_{i}, K\right)$ is bipartite. It is natural to ask whether we can characterize $P_{i-1}-P_{i}$ perfect graphs in terms of forbidden subgraphs. Such a characterization must exist for any $F-G$ perfection since any subgraph of a $F-G$ perfect graph is $F-G$ perfect and hence there is a unique set of minimal $F-G$ imperfect graphs. From Theorem 4.1 we see that the set of forbidden subgraphs for $K_{1, i-1}-K_{1, i}$ perfection is $\left\{C_{2 n+1}\right.$ :
$n \geq 1\}$ if $i=1,\left\{C_{2 n+1}: n \geq 1\right\} \cup\left\{K_{1,3}\right\}$ if $i=2$ and $\left\{K_{1, i+1}\right\}$ if $i \geq 3$. For $P_{i-1}-P_{i}$ perfection we state the following conjecture:

Conjecture 4.2. For all $i \neq 3, K$ is $P_{i-1}-P_{i}$ perfect if and only if $K$ does not contain an odd cycle of length $\geq i$.

It is clear that any graph $K$ that contains an odd cycle of length $\geq i$ is not $P_{i-1}-P_{i}$ perfect since $H\left(P_{i-1}, P_{i}, K\right)$ contains an odd cycle of the same length. Since $H\left(P_{1}, P_{2}, K\right) \cong K$, the conjecture holds for $i=2$; however, it fails for $i=3$ by our characterization of $K_{1,1}-K_{1,2}$ perfect graphs (for example $K_{1,3}$ is cycle-free but not $P_{2}-P_{3}$ perfect). We remark that we can show for any even $i \geq 2$, every bipartite graph is $P_{i-1}-P_{i}$ perfect.

Finally, we examine $P_{3}-P_{4}$ perfection. In Sections 2 and 3 we saw that the graph in Figure 1 is not $P_{3}-C_{4}$ perfect because of the 3-cycle in $H_{s}\left(P_{3}, C_{4}, K\right)$ where $S=\{\{b c, c f, f e, b e\},\{e f, f i, i g, e g\},\{i g, g b, b c, c i\}\}$. This example leads to the following characterization of $P_{3}-C_{4}$ perfect graphs.

Theorem 4.3. $K$ is a $P_{3}-C_{4}$ perfect graph if and only if $K_{2,3}$ is not a partial subgraph of $K$.

Proof. From Theorem 3.6 we need only be concerned with odd cycles in $H\left(P_{3}, C_{4}, K\right)$. If $K_{2,3}$ is a partial subgraph of $K$ we immediately have a 3-cycle in $H\left(P_{3}, C_{4}, K\right)$. Now assume that $K$ is not $P_{3}-C_{4}$ perfect and thus that there is an odd cycle in $H\left(P_{3}, C_{4}, K\right)$. Since there is such a cycle we know that there exists pairs of $C_{4}$ s in $K$ that overlap in a $P_{3}$. Since no two $C_{4} s$ as partial subgraphs of $K_{4}$ can share a $P_{3}$ we have the situation depicted in Figure 5, where cycles $\{a, b, c, d\}$ and $\{a, b, c, e\}$ share the $P_{3}\{a, b, c\}$. This, however, forces the third cycle $\{a, e, c, d\}$ and in fact there exists a 3-cycle in $H\left(P_{3}, C_{4}, K\right)$.

We also see that $K$ is $P_{3}-C_{4}$ perfect if and only if no two $C_{4} \mathrm{~s}$ in $K$ overlap in a $P_{3}$ (i.e., $H\left(P_{3}, C_{4}, K\right)$ is a matching).


FIGURE 5. $K_{2.3}$.

## 5. CONCLUDING REMARKS

In the previous three sections we have assumed that $F$ is contained as a partial subgraph in $G$. Alternatively, we could replace partial containment with induced containment. An examination of the development in Sections 2 and 3 shows that very similar results hold for the induced form of $F-G$ perfection.

Finally, we mention one further extension to $F-G$ perfection. Let $\mathscr{F}$ and $\mathscr{G}$ be families of graphs with the property that every $G \in \mathscr{G}$ contains as a subgraph (either defined to be partial or induced) some $F \in \mathscr{F}$. For a graph $K$ we define $\mathscr{F}(K)$ to be all subgraphs of $K$ isomorphic to some $F \in \mathscr{F}$ and define $\mathscr{G}(K)$ analogously. For a collection $S \subseteq \mathscr{G}(K)$ an $\mathscr{F}-\mathscr{G}$ cover of $S$ is a collection $T \subseteq \mathscr{F}(K)$ such that every subgraph in $S$ contains as a subgraph a member of $T$; an $\mathscr{F}-\mathscr{G}$ packing of $S$ is a subcollection $S^{\prime} \subseteq S$ such that no two members of $S^{\prime}$ both contain a subgraph $F^{\prime} \in \mathscr{F}(K) . K$ is said to be $\mathscr{F}-\mathscr{G}$ perfect if for all $S \subseteq \mathscr{G}(K)$, the minimum cardinality of an $\mathscr{F}-\mathscr{G}$ cover of $S$ equals the maximum cardinality of an $\mathscr{F}-\mathscr{G}$ packing of $S$. Results similar to those developed in Sections 2 and 3 may be obtained for $\mathscr{F}-\mathscr{G}$ perfection.

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