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FACETS OF THE BIPARTITE SUBGRAPH POLYTOPE*

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The bipartite subgraph polytope $P_B(G)$ of a graph $G = [V, E]$ is the convex hull of the incidence vectors of all edge sets of bipartite subgraphs of G. We show that all complete subgraphs of G of odd order and all so-called odd bicycle wheels contained in G induce facets of $P_B(G)$. Moreover, we describe several methods with which new facet defining inequalities of $P_B(G)$ can be constructed from known ones. Examples of these methods are contraction of node sets in odd complete subgraphs, odd subdivision of edges, certain splittings of nodes, and subdivision of all edges of a cut. Using these methods we can construct facet defining inequalities of $P_B(G)$ having coefficients of order $|V|^2$.

1. Introduction and notation. In this paper we study the polytope associated with the bipartite subgraphs of a graph. We shall exhibit various new classes of facets of this polytope; in particular, we describe several methods with which new classes of facets can be derived from known ones. Using these constructions we obtain facet defining inequalities with integral coefficients in the interval $[0, n^2/4]$ where n is the order of the given graph.

Results about the bipartite subgraph polytope have been used by Grötschel and Pulleyblank (1981) for the design of polynomial time algorithms for special cases of the max-cut problem. These ideas have been successfully implemented by Barahona and Maccioni (1982) for the solution of medium sized real-world problems of this type. We hope that our polyhedral results may lead to the design of new efficient LP-based cutting plane algorithms for the max-cut problem.

The graphs we consider are finite and undirected. Loops and multiple edges do not play a role in what follows, so we assume that our graphs have no multiple edges and no loops. We denote a graph by $G = [V, E]$, where V is the *node set* and E the *edge set* of G. The cardinality of V is called the *order* of G. If $e \in E$ is an edge with endnodes i and j we also write ij to denote the edge e. If $H = [W, F]$ is a graph with $W \subseteq V$ and $F \subseteq E$ then H is called a *subgraph* of G. We also say that H is contained in G and that G contains H.

If $G = [V, E]$ is a graph and $F \subseteq E$, then $V(F)$ denotes the set of nodes of V which occur at least once as an endnode of an edge in F. Similarly, for $W \subseteq V$, $E(W)$ denotes the set of all edges of G with both endnodes in W.

A graph G is called *complete* if every two different nodes of G are linked by an edge. The complete graph with n nodes is denoted by K_n. A graph is called *bipartite* if its node set can be partitioned into two nonempty, disjoint sets V_1 and V_2 such that no two nodes in V_1 and no two nodes in V_2 are linked by an edge. We call V_1, V_2 a *bipartition* of V.

If $W \subseteq V$ then $\delta(W)$ is the set of edges with one endnode in W and the other in $V \setminus W$. The edge set $\delta(W)$ is called a *cut*. Clearly, $[V, \delta(W)]$ is a bipartite subgraph of G. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$ and call $\delta(v)$ the *star* of v. For a node $v \in V$ the cardinality of $\delta(v)$ is called the *degree* of v. For $W \subseteq V$ the set of nodes in $V \setminus W$ which are adjacent to a node in W is denoted by $\Gamma(W)$.

If U, W are disjoint subsets of V, then $[U : W]$ denotes the set of edges of G which have one endnode in U and the other endnode in W, for instance $[\{v\} : V \setminus \{v\}] = \delta(v)$.

A *path* P in $G = [V, E]$ is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1 = v_0v_1$, $e_2 = v_1v_2, \ldots, e_k = v_{k-1}v_k$ and such that $v_i \neq v_j$ for $i \neq j$. The nodes v_0 and v_k are the endnodes of P and we say that P links v_0 and v_k or goes from v_0 to v_k. The number k of edges of P is called the *length* of P. If $P = e_1, e_2, \ldots, e_k$ is a path linking v_0 and v_k and $e_{k+1} = v_0v_k \in E$, then the sequence $e_1, e_2, \ldots, e_k, e_{k+1}$ is called a *cycle* of length $k + 1$. A cycle (path) is called *odd* if its length is odd, otherwise it is called *even*. If P is a cycle or a path and uv an edge of $E \setminus P$ with $u, v \in V(P)$, then uv is called a *chord* of P.

If v is a node of a graph G, then $G - v$ denotes the subgraph of G obtained by removing node v and all edges incident to v from G.

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces in $\mathbb{R}^m$. A *polytope* is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, denoted by $\dim P$, is the maximum number of affinely independent points in P minus one.

If $a \in \mathbb{R}^m \setminus \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^Tx \leqslant a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^Tx \leqslant a_0\}$. We say that a valid inquality $a^Tx \leqslant a_0$ *supports* P or defines a *face* of P if $\emptyset \neq P \cap \{x \mid a^Tx = a_0\} \neq P$. A valid inquality $a^Tx \leqslant a_0$ defines a *facet* of P if it defines a face of P and if there exist $\dim P$ affinely independent points in $P \cap \{x \mid a^Tx = a_0\}$. Two face-defining inequalities $a^Tx \leqslant a_0$, $b^Tx \leqslant b_0$ are called *equivalent* if $P \cap \{x \mid a^Tx = a_0\} = P \cap \{x \mid b^Tx = b_0\}$.

If $P = \{x \mid Ax \leqslant b\}$, then $Ax \leqslant b$ is called *nonredundant* if $P \neq \{x \mid A'x \leqslant b'\}$ for all inequality systems $A'x \leqslant b'$ where $A'x \leqslant b'$ arises from $Ax \leqslant b$ by deleting one inequality.

For every facet F of a *full-dimensional* polyhedron $P \subseteq \mathbb{R}^m$, i.e. $\dim P = m$, there exists a unique (up to multiplication by a positive constant) valid inequality $a^Tx \leqslant a_0$ such that $F = \{x \in P \mid a^Tx = a_0\}$. This implies that (up to multiplication by a positive constant) there is a unique complete and nonredundant inequality system $Ax \leqslant b$ such that $P = \{x \in \mathbb{R}^m \mid Ax \leqslant b\}$. We shall present a quite large, but partial, nonredundant system of inequalities for the polyhedron associated with the bipartite subgraphs of a graph.

If x is a real number, then $\lceil x \rceil$ resp. $\lfloor x \rfloor$ denotes the smallest resp. largest integer not smaller resp. larger than x.

2. The bipartite subgraph polytope and the max-cut problem. Given a graph $G = [V, E]$ with edge weights $c_e > 0$ for all $e \in E$, then the *max-cut problem* can be stated as follows: Find a cut $\delta(W)$ in G such that $c(\delta(W)) := \sum_{e \in \delta(W)} c_e$ is as large as possible. This problem can be approached from a polyhedral point of view in the following way.

If $G = [V, E]$ is a graph and $F \subseteq E$ an edge set then the 0/1-vector $x^F \in \mathbb{R}^E$ with $x_e^F = 1$ if $e \in F$, and $x_e^F = 0$ if $e \notin F$ is called the *incidence vector* of F. The convex hull $P_B(G)$ of the incidence vectors of all edge sets of bipartite subgraphs of G is called the *bipartite subgraph polytope* of G, i.e.

$$P_B(G) = \operatorname{conv}\{x^F \in \mathbb{R}^E \mid [V, F] \text{ is a bipartite subgraph of } G\}. \qquad (2.1)$$

Obviously, every edge set of a bipartite subgraph of G is contained in a cut of G. This

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implies that for positive edge weights c_e, $e \in E$, every optimum basic solution of the linear program

$$\max c^T x, \qquad x \in P_B(G), \tag{2.2}$$

is the incidence vector of a cut of G. Hence, whenever the linear program (2.2) can be solved in polynomial time, the max-cut problem can be solved in polynomial time (this implication also holds the other way around).

Since the max-cut problem is NP-complete we cannot expect to find a complete explicit characterization of $P_B(G)$ for all graphs G. It may however be that for certain classes of graphs G, $P_B(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities polynomial time separation algorithms can be designed, so that the max-cut problem for these graphs is solvable in polynomial time. Examples of such classes of graphs are planar graphs, cf. Barahona (1980), and weakly bipartite graphs, cf. Grötschel and Pulleyblank (1981).

It is trivial to see that $P_B(G)$ is full-dimensional, which implies that $P_B(G)$ has a unique (up to positive scaling) nonredundant defining linear inequality system. Barahona (1980) showed that the following inequalities

$$0 \leqslant x_e \leqslant 1 \qquad \text{for all} \quad e \in E, \tag{2.3}$$

$$x(C) \leqslant |C| - 1 \qquad \text{for all odd cycles } C \text{ in } G, \tag{2.4}$$

define facets of $P_B(G)$. (We call the inequalities (2.3) *trivial* and the inequalities (2.4) *odd cycle constraints*.) Using matching techniques Grötschel and Pulleyblank (1981) devised a polynomial time separation algorithm for the system (2.3), (2.4) (i.e. an algorithm which decides in polynomial time whether a given vector $y \in \mathbf{Q}^E$ satisfies (2.3) and (2.4), and if not, finds a violated inequality). Thus, the ellipsoid method implies that there is a polynomial time algorithm for the solution of (2.2) whenever $P_B(G)$ is completely determined by the trivial inequalities (2.3) and the odd cycle constraints (2.4). Such graphs are called *weakly bipartite*. It is for instance known that all planar graphs are weakly bipartite, cf. Barahona (1980). In the next two paragraphs we shall introduce various new classes of inequalities defining facets of $P_B(G)$.

To simplify technical details in subsequent proofs we now state a lemma which we shall use frequently in the following sections.

LEMMA 2.5. *Let $b^Tx \leqslant \beta$ be a valid inequality with respect to $P_B(G)$. Let Y, Y', Z and Z' be mutually disjoint subsets of V and let $v \in V \setminus (Y \cup Y' \cup Z \cup Z')$. Suppose that the incidence vectors of the following edge sets of bipartite subgraphs*

$$B_1 := [Y' \cup Y \cup \{v\} : Z' \cup Z], \qquad B_2 := [Y' \cup Y : Z' \cup Z \cup \{v\}],$$

$$B_3 := [Y' \cup Z \cup \{v\} : Z' \cup Y], \qquad B_4 := [Y' \cup Z : Z' \cup Y \cup \{v\}],$$

satisfy $b^Tx \leqslant \beta$ with equality. Then $\sum_{z \in Z} b_{zv} = \sum_{y \in Y} b_{yv}$.

PROOF. Clearly,

$$0 = \beta - \beta = b^Tx^{B_1} - b^Tx^{B_3} = \sum_{z \in Z}\Big(b_{zv} + \sum_{y' \in Y'} b_{y'z}\Big) + \sum_{y \in Y}\sum_{z' \in Z'} b_{yz'}$$

$$- \sum_{z \in Z}\sum_{z' \in Z'} b_{zz'} - \sum_{y \in Y}\Big(b_{yv} + \sum_{y' \in Y'} b_{yy'}\Big).$$

$$0 = \beta - \beta = b^Tx^{B_2} - b^Tx^{B_4} = \sum_{z \in Z}\sum_{y' \in Y'} b_{y'z} + \sum_{y \in Y}\Big(\sum_{z' \in Z'} b_{yz'} + b_{yv}\Big)$$

$$- \sum_{z \in Z}\Big(\sum_{z' \in Z'} b_{zz'} + b_{zv}\Big) - \sum_{y \in Y}\sum_{y' \in Y'} b_{yy'}.$$

Thus, subtracting the two equations from each other we obtain $0 = 2\sum_{z \in Z} b_{zv} - 2\sum_{y \in Y} b_{yv}$ which proves our claim. ∎

3. **Further rank facets of $P_B(G)$.** For a graph $G = [V, E]$ let

$$\mathscr{B}(G) := \{B \subseteq E \mid [V, B] \text{ is bipartite}\} \tag{3.1}$$

denote the set of edge sets of bipartite subgraphs of G. Clearly, $\mathscr{B}(G)$ is an independence system on E, i.e. $\mathscr{B}(G)$ satisfies $\emptyset \in \mathscr{B}(G)$ and $B' \subseteq B \in \mathscr{B}(G) \Rightarrow B' \in \mathscr{B}(G)$. For every set $F \subseteq E$ the number $r(F) := \max\{|B| : B \in \mathscr{B}(G),\ B \subseteq F\}$ is called the *rank* of F (with respect to $\mathscr{B}(G)$). If $F \subseteq E$ then the 0/1-inequality

$$x(F) = \sum_{e \in F} x_e \leqslant r(F) \tag{3.2}$$

is by definition a supporting inequality of $P_B(G)$. The inequalities of type (3.2) are called *rank inequalities*, e.g. the odd cycle constraints (2.4) are rank inequalities.

Obviously, every facet of $P_B(G)$ defined by a 0/1-inequality is a rank inequality, and it is a very interesting open problem to characterize those rank inequalities of $P_B(G)$ that define facets of $P_B(G)$.

In this section we introduce two new "basic" classes of facet defining rank inequalities, further such inequalities will be derived in the next section.

THEOREM 3.3. *Let $G = [V, E]$ be a graph and let $[W, E(W)]$ be a complete subgraph of order $p \geqslant 3$ of G. Then*

$$x(E(W)) \leqslant \lceil p/2 \rceil \lfloor p/2 \rfloor \tag{3.4}$$

is a valid inequality with respect to $P_B(G)$. (3.4) defines a facet of $P_B(G)$ if and only if p is odd.

PROOF. First of all, it is trivial to see that the right-hand side of (3.4) gives the rank of a complete subgraph with respect to $\mathscr{B}(G)$.

We now show that for p even, inequality (3.4) is the sum of p (positively scaled) inequalities (3.4) for which $|W|$ is odd. Namely, if $[W, E(W)]$ is a complete subgraph of G with $p = |W|$ even, then

$$x(E(W)) = \frac{1}{p-2}\sum_{v \in W} x(E(W \setminus \{v\})) \leqslant \frac{1}{p-2}\,p\left\lceil \frac{p-1}{2} \right\rceil \left\lfloor \frac{p-1}{2} \right\rfloor = \frac{p^2}{4} = \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor.$$

This implies that (3.4) does not define a facet of $P_B(G)$ in case p is even.

Now let us assume that $[W, E(W)]$ is a complete subgraph of G with $|W| = 2k+1$, $k \geqslant 1$, and let us denote the corresponding inequality $x(E(W)) \leqslant k(k+1)$ by $a^Tx \leqslant \alpha$. Assume, moreover, that $b^Tx \leqslant \beta$ is a facet defining inequality for $P_B(G)$ which has the following property. If a vertex $x \in P_B(G)$ satisfies $a^Tx = \alpha$ then x also satisfies $b^Tx = \beta$. If we can prove that $b = \rho a$ for some $\rho > 0$ then we can conclude that $a^Tx \leqslant \alpha$ is equivalent to $b^Tx \leqslant \beta$, i.e. defines a facet of $P_B(G)$, and we are done.

To prove the claim above we first show that for any three different nodes $r, s, t \in W$ we have $b_{rt} = b_{st}$. To prove this, we take two disjoint node sets $U, U' \subseteq W \setminus \{r, s, t\}$ of cardinality $k-1$, and apply Lemma (2.5) with $Y' = U$, $Z' = U'$, $Y = \{r\}$, $Z = \{s\}$, and $v = t$.

Since r, s, t were arbitrarily chosen and since $[W, E(W)]$ is connected we can conclude that for some $\rho \in \mathbb{R}$ $b_{ij} = \rho$ for all $ij \in E(W)$. It is a trivial matter to prove that $b_{ij} = 0$ for all $ij \in E \setminus E(W)$. Moreover, it is easy to see that for every edge e in G there is a set $B \in \mathscr{B}(G)$ with $e \in B$ and $a^Tx^B = \alpha$. This implies that the face defined by $a^Tx \leqslant \alpha$ is not contained in the trivial facet $\{x \in P_B(G) \mid x_e = 0\}$ for some $e \in E$. Hence $b^Tx \leqslant \beta$ defines a nontrivial facet of $P_B(G)$ which by the monotonicity (i.e.

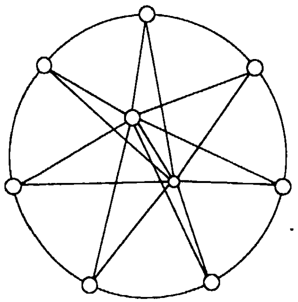

FIGURE 3.1

$0 \leq x \leq y \in P_B(G)$ implies $x \in P_B(G)$) of $P_B(G)$ implies that all coefficients of b are nonnegative and at least one is positive. Hence we can conclude that $\rho > 0$ and, thus, $b = \rho a$ holds which proves our claim. ∎

In the following we shall refer to the facet defining inequalities introduced in (3.3) as *odd complete subgraph inequalities* or as the *K_{2k+1} – inequalities*.

Note that Theorem (3.3) can be used to prove the following

REMARK. For every graph $G = [V, E]$ the cut polytope $\operatorname{conv}\{x^{\delta(W)} \in \mathbb{R}^E \mid W \subseteq V\}$ has full dimension $|E|$.

Namely, we embed G into a complete graph K_{2k+1}. Then we know that the K_{2k+1}-inequality defines a facet of $P_B(K_{2k+1})$. We therefore can make up a nonsingular $(\binom{2k+1}{2}, \binom{2k+1}{2})$-matrix M whose rows are incidence vectors of bipartite edge sets in K_{2k+1} satisfying the K_{2k+1}-inequality with equality. Note that each of these bipartite edge sets necessarily is a cut of K_{2k+1}. Now we remove all columns from M corresponding to edges of K_{2k+1} which are not in G. The resulting matrix M' has full rank $|E|$ and the rows of M' correspond to cuts of G. Thus, there are $|E|$ cuts of G whose incidence vectors are linearly independent. Adding the zero vector (which is the incidence vector of $\delta(V)$) we obtain a set of $|E| + 1$ affinely independent points in the cut polytope of G.

A graph $G = [V, E]$ is called a *bicycle p-wheel* if G consists of a cycle of length p and two (so-called *universal*) nodes which are adjacent to each other and to every node in the cycle. A bicycle p-wheel is called *odd* if p is odd. Figure 3.1 shows a bicycle 7-wheel.

THEOREM 3.5. *Let $G = [V, E]$ be a graph and let $[W, F]$ be a bicycle $(2k + 1)$-wheel, $k \geq 1$, contained in G. Then the inequality*

$$x(F) \leq 2(2k + 1) \tag{3.6}$$

defines a facet of $P_B(G)$.

PROOF. Let v, w be the two universal nodes of the bicycle $(2k + 1)$-wheel $[W, F]$ in G, and let the cycle C of $[W, F]$ be formed by the edges $u_1u_2, u_2u_3, u_3u_4, \ldots, u_{2k}u_{2k+1}, u_{2k+1}u_1$. $[W, F]$ contains the following triangles

$$T_i := \{vw, vu_i, wu_i\}, \qquad i = 1, \ldots, 2k + 1,$$

$$T_i^v := \{vu_i, vu_{i+1}, u_iu_{i+1}\}, \qquad i = 1, \ldots, 2k, \quad T_{2k+1}^v := \{vu_1, vu_{2k+1}, u_1u_{2k+1}\},$$

$$T_i^w := \{wu_i, wu_{i+1}, u_iu_{i+1}\}, \qquad i = 1, \ldots, 2k, \quad T_{2k+1}^w := \{wu_1, wu_{2k+1}, u_1u_{2k+1}\}.$$

By summing up appropriately we get

$$\begin{aligned}(2k + 1)x(F) &= \sum_{i=1}^{2k+1} (x(T_i) + kx(T_i^v) + kx(T_i^w)) + x(C)\\ &\leq (2k + 1)(2 + 2k + 2k) + 2k\\ &= (2k + 1)2(2k + 1) + 2k.\end{aligned}$$

Dividing through $2k + 1$ we obtain $x(F) \leq 2(2k + 1) + 2k/(2k + 1)$. Since for every vertex of $P_B(G)$ the left-hand side of the above inequality is an integer which, by the validity of the odd cycle inequalities, does not exceed the right-hand side, we can round the right-hand side down to the next integer and obtain that (3.6) is valid for $P_B(G)$.

Let us denote our inequality $x(F) \leq 2(2k + 1)$ by $a^Tx \leq \alpha$ and let us assume (as in the proof before) that $b^Tx \leq \beta$ is a facet defining inequality such that $\{x \in P_B(G) \mid a^Tx = \alpha\} \subseteq \{x \in P_B(G) \mid b^Tx = \beta\}$. We are done if we can prove that there is $\rho > 0$ such that $b = \rho a$.

First we show that for every node u_i, $1 \leq i \leq 2k + 1$, $b_{vu_i} = b_{vw}$ holds. We simply pick an arbitrary node of the cycle C, say u_1, and apply Lemma (2.5) with

$$Y' := \{u_3, u_5, \ldots, u_{2k+1}\}, \qquad Z' := \{u_2, u_4, \ldots, u_{2k}\},$$

$$Y := \{u_1\}, \qquad Z := \{w\} \quad \text{and} \quad v := v.$$

By symmetry we obtain $b_{u_iw} = b_{vw}$ for $i = 1, \ldots, 2k + 1$. Thus, if we set $\rho := b_{vw}$ we can conclude that $b_{vu_i} = b_{wu_i} = \rho$, $i = 1, \ldots, 2k + 1$.

In the next step we show that the b-values of the cycle edges are equal. Set $B_1 := [Y' \cup Z : Z' \cup \{u_1, v\}]$, $B_2 := [Y' \cup \{w, u_1\} : Z' \cup \{v\}]$ then again $B_j \in \mathscr{B}(G)$ and $a^Tx^{B_j} = \alpha$, $j = 1, 2$. So

$$0 = b^Tx^{B_2} - b^Tx^{B_1} = b_{u_1u_2} + b_{u_1v} - b_{u_1u_{2k+1}} - b_{u_1w} = b_{u_1u_2} - b_{u_1u_{2k+1}}$$

which implies $b_{u_1u_{2k+1}} = b_{u_iu_{i+1}} = \rho'$, $i = 1, \ldots, 2k$ for some $\rho' \in \mathbb{R}$.

To prove that $\rho = \rho'$ consider the sets

$$B_3 := [Y' \cup \{u_1, u_2\} : (Z' \setminus \{u_2\}) \cup \{v, w\}],$$

$$B_4 := [Y' \cup \{u_2\} : (Z' \setminus \{u_2\}) \cup \{v, w, u_1\}]$$

which are in $\mathscr{B}(G)$ and for which $a^Tx^{B_3} = a^Tx^{B_4} = \alpha$. We get $0 = b^Tx^{B_4} - b^Tx^{B_3} = b_{u_1u_2} + b_{u_1u_{2k+1}} - b_{u_1v} - b_{u_1w} = 2(\rho' - \rho)$. Hence we have shown $b_{ij} = \rho$ for all $ij \in F$.

For every pair of nodes u_i, u_j which are not neighbours on the cycle C, it is easy to construct a bipartite subgraph $[W, B]$ of $[W, F]$ with bipartition W_1, W_2 such that $u_i \in W_1$, $u_j \in W_2$ and such that $a^Tx^B = \alpha$. If $u_iu_j \in E$, then of course $B' = B \cup \{u_iu_j\} \in \mathscr{B}(G)$ and $a^Tx^{B'} = \alpha$. Hence $0 = b^Tx^{B'} - b^Tx^B = b_{u_iu_j}$.

If ij is an edge of E with one endnode not in W, then it is trivial to construct $B \in \mathscr{B}(G)$ such that $B \cup \{ij\} \in \mathscr{B}(G)$ and $a^Tx^B = \alpha$, which implies $b_{ij} = 0$. Altogether we have now shown

$$b_{ij} = \rho \quad \text{for all} \quad ij \in F,$$

$$b_{ij} = 0 \quad \text{for all} \quad ij \in E \setminus F.$$

As in the proof of (3.3) it follows that $\rho > 0$ and thus that $b = \rho a$ which finishes the proof of (3.5). ∎

In the following we shall call the inequalities (3.6) *odd bicycle wheel inequalities*. Of course, we could have considered even bicycle wheel inequalities as well, but as in (3.3) these "even inequalities" do not define facets of $P_B(G)$.

Note that the bicycle 3-inequalities (3.6) are the same as the K_5-inequalities (3.4). But this is the only overlapping of these two classes of inequalities.

Let us denote the polytope defined by the trivial inequalities (2.3) and the odd cycle inequalities (2.4) by $P_C(G)$. Barahona (1980) has shown that the max-cut problem for the class of graphs G which have the following property "there exists a node v such that the $G - v$ is planar" is NP-complete. Since linear programs over $P_C(G)$ are solvable in polynomial time, we can conclude (unless $P = NP$) that $P_C(G)$ does not equal $P_B(G)$. This observation lead us to the discovery of the odd bicycle wheel inequalities. Namely, if one of the universal nodes of a bicycle wheel is removed then the remaining graph is planar.

In fact, Barahona (1983) has shown that the max-cut problem is NP-complete for the following class of graphs G: "G contains a node v such that $G - v$ is planar and all nodes in $G - v$ have degree at most three". The odd bicycle wheels (except $p = 3$) do not belong to this class. We shall, however, give a construction (node-splitting) with which facets of $P_B(G)$ for graphs G of this type can sometimes be constructed from odd bicycle wheel inequalities. These inequalities however are not rank inequalities, since they have coefficients larger than one (see §6 for an example).

It is obvious that $P_B(G) = \operatorname{conv}\{x \in P_C(G) \mid x \text{ integer}\}$, so one may ask what is the Chvátal rank, cf. Chvátal (1973), for a class of facets of $P_B(G)$ with respect to the trivial inequalities (2.3) and the odd cycle constraints (2.4). The validity proof for the odd bicycle wheel inequalities given in (3.5) shows that these inequalities are of Chvátal rank one. This is, however, not the case for the K_{2k+1}-inequalities. For instance, one can easily show that the K_7-inequalities are of Chvátal rank 2, and we conjecture the Chvátal rank (with respect to $P_C(G)$) of K_{2k+1}-inequalities grows linearly with k.

4. Contraction and splitting of nodes. In this section we shall describe three methods for constructing "facets from facets". The first one called "node set contraction" can only be applied to the odd complete subgraph inequalities introduced in §3, while the other two, named "odd subdivision of edges" and "node splitting", are universal and can be applied to any facet defining inequality. The latter two operations can also be reversed.

THEOREM 4.1 (Node Set Contraction). *Let $[W, E(W)]$ be a complete subgraph of order $2k+1$, $k \geq 3$, of a graph $G = [V, E]$ and let $W_1, W_2, \ldots, W_s$ be a partition of W into nonempty mutually disjoint subsets such that $\sum_{i=1, |W_i|>1}^{s} |W_i| \leq k - 1$.*

Let $G' = [V', E']$ denote the graph obtained from G by shrinking the nodes in W_i into a single node w_i, $i = 1, \ldots, s$ (i.e. all new nodes w_i are adjacent in G', and a new node w_i is adjacent in G' to an old node $v \in V \setminus W$ iff W_i contains a node adjacent to v in G). Set

$$a_{w_i w_j} := |W_i|\,|W_j| \qquad \text{for all} \quad 1 \leq i < j \leq s,$$

$$a_{vw} := 0 \qquad \text{for all other} \quad vw \in E',$$

then

$$a^T x \leq \alpha := k(k+1) \tag{4.2}$$

defines a facet of $P_B(G')$.

PROOF. Denoting by $\bar{a}^T x := x(E(W)) \leq k(k+1) =: \alpha$ the odd complete subgraph inequality of $[W, E(W)]$ with respect to $P_B(G)$ we immediately see that a nonzero coefficient $a_{w_i w_j}$ of the new inequality (4.2) is equal to $\sum_{v \in W_i} \sum_{w \in W_j} \bar{a}_{vw}$. If we have a bipartition V_1', V_2' of V' we can expand the nodes of V' into nodes of V to obtain the corresponding bipartition V_1, V_2 of V. Then it immediately follows that for the edge

sets $B' = [V_1' : V_2']$ and $B = [V_1 : V_2]$ we have $\bar{a}^T x^B = a^T x^{B'}$ which implies that $a^T x \leq \alpha$ is valid with respect to $P_B(G')$.

Assume as in the previous proofs of this type that $b^T x \leq \beta$ is a facet defining inequality for $P_B(G')$ with $\{x \in P_B(G') \mid a^T x = \alpha\} \subseteq \{x \in P_B(G') \mid b^T x = \beta\}$. To prove our theorem we have to show that $b = \rho a$ holds for some $\rho > 0$.

Let W' denote the subset of V' obtained by shrinking the node sets $W_1, \ldots, W_s$. We may assume that $|W_i| > 1$ for $i = 1, \ldots, r$ and $|W_i| = 1$ for $i = r+1, \ldots, s$. (If no set W_i has cardinality more than one, there is nothing to prove.) Note that $s - r \geq k + 2$ because of the assumption $q := |W_1| + \cdots + |W_r| \leq k - 1$. Set $S := \{w_1, \ldots, w_r\}$, $T := W' \setminus S$.

Let u, v, w be three different nodes in T. Let U be any subset of $T \setminus \{u, v, w\}$ of cardinality $k - 1$ and $S' := T \setminus (U \cup \{u, v, w\})$. Set $Y := \{u\}$, $Z := \{w\}$, $Y' := U$, $Z' := S \cup S'$, then the incidence vectors of $B_1, \ldots, B_4$ as defined in (2.5) satisfy $a^T x \leq \alpha$ with equality, and hence $b^T x \leq \beta$ also with equality. Therefore Lemma (2.5) implies $b_{uv} = b_{wv}$ and we can conclude that there is $\rho \in \mathbb{R}$ with $b_{uv} = \rho$ for all $u, v \in T$.

Now we choose any $w_i \in S$ and any $v \in T$. Set $Y := \{w_i\}$, let Z be any subset of $T \setminus \{v\}$ of cardinality $|W_i|$, Z' be any subset of $T \setminus (Z \cup \{v\})$ of cardinality $k - |W_i|$ and $Y' := (S \cup T) \setminus (Z \cup Z' \cup \{v\} \cup \{w_i\})$. Again, the incidence vectors of $B_1, \ldots, B_4$ as defined in (2.5) satisfy $a^T x \leq \alpha$ and hence $b^T x \leq \beta$ with equality, so we can conclude from (2.5) that $b_{vw_i} = \sum_{z \in Z} b_{zv} = \rho |W_i|$ thus $b_{vw_i} = \rho |W_i|$ for all $w_i \in S$ and all $v \in T$.

Let S_1 be any subset of T of cardinality $k + 1 - q$ and set $S_2 := T \setminus S_1$. Then $B = [S \cup S_1 : S_2]$ is an edge set in $\mathscr{B}(G)$ satisfying $a^T x = \alpha$. Hence $\beta = b^T x^B = \sum_{w_i \in S} \sum_{j \in S_2} b_{jw_i} + \sum_{i \in S_1} \sum_{j \in S_2} b_{ij} = \rho k(k+1)$.

To determine the coefficients $b_{w_i w_j}$ for $w_i, w_j \in S$ (in case $|S| > 1$) we use the following argument. Let S_1, S_2 be any bipartition of S, then by our assumptions we have that $k_1 := \sum_{w_i \in S_1} |W_i| \leq k - 3$ and $k_2 := \sum_{w_j \in S_2} |W_j| \leq k - 3$.

Choosing any subset S_1' of T of cardinality $k + 1 - k_1$ and setting $S_2' := T \setminus S_1'$ we see that $B := [S_1 \cup S_1' : S_2 \cup S_2']$ is the edge set of a bipartite subgraph of G' such that $a^T x^B = \alpha$ and therefore $b^T x^B = \beta$ holds. Thus, for every bipartition S_1, S_2 of S we can find an edge set $B \in \mathscr{B}(G')$ with $B \subseteq E(W')$ such that $b^T x^B = \beta$.

Observe that for all edges $ij \in E' \setminus E(W')$ the components x_{ij}^B of the incidence vectors x^B of all these edge sets B are zero by construction. Let $\bar{y} \in \mathbb{R}^{E(S)}$ (resp. $\overline{\overline{y}} \in \mathbb{R}^{E(W') \setminus E(S)}$) denote the vector obtained from a vector $y \in \mathbb{R}^{E'}$ by dropping the components corresponding to edges $ij \in E' \setminus E(S)$ (resp. edges $ij \in E' \setminus (E(W') \setminus E(S))$). Thus for every edge set $B \in \mathscr{B}(G)$ as constructed above we have $\beta = b^T x^B = \bar{b}^T \bar{x}^B + \overline{\overline{b}}^T \overline{\overline{x}}^B$. Since we know already all coefficients b_{ij} for $ij \in E(W') \setminus E(S)$, we get

$$\begin{aligned}
\overline{\overline{b}}^T \overline{\overline{x}}^B &= \sum_{w_i \in S_1} \sum_{j \in S_2'} b_{jw_i} + \sum_{w_j \in S_2} \sum_{i \in S_1'} b_{iw_j} + \sum_{i \in S_1'} \sum_{j \in S_2'} b_{ij} \\
&= \sum_{w_i \in S_1} \rho |S_2'|\,|W_i| + \sum_{w_j \in S_2} \rho |S_1'|\,|W_j| + \rho |S_1'|\,|S_2'| \\
&= \rho(k_1 |S_2'| + k_2 |S_1'| + |S_1'|\,|S_2'|) \\
&= \rho(k_1(k - k_2) + k_2(k + 1 - k_1) + (k - k_2)(k + 1 - k_1)) \\
&= \rho(k(k+1) - k_1 k_2)
\end{aligned}$$

which implies that for every set $B \in \mathscr{B}(G)$ constructed from a bipartition S_1, S_2 of S as above we obtain an equation

$$\bar{b}^T \bar{x}^B = \beta - \rho(k(k+1) - k_1 k_2) = \rho k_1 k_2 .$$

Let us write this equation system in the form $A\bar{b} = \bar{\beta}$, i.e. we have $|E(S)|$ unknowns b_{ij}, $ij \in E(S)$, and the matrix A is an $(2^{|S|-1} - 1, |E(S)|)$-matrix whose rows are the incidence vectors of the cuts $\delta(S_1)$ of $[S, E(S)]$, $\emptyset \neq S_1 \neq S$, $S_1 \subseteq S$, and the right-hand side $\bar{\beta}$ is given as calculated above. This matrix A has full rank $|E(S)|$ (see Remark after Theorem (3.3)), so, if the system $A\bar{b} = \bar{\beta}$ has a solution, it has a unique solution. Now we simply set $b_{w_i w_j} = \rho|W_i||W_j|$ for all $w_i, w_j \in S$ and can easily see that this stipulation in fact gives a solution of our system. Hence we also know the coefficients of b corresponding to edges in $E(S)$.

It is trivial to show that $b_{ij} = 0$ for all $ij \in E' \setminus E(W')$, thus we have proved that $b = \rho\alpha$ holds. ∎

Theorem (4.1) shows that for any K_{2k+1}-inequality, $k \geq 3$, one can shrink one or more subsets of nodes (subject to some cardinality conditions) to obtain new facet defining inequalities with coefficients which are not only zero and one. Moreover the *supports* of these inequalities $a^T x \leq \alpha$ (i.e. the edge sets $F_a := \{ij \in E' \mid a_{ij} > 0\}$) may also be the edge sets of complete subgraphs of G' of even order.

COROLLARY 4.3. *Let $G = [V, E]$ be a graph containing a complete subgraph $[W, E(W)]$ of order $k+4$, $k \geq 2$. Then $P_B(G)$ has a facet defining inequality $a^T x \leq k(k+1)$ whose support is $E(W)$ and whose coefficients are $0, 1, \frac{1}{2}(k-1), \frac{1}{4}(k-1)^2$ in case k is odd, resp. $0, 1, (k-2)/2, k/2, \frac{1}{4}k(k-2)$ in case k is even.* ∎

In particular, if G itself is the complete graph of order $n = k+4$, Corollary (4.3) shows that the coefficients of facets of $P_B(G)$ may be of order n^2.

We now introduce a facet construction method which will turn out to be a special case of another such method but which is of particular interest and therefore gets its own name.

THEOREM 4.4. (a) (Odd Edge Subdivision). *Let $G = [V, E]$ be a graph and $a^T x \leq \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $ij \in E$ be an edge with $a_{ij} > 0$. Let $G' = [V', E']$ be a graph obtained from G in the following way. An even number of new nodes $i_1, i_2, \ldots, i_k$ is added. The edge set $P = ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj$ (i.e. a path of odd length) is added. Any further edge having at least one of the new nodes $i_1, \ldots, i_k$ as an endnode may be added. The edge ij may be removed. Let $\bar{a} \in \mathbb{R}^{E'}$ be defined as follows*

$$\begin{aligned} \bar{a}_{uv} &:= a_{uv} && \text{for all } uv \in (E \cap E') \setminus \{ij\}, \\ \bar{a}_{uv} &:= a_{ij} && \text{for all } uv \in P, \\ \bar{a}_{uv} &:= 0 && \text{otherwise.} \end{aligned}$$

Then $\bar{a}^T x \leq \alpha + k a_{ij}$ defines a facet of $P_B(G')$.

(b) (Replacing an odd path by an edge). *Let $G' = [V', E']$ be a graph and $\bar{a}^T x \leq \bar{\alpha}$ be a facet defining inequality for $P_B(G')$. Suppose the support of $\bar{a}$ contains a path $P = \{ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj\}$ of odd length $k+1 \geq 3$ with $a_{uv} = \alpha'$ for all $uv \in P$ and where the degree of all nodes $i_1, \ldots, i_k$ in the support of $\bar{a}$ is 2 and ij is not in the support of $\bar{a}$. Let $G = [V, E]$ be the graph obtained from G' by removing the nodes $i_1, \ldots, i_k$ and adding the edge ij (if ij is not already contained in G'). Let $a \in \mathbb{R}^E$ be defined as follows:*

$$\begin{aligned} a_{uv} &:= \bar{a}_{uv} && \text{for all } uv \in E \cap E', \\ a_{ij} &:= \alpha'. \end{aligned}$$

Then $a^T x \leq \bar{\alpha} - k\alpha'$ defines a facet of $P_B(G)$. ∎

Thus Theorem (4.4)(a) shows that we can subdivide "positive edges" of a facet defining inequality an odd number of times and that we will get another facet defining

inequality this way. Theorem (4.4)(b) tells us that this operation can also be reversed without losing the desired properties. Theorem (4.4) is a corollary of the following more general result (set $F = \{ij\}$ and apply (4.5) repeatedly).

THEOREM 4.5. (a) (Node Splitting). *Let $G = [V, E]$ be a graph and $a^T x \leq \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let E_a be the support of a, and let v be a node in $V(E_a)$. Let V_1, V_2 be any bipartition of V such that $a^T x^{\delta(V_1)} = \alpha$ and assume that $v \in V_1$. Choose any nonempty subset $F \subseteq \delta(v) \cap E_a \cap E(V_1)$ and construct a new graph $G' = [V', E']$ from G as follows. Split node v into two nodes v_1, v_2 such that v_1 is incident to all edges contained in F and v_2 is incident to all edges in $\delta(v) \setminus F$. The edge v_1v_2 may be added, as well as any edge v_2u and $vu \in F$. Moreover, add a new node v_3 adjacent to v_1 and v_2; in addition any further edge having v_3 or v_1 as one endnode may be added. The other parts of G remain unchanged. Set*

$$\begin{aligned} \bar{a}_{ij} &:= a_{ij} && \text{for all } ij \in E \setminus \delta(v), \\ \bar{a}_{v_1u} &:= a_{vu} && \text{for all } v_1u \in E' \text{ with } vu \in F, \\ \bar{a}_{v_2u} &:= a_{vu} && \text{for all } v_2u \in E' \text{ with } vu \in (\delta(v) \cap E_a) \setminus F, \\ \bar{a}_{v_3v_1} &:= \bar{a}_{v_3v_2} := \gamma := \sum_{ij \in F} a_{ij}, \\ \bar{a}_{ij} &:= 0 && \text{otherwise} \\ \bar{\alpha} &:= \alpha + 2\gamma, \end{aligned}$$

then $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $P_B(G')$.

(b) (Contraction of a path of length two). *Let $G' = [V', E']$ be a graph and $\bar{a}^T x \leq \bar{\alpha}$ be an inequality defining a facet of $P_B(G')$. Suppose G' contains a node v_3 contained in exactly two edges, say v_1v_3 and v_2v_3, of the support $F_{\bar{a}}$ of $\bar{a}$, that $v_1v_2 \notin F_{\bar{a}}$, that v_1 and v_2 have no common neighbour in $[V', F_{\bar{a}}]$ except v_3, and that $\bar{a}_{v_1v_3} = \bar{a}_{v_2v_3} = \bar{a}(\delta(v_1) \setminus \{v_1v_3\}) \leq \bar{a}(\delta(v_2) \setminus \{v_2v_3\})$. Let $G = [V, E]$ be the graph obtained from G' by removing the nodes v_1, v_2, v_3 adding a new node v and the edges $\{vj \mid \bar{a}_{v_1j} > 0\} \cup \{vj \mid v_2j \in E'\}$. Set*

$$\begin{aligned} a_{ij} &:= \bar{a}_{ij} && \text{for all } ij \in E \cap E', \\ a_{vj} &:= \bar{a}_{v_1j} && \text{if } \bar{a}_{v_1j} > 0, \\ a_{vj} &:= \bar{a}_{v_2j} && \text{if } v_2j \in E' \text{ unless } \bar{a}_{v_1j} > 0 \\ \alpha &:= \bar{\alpha} - 2\bar{a}_{v_1v_3}, \end{aligned}$$

then $a^T x \leq \alpha$ defines a facet of $P_B(G)$.

PROOF. The validity of the new inequalities defined in (a) and (b) follows by elementary construction. It remains to prove that the faces defined by the inequalities contain sufficiently many affinely independent points.

(a) Since $a^T x \leq \alpha$ is nontrivial, there are $m = |E|$ bipartite edge sets $B_1, \ldots, B_m \subseteq E$ whose incidence vectors are linearly independent and satisfy $a^T x^{B_i} = \alpha$, $i = 1, \ldots, m$. Set

$$B_i' := (B_i \setminus \delta_G(v)) \cup \{v_1v_3, v_2v_3\} \cup \{v_1u \mid vu \in B_i \cap F\} \cup \{v_2u \mid vu \in B_i \setminus F\},$$

$i = 1, \ldots, m$, then B_i' is a bipartite edge set in G' and $\bar{a}^T x^{B_i'} = \alpha + 2\gamma$. Moreover, the incidence vectors $x^{B_i'} \in \mathbb{R}^{E'}$, $i, \ldots m$ are linearly independent.

Let $F_{\bar{a}} \subseteq E'$ denote the support of $\bar{a}$. Using the assumption that $a^T x^{\delta(V_1)} = \alpha$ and $v \in V_1$ we immediately see that the incidence vectors of the bipartite edge sets $B_{m+1} := \delta_{G'}((V_1 \setminus \{v\}) \cup \{v_2, v_3\}) \cap F_{\bar{a}}$ and $B_{m+2} := \delta_{G'}((V_1 \setminus \{v\}) \cup \{v_2\}) \cap F_{\bar{a}}$ in G' satisfy $\bar{a}^T x^{B_i} = \alpha + 2\gamma$, $i = m+1, m+2$. It is obvious that each of the vectors $x^{B_{m+1}}$

and $x^{B_{m+2}}$ is linearly independent from the vectors x^{B_i}, $i = 1, \ldots, m$, and it is an easy exercise in linear algebra to prove that in fact all the vectors x^{B_i}, $i = 1, \ldots, m+2$ are linearly independent.

Moreover, consider the incidence vectors of the following bipartite edge sets. If $v_1v_2 \in E'$ set $B_{v_1v_2} = B_{m+2} \cup \{v_1v_2\}$. If $v_3u \in E'$, $v_1 \neq u \neq v_2$, then, if $u \in V_1$ set $B_{v_3u} = B_{m+2} \cup \{v_3u\}$, if $u \in V \setminus V_1$ set $B_{v_3u} = B_{m+1} \cup \{v_3u\}$. If $v_1u \in E'$ such that $vu \notin F$, then, if $u \in V_1$ set $B_{v_1u} := B_{m+2} \cup \{v_1u\}$, if $u \notin V_1$ then set $B_{v_1u} := (B_{m+2} \setminus \delta(v_1)) \cup \{v_1v_3, v_1u\}$. If $v_2u \in E'$ with $vu \in F$, then because $a^Tx \leqslant \alpha$ defines a nontrivial facet of $P_B(G)$ there exists a bipartite edge set $B_u \subseteq E$ with $vu \in B_u$ and $a^Tx^{B_u} = \alpha$, and we set

$$B_{v_2u} = (B_u \setminus \delta_G(v)) \cup \{v_1v_3, v_2v_3\} \cup \{v_1j \mid vj \in B_u \cap F\} \cup \{v_2j \mid vj \in B_u \setminus F\} \cup \{v_2u\}.$$

Clearly, these incidence vectors satisfy $\bar{a}^Tx \leqslant \alpha + 2\gamma$ with equality and the set of $|E'|$ vectors constructed above is linearly independent. Thus $P_B(G') \cap \{x \mid \bar{a}^Tx = \alpha + 2\gamma\}$ contains $|E'|$ linearly independent points which proves that $a^Tx \leqslant \alpha + 2\gamma$ defines a facet of $P_B(G')$.

(b) Let $Q = \{v_3u \in E' \mid v_1 \neq u \neq v_2\} \cup \{v_1u \in E' \mid \bar{a}_{v_1u} = 0\} \cup \{v_2u \in E' \mid \bar{a}_{v_1u} > 0\}$. Then by assumption $\bar{a}_{ij} = 0$ for all $ij \in Q$. If $\bar{a}^Tx \leqslant \bar{\alpha}$ defines a facet of $P_B(G')$ then the inequality $\sum_{ij \in E' \setminus Q} \bar{a}_{ij}x_{ij} \leqslant \bar{\alpha}$ clearly defines a facet of $P_B(G' - Q)$. Thus we may assume without loss of generality that $Q = \emptyset$, i.e. that in G' no node in $V' \setminus \{v_1, v_2, v_3\}$ is adjacent to v_3 and that no such node is adjacent to both v_1 and v_2.

By assumption, $\bar{a}^Tx \leqslant \bar{\alpha}$ defines a nontrivial facet of $P_B(G')$, so there are $m := |E'|$ bipartite edge sets $B'_1, B'_2, \ldots, B'_m$ whose incidence vectors are linearly independent and satisfy $\bar{a}^Tx^{B_i} = \bar{\alpha}$, $i = 1, \ldots, m$. We may assume that $N \subseteq V'$ is the set of neighbours of v_1 in $[V', F_{\bar{a}}]$ different from v_3, and that $k := |N|$.

Now let M be the (m, m)-matrix whose rows are the incidence vectors $x^{B'_1}, \ldots, x^{B'_m}$. We may assume that the last $k+2$ columns correspond to the edges v_1i, $i \in N$, and v_1v_3, v_2v_3. Moreover, we assume that the sets $B'_1, \ldots, B'_m$ are ordered in such a way that the sets $B'_1, \ldots B'_r$ contain both edges v_1v_3, v_2v_3, the sets $B'_{r+1}, \ldots, B'_s$ contain v_1v_3 and not v_2v_3 and the sets $B'_{s+1}, \ldots, B'_m$ contain v_2v_3 but not v_1v_3.

Note that the assumptions of the theorem imply that if $\bar{a}^Tx^{B'} = \bar{\alpha}$, then B' contains at least one of the edges v_1v_3, v_2v_3, and moreover, if B' contains only one of the edges v_1v_3, v_3v_3 then B' necessarily contains all edges v_1i, $i \in N$, otherwise $B'' = (B' \setminus \{v_1i \mid i \in N\}) \cup \{v_1v_3, v_2v_3\}$ would be a bipartite edge set with $\bar{a}^Tx^{B''} > \bar{\alpha}$. Thus, our matrix M looks as follows

$$\begin{array}{c} \\ r\;\{ \\ s-r\;\{ \\ m-s\;\{ \end{array} \begin{array}{cccc} m-k-2 & k & \multicolumn{2}{c}{2} \\ \hline \multicolumn{1}{|c|}{M_1} & \multicolumn{1}{c|}{M_2} & \multicolumn{1}{c|}{\mathbb{1}} & \multicolumn{1}{c|}{\mathbb{1}} \\ \hline \multicolumn{1}{|c|}{M_3} & \multicolumn{1}{c|}{M_5} & \multicolumn{1}{c|}{\mathbb{1}} & \multicolumn{1}{c|}{0} \\ \hline \multicolumn{1}{|c|}{M_4} & \multicolumn{1}{c|}{M_6} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{\mathbb{1}} \\ \hline \end{array} = M$$

where M_1, M_2, M_3, M_4 are some 0/1-matrices, the matrices M_5 and M_6 contain ones only and the last two columns are as indicated, where $\mathbb{1}$ denotes a column of ones.

Now we transform the edge sets $B'_i \subseteq E'$ into bipartite edge sets $B_i \subseteq E$, $i = 1, \ldots, m$ as follows

$$B_i := (B'_i \setminus (\delta_{G'}(v_1) \cup \delta_{G'}(v_2))) \cup \{vj \mid v_2j \in B'_i, j \neq v_3\} \cup \{vj \mid v_1j \in B'_i, j \neq v_3\}.$$

Note that this transformation corresponds to a shrinking of the nodes v_1, v_2, v_3 into the node v. (By our assumption $Q = \emptyset$, no multiple edges will occur.) It follows from our

remarks above that $a^Tx^{B_i} = \alpha$ for $i = 1, \ldots, m$. If A is the $(m, m-2)$-matrix whose rows are the incidence vectors x^{B_i}, $i = 1, \ldots, m$, then A looks as follows

$$\begin{array}{c} \\ r\;\{ \\ s-r\;\{ \\ m-s\;\{ \end{array} \begin{array}{cc} m-k-2 & k \\ \hline \multicolumn{1}{|c|}{A_1} & \multicolumn{1}{c|}{A_2} \\ \hline \multicolumn{1}{|c|}{A_3} & \multicolumn{1}{c|}{A_5} \\ \hline \multicolumn{1}{|c|}{A_4} & \multicolumn{1}{c|}{A_6} \\ \hline \end{array} = A$$

where by construction the following holds: $A_i = M_i$, $i = 1, \ldots, 4$, and A_5, A_6 are matrices containing zeros only.

To prove the theorem we now show that the rank of A equals $m-2$.

First we add to M a $(m+1)$st column which contains twos only and then we add to this matrix a $(m+1)$st row having zeros everywhere except in positions $m-k-1, \ldots, m-2$ and $m+1$ which contain ones. Let us denote this matrix by M':

$$\left[\begin{array}{cc|c|c|c} M_1 & M_2 & \mathbb{1} & \mathbb{1} & 2 \\ M_3 & M_5 & \mathbb{1} & 0 & \vdots \\ M_4 & M_6 & 0 & \mathbb{1} & 2 \\ \hline & 1 \ldots 1 & 0 & 0 & 1 \end{array}\right] = M'.$$

Since M is nonsingular the equation system $M\lambda = \underline{2}$ has a unique solution which, in fact, is nothing but a multiple of the normal vector $\bar{a}$ of the given facet, i.e. we obtain $\lambda_{ij} = 2\bar{a}_{ij}/\bar{\alpha}$ for all $ij \in E'$.

Suppose the last column of M' is dependent of the first m columns of M', then

$$\sum_{i \in N} \lambda_{v_1i} = \sum_{i \in N} \frac{2\bar{a}_{v_1i}}{\bar{\alpha}} = \frac{2\bar{a}_{v_1v_3}}{\bar{\alpha}} = 1$$

has to hold. But this implies that $2\bar{a}_{v_1v_3} = \bar{\alpha}$, hence the incidence vector of the bipartite edge set $\{v_1v_3, v_2v_3\}$ satisfies $\bar{a}^Tx \leqslant \bar{\alpha}$ with equality which is impossible by our assumptions. Therefore we can conclude that M' is a nonsingular matrix.

Now we perform the following operations on M'. To each of the columns $m-k-1, \ldots, m-2$, i.e. the columns corresponding to the edges v_1i, $i \in N$, we add the column $m-1$ and the column m and subtract the column $m+1$. It is easy to see that the matrix M'' we obtain this way looks as follows:

$$\left[\begin{array}{cc|c|c|c} A_1 & A_2 & \mathbb{1} & \mathbb{1} & 2 \\ A_3 & A_5 & \mathbb{1} & 0 & \vdots \\ A_4 & A_6 & 0 & \mathbb{1} & 2 \\ \hline 0 & & & & 1 \end{array}\right] = M''.$$

M'' is nonsingular, since M' is nonsingular. Deleting the last column and the last row from M'' we again obtain a nonsingular matrix M'''. Deleting the last two columns

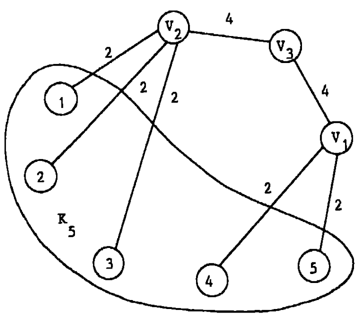

FIGURE 4.1

from M''' we obtain a matrix of full column rank $m-2$. But this matrix is the matrix A, and our proof is complete. ∎

To illustrate the methods of facet construction introduced before we now give an example. Let $[V, E_7]$ be the complete graph K_7, then $x(E_7) \leqslant 12$ defines a facet of $P_B(K_7)$ by Theorem (3.3). We pick the nodes 6 and 7 of K_7 and shrink them by node set contraction into a new node 6' to obtain the complete graph K_6. By Theorem (4.1) the inequality $2x(\delta_{K_6}(6')) + x(E(\{1,2,\dots,5\})) \leqslant 12$ defines a facet of $P_B(K_6)$. $\delta_{K_6}(V_1)$ with $V_1 = \{4, 5, 6'\}$ is a bipartite edge set whose incidence vector satisfies the above inequality with equation. Choosing $F = \{46', 56'\}$ we construct the following new graph G' (with 8 nodes) by splitting node 6' into v_1 and v_2 and adding a new node v_3 with the method described in Theorem (4.5)(a), cf. Figure 4.1.

The inequality $\bar{a}^T x \leqslant 20$ defines a facet of $P_B(G')$ where the coefficients $\bar{a}_{ij}$ are equal to one for ij with $1 \leqslant i < j \leqslant 5$, and the other coefficients have the value indicated on the corresponding edges in Figure 4.1. Moreover, this inequality defines a facet of $P_B(H)$ for all graphs H containing G' as a subgraph.

5. Cut subdivisions. We now describe how a cut of the support of a facet defining inequality can be subdivided to get a new facet. We start with a special case.

THEOREM 5.1 (Subdivision of a Star). *Let $G = [V, E]$ be a graph and $a^T x \leqslant \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $v \in V$ be any node and suppose the star $\delta(v)$ of v consists of the edges $vi_1, \dots, vi_k$. Construct a new graph $G' = [V', E']$ from G as follows. Subdivide each edge of the star $\delta(v)$, i.e. add k new nodes $j_1, \dots, j_k$ and replace the edges $vi_1, vi_2, \dots, vi_k$ by the edges $vj_1, j_1i_1; vj_2, j_2i_2; \dots; vj_k, j_ki_k$. Keep all other nodes and edges, and, moreover, any further edge linking a new node j_s to any other node in V', and any edge vi_s, $s = 1, \dots, k$ may be added. Set*

$$\bar{a}_{ij} := a_{ij} \quad \text{for all} \quad ij \in E \cap E',$$

$$\bar{a}_{vj_s} := \bar{a}_{j_si_s} := a_{vi_s} \quad \text{for} \quad s = 1, \dots, k,$$

$$\bar{a}_{ij} := 0 \quad \text{else},$$

$$\bar{\alpha} := \alpha + \sum_{s=1}^{k} a_{vi_s},$$

then $\bar{a}^T x \leqslant \bar{\alpha}$ defines a facet of $P_B(G')$.

PROOF. It should be clear that $\bar{a}^T x \leqslant \bar{\alpha}$ is valid with respect to $P_B(G')$. Since $a^T x \leqslant \alpha$ is a nontrivial facet of $P_B(G)$ there are bipartite edge sets $B_1, \dots, B_m \subseteq E$, $m := |E|$, whose incidence vectors are linearly independent and satisfy the inequality with equality.

We claim that there is an injective mapping $\phi: \{1, \dots, k\} \to \{1, \dots, m\}$ such that $B_{\phi(s)}$ does not contain the edge vi_s, $1 \leqslant s \leqslant k$, i.e. that we may assume that B_s does not contain vi_s, $1 \leqslant s \leqslant k$. Namely, first of all for each edge vi_s there is a set B_j not containing vi_s, otherwise $a^T x \leqslant \alpha$ would define the (trivial) facet $\{x \in P_B(G) \mid x_{vi_s} = 1\}$. Secondly, if there were two edges vi_s, vi_r such that each set B_j not containing vi_s does not contain vi_r and vice versa, then the nonsingular (m, m)-matrix M whose rows are the incidence vectors of the sets B_j would contain two identical columns, a contradiction. We may also assume that the last k columns of M correspond to the edges vi_s, $1 \leqslant s \leqslant k$, i.e. that M has the following form

$$k\left\{\begin{array}{|c|c|} \hline M_1 & \overbrace{\begin{matrix} 0 & & \\ & 0 & \cdot \\ \cdot & \cdot & 0 \end{matrix}}^{k} \\ \hline M_2 & M_3 \\ \hline \end{array}\right. = M$$

where M_1, M_2, M_3 are some 0/1-matrices and the (k,k)-submatrix in the upper right-hand corner is a 0/1-matrix M_4 with zeros on the main diagonal.

Let $m' := |E'|$. We now construct edge sets B'_t in G' as follows:

$$B'_t := (B_t \setminus \delta_G(v)) \cup \{vj_s \mid s = 1, \dots, k\} \cup \{j_si_s \mid vi_s \in B_t\}, \qquad t = 1, \dots, m,$$

$$B'_{m+s} := (B'_s \setminus \{vj_s\}) \cup \{j_si_s\}, \qquad s = 1, \dots, k,$$

and for the k', say, edges $e_{m+k+1}, \dots, e_{m+k+k'}$ in G' for which one of the endnodes is one of the nodes j_s, $s \in \{1, \dots, k\}$, and the other endnode is a node in $V' \setminus \{j_s, i_s, v\}$, say $e_{m+k+t} = j_su$, we define B'_{m+k+t}, $t = 1, \dots, k'$ to be the bipartite edge set among the two sets $B'_{m+s} \cup \{j_su\}$ or $B'_s \cup \{j_su\}$. Moreover, for the (possibly existing) further $m' - (m + k + k')$ edges $e_{m+k+k'+1}, \dots, e_{m'}$ in G' for which one of the endnodes is v and the other is one of the endnodes i_s, $s \in \{1, \dots k\}$, say $e_{m+k+k'+t} = vi_r$, we set

$$B'_{m+k+k'+t} := (B_{l_r} \setminus \delta_G(v)) \cup \{vj_s \mid s = 1, \dots, k\} \cup \{j_si_s \mid vi_s \in B_{l_r}\} \cup \{vi_r\}$$

where B_{l_r} is a bipartite edge set in G such that v and i_r are in the same set of the corresponding bipartition of V and such that $a^T x^{B_{l_r}} = \alpha$. Note that such a set B_{l_r} exists because $a^T x \leqslant \alpha$ is nontrivial.

It is clear from the construction that each of the sets B'_t, $t = 1, \dots, m'$, is the edge set of a bipartite graph and that its incidence vector satisfies $\bar{a}^T x \leqslant \bar{\alpha}$ with equality.

Consider now the (m', m')-matrix M' whose rows are the incidence vectors of the sets B'_t, $t = 1, \dots, m'$. We may assume that the columns of M' corresponding to the edges in $E \cap E'$ are ordered in the same way as in M, that the k columns $m - k + 1, \dots, m$ of M' correspond to the edges j_si_s, $s = 1, \dots, k$, that the next k columns $m + 1, \dots, m + k$ correspond to the edges $vj_1, \dots, vj_k$ and that the last $m' - (m + k)$ columns correspond to the further edges $e_{m+k+1}, \dots, e_{m'}$ of G'. Then M' looks as follows

$$
m\left\{\begin{array}{c|c|c|c}
 & \overbrace{\qquad\qquad\qquad}^{m} & & \\
M_1 & \begin{matrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{matrix} & 1 & 0 \\
\hline
M_2 & M_3 & & \\
\hline
M_5 & \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} & \begin{matrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{matrix} & 0 \\
\hline
M_6 & M_7 & M_8 & I
\end{array}\right. = M'
$$

where the principal (m,m)-submatrix is equal to M, M_5, M_6, M_7, M_8 are $0/1$-matrices and the other matrices are of the indicated type. It is not too difficult to prove that M' is nonsingular. This finishes our proof. ∎

Using the above construction and Theorem (4.4) we get a more general method.

THEOREM 5.2 (a) (Subdivision of a Cut). *Let $G = [V, E]$ be a graph and $a^T x \leqslant \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $W \subseteq V$ and suppose that $\delta(W) = \{w_1u_1, \ldots, w_ku_k\}$, with $w_i \in W$, $i = 1, \ldots, k$ (possibly $w_i = w_j$ for $i \neq j$ or $u_i = u_j$ for $i \neq j$). Construct a new graph $G' = [V', E']$ from G as follows. Subdivide each edge of $\delta(W)$, i.e. add k new nodes $j_1, \ldots, j_k$ and replace the edges $w_1u_1, w_2u_2, \ldots, w_ku_k$ by the edges $w_1j_1, j_1u_1; w_2j_2, j_2u_2; \ldots; w_kj_k, j_ku_k$. Keep all other nodes and edges, and moreover, any further edge linking a new node j_s to any other node $v \in V' \setminus \{j_s, u_s, w_s\}$ may be added as well as any of the edges $w_1u_1, \ldots, w_ku_k$. Set*

$$\bar{a}_{ij} := a_{ij} \quad \text{for all} \quad ij \in E \cap E',$$

$$\bar{a}_{w_sj_s} := \bar{a}_{u_sj_s} := a_{w_su_s}, \quad s = 1, \ldots, k,$$

$$\bar{a}_{ij} := 0 \quad \text{otherwise},$$

$$\bar{\alpha} := \alpha + \sum_{s=1}^{k} a_{w_su_s},$$

then $\bar{a}^T x \leqslant \bar{\alpha}$ defines a facet of $P_B(G')$.

(b) (Contraction of a Cut). *Let $G' = [V', E']$ be a graph and $\bar{a}^T x \leqslant \bar{\alpha}$ be a facet of $P_B(G')$. Let $W \subseteq V'$ be a node set, and let $W' \subseteq V' \setminus W$ be the set of nodes which are linked to a node in W by an edge contained in the support $F_{\bar{a}}$ of $\bar{a}$. Suppose that $E'(W') \cap F_{\bar{a}} = \emptyset$, and that every node $j \in W'$ is contained in exactly two edges, say w_jj, u_jj where $w_j \in W$ and $u_j \in V \setminus (W \cup W')$, of the support of $\bar{a}$ which in addition satisfy $\bar{a}_{u_jj} = \bar{a}_{w_jj}$. Construct a new graph $G = [V, E]$ from G' as follows. Remove the node set W'. Remove all edges of $\delta_{G'}(W)$. Keep all other nodes and edges and add the edges u_jw_j for all $j \in W'$ (taking care that no multiple edges occur). Set*

$$a_{ij} := \bar{a}_{ij} \quad \text{for all} \quad ij \in E \cap E',$$

$$a_{u_jw_j} := \bar{a}_{u_jj} \quad \text{for all} \quad j \in W',$$

$$\alpha := \bar{\alpha} - \bar{a}(\delta(W)),$$

then the inequality $a^T x \leqslant \alpha$ defines a facet of $P_B(G)$.

PROOF. It is clear that the new inequalities defined in (a) and (b) are valid.

(a) For every node $w \in W$ we perform the subdivision of its cut $\delta(w)$ as described in Theorem (5.1) (and add the additional edges we want to have). This gives a facet defining inequality where each edge of the cut $\delta(w)$ is subdivided as described in the theorem and each edge $vw \in E(W)$ is replaced by a path of length three. Using Theorem (4.4)(b) we replace all these paths by the (original) edges vw finishing the proof.

(b) First we remove all edges ij from G' which have (at least) one endnode in W' and for which $\bar{a}_{ij} = 0$ to obtain a graph G'' where all nodes of W' have degree two. Clearly, the projected inequality $\bar{a}^T x \leqslant \bar{\alpha}$ (leaving out the deleted coefficients) defines a facet of $P_B(G'')$. Now we apply part (a) of the theorem to subdivide the cut $\delta(W)$. This way we obtain paths of length three where in each path all edges have the same positive $\bar{a}$-coefficient. We replace these paths by a single edge using Theorem (4.4)(b) to obtain a facet defining inequality for $P_B(G)$. ∎

6. Examples and final remarks. The constructions defined in §§4 and 5 can be used to obtain an incredibly large number of quite "wild" facets from the facets described in §3. We shall discuss some examples.

Taking the K_{2k+1}-inequalities (3.3) and node splitting (4.5) we can generate facet defining inequalities for $P_B(G)$ where G is a graph of order $4k$ such that all integers $0, 1, \ldots, k$ occur as coefficients. For example, consider the graph K_7, i.e. $k = 3$. The inequality $x(E_7) \leqslant \alpha = 12$ defines a facet of $P_B(K_7)$ by (3.3). Now we pick a node, say 7, take any set of three edges incident to 7, say 47, 57, 67 as set F and apply the node splitting described in (4.5). We obtain the graph shown in Figure 6.1 (coefficients of $\bar{a}$ larger than one are written on the corresponding edges). Now we split on node 9 taking 19, 29 as F to get the graph of Figure 6.2. Weighting the edges of the graph G shown in Figure 6.2 as indicated we get a facet of $P_B(G)$ with coefficients 1, 2, 3 (zero coefficients can be obtained by adding further edges).

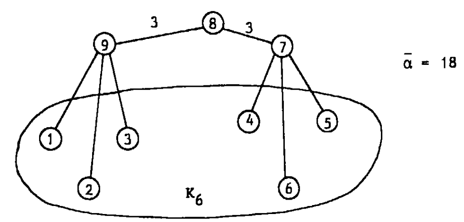

FIGURE 6.1

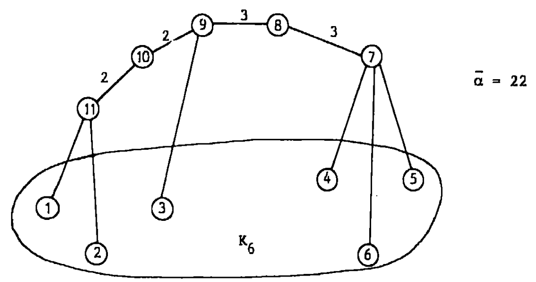

FIGURE 6.2

A graph—of interest in all parts of graph theory—is the Petersen graph P. We display a facet of $P_B(P)$ with coefficients 1 and 2 and right-hand side 16 which is obtained by a metamorphosis of K_5 (showing this way that P is not weakly bipartite). We start from $x(E_5) \leq 6$ and then apply various constructions. We indicate the constructions on the arrows of Figure 6.3, a double line states that the coefficient of the edge in the corresponding inequality is two. The right-hand sides, denoted by RS, of the inequalities corresponding to the graphs shown in Figure 6.3 are also indicated.

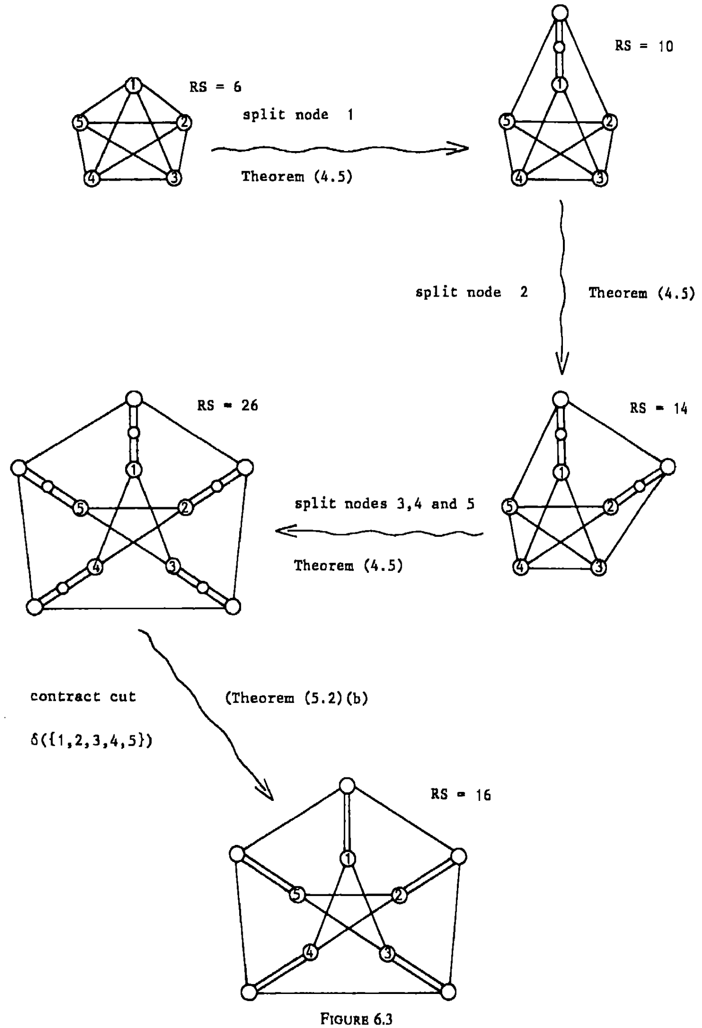

FIGURE 6.3

By repeated splitting of one of the universal nodes of a bicycle p-wheel we can construct graphs which have the property that they contain a node v (namely the remaining universal node) whose removal results in a planar graph where all nodes have degree at most three. Figure 6.4 shows such a construction for a bicycle 5-wheel. (The method, however, does not work for all bicycle wheels.) The final graph G of the sequence of splittings shown in Figure 6.4 has the property that $G - v$ is planar and all nodes have degree at most three. The inequality derived from this graph (edges drawn with a double line get coefficient 2, all other coefficient 1) with right-hand side 18 defines a facet of $P_B(G)$.

The algorithmic aspects of our theoretical analysis are still unexplored. We would like to know whether there are polynomial time algorithms for the separation problems involved with the classes of facets of $P_B(G)$ described here.

For instance, for s fixed, one can check whether all K_{2k+1}-inequalities and all bicycle $(2k+1)$-wheel inequalities, $2k+1 \leq s$, are satisfied by brute force enumeration in polynomial time. But even for K_5 this takes $O(n^5)$ time. We do not know whether there is a polynomial time algorithm for checking all odd complete subgraph inequalities or all odd bicycle wheel inequalities. (Note added in proof: See the paper of B. Gerards in this issue of *Mathematics of Operations Research* on this subject.)

It seems quite hopeless to find reasonable algorithms that solve the separation problems for the classes of facets obtained from the odd complete subgraph inequalities or odd bicycle wheel inequalities by any of the constructions described in §§4 and 5.

Even the simplest case is open, the graphs obtained from K_5 by odd subdivision of edges (Theorem (4.5)(a)). Let us call these graphs *odd homeomorphs* of K_5. What one has to solve here is the following. Given a graph $G = [V, E]$ and a point $y \in \mathbf{Q}^E$, say

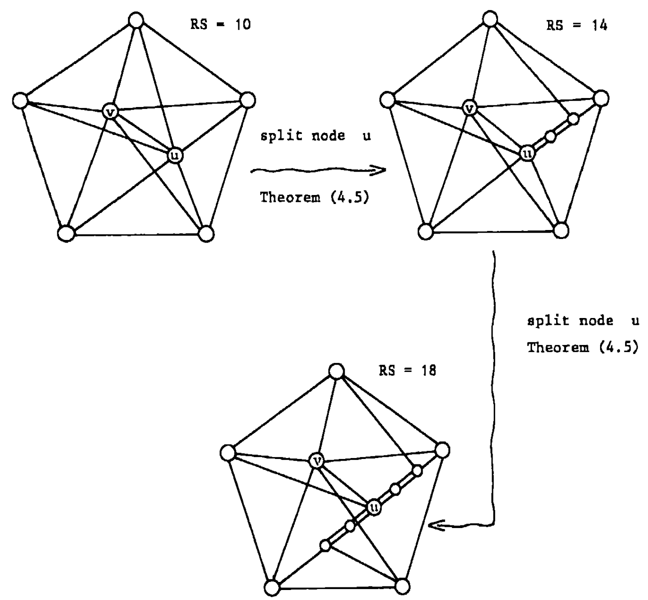

FIGURE 6.4

$0 \leqslant y_e \leqslant 1$ for all $e \in E$. Check whether there is a subgraph $H = [W, F]$ of G obtained from K_5 by odd subdivision of edges such that the corresponding facet defining inequality of $P_B(G)$ is violated.

For each odd homeomorph $H = [W, F]$ of K_5 the corresponding inequality can be written as

$$x(F) \leqslant |F| - 4 \tag{6.1}$$

see (4.4)(a) and (3.3). If y is given as above then we set $w_e := 1 - y_e$ for all $e \in E$. If we can solve in polynomial time

$$\min\{w(F) \mid [V(F), F] \text{ odd homeomorph of } K_5\} \tag{6.2}$$

then we can solve the separation problem for the inequalities (6.1) in polynomial time. Namely, if the minimum in (6.2) is 4 or larger then y satisfies all inequalities (6.1), otherwise for the minimum F^* $y(F^*) > |F^*| - 4$ holds and a violated inequality is found.

Of course, this question can be asked for the odd homeomorphs of K_7, K_9 etc. as well. We have the feeling that at least for the odd homeomorphs of K_5 a polynomial separation algorithm can be found.

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TESTING THE ODD BICYCLE WHEEL INEQUALITIES FOR THE BIPARTITE SUBGRAPH POLYTOPE*†

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In their paper on facets of the bipartite subgraph polytope Barahona, Grötschel and Mahjoub pose the question: Is there a polynomial time algorithm testing the odd bicycle wheel inequalities? This note describes such an algorithm.

1. Introduction. Throughout this text $G = [V, E]$ denotes an undirected graph and $P_B(G)$ its bipartite subgraph polytope (see [2, (2.1)]). A bicycle p-wheel in G consists of a cycle C, $|C| = p$, and an edge uv, with $u, v \notin C$, together with all the edges uw and vw with $w \in C$. We call uv the axle of the bicycle wheel. In [2], Barahona, Grötschel and Mahjoub state that the following inequalities induce facets of $P_B(G)$:

$$x(F) \leqslant 2(2k+1), \qquad [W, F] \text{ is a bicycle } 2k+1\text{-wheel in } G. \tag{1.1}$$

In §6 of their paper they say that no polynomial time algorithm is known which checks all the inequalities (1.1). In the next section of this note we shall describe such an algorithm. We use as a subroutine an algorithm of Grötschel and Pulleyblank, a version of which tests if a given $x \in \mathbb{R}^E$ satisfies the system (1.2), and if not, finds a violated inequality [3, §5].

$$\begin{aligned} 0 \leqslant x(e) \leqslant 1, &\qquad e \in E, \\ x(C) \leqslant |C| - 1, &\qquad C \text{ is an odd cycle in } G. \end{aligned} \tag{1.2}$$

All the inequalities in (1.2) induce facets of $P_B(G)$ (Barahona [1]). The algorithm of §2 checks the system (1.1) and (1.2) together.

2. The algorithm. If we have an element $x \in \mathbb{R}^E$ and we want to know whether it satisfies (1.1) and (1.2), then we first check (1.2) with the aid of the efficient algorithm given in [3, §5]. If (1.2) is violated we are finished. If not, for each $uv \in E$ we proceed as follows:

Define $V_{uv} := \{w \in V \mid uw,\ vw \in E\}$, $E_{uv} := \{ww' \in E \mid w,\ w' \in V_{uv}\}$ and for each $ww' \in E_{uv}$: $y(ww') := 2 - x(ww') - \frac{1}{2}[x(uw) + x(vw) + x(uw') + x(vw')]$. It is straightforward to see that there is an odd bicycle wheel with axle uv, for which the inequality of (1.1) does not hold iff the minimum weight odd cycle in the graph $[V_{uv}, E_{uv}]$ with edge weights y has weight less than $x(uv)$. Moreover $2y(ww') = 4 - x(\{ww', wu, w'u\}) - x(\{ww', wv, w'v\}) \geqslant 0$, since x satisfies (1.2), and so also the 3-cycle inequalities. Since in [3, §5] we find an efficient algorithm which finds a minimum weight odd cycle in the graph with nonnegative weights on the edges, we hereby have a polynomial time algorithm to test the system of inequalities (1.1) and (1.2).

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