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Steiner trees and polyhedra ☆

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Abstract

In this paper we study the dominant of the Steiner tree polytope. We introduce a new class of valid inequalities that generalizes the so-called odd hole, wheel, bipartite, anti-hole and Steiner partition inequalities introduced by Chopra and Rao (Math. Programming 64 (1994) 209–229, 231–246), and we give sufficient conditions for these inequalities to define facets. We describe some procedures that permit to construct facets from known ones for the dominant of the Steiner tree polytope and the closely related Steiner connected subgraph polytope. Using these methods we give a counterexample to a conjecture of Chopra and Rao on the dominant of the Steiner tree polytope on 2-trees. We also describe the dominant of the Steiner tree polytope and the Steiner connected subgraph polytope on special classes of graphs. In particular, we show that if the underlying graph is series–parallel and the terminals satisfy certain conditions, then both polyhedra are given by the trivial inequalities and the Steiner partition inequalities. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Steiner tree; Steiner connected subgraph; Polytope; Facet; Series–parallel graph

1. Introduction

Given a graph $G=(V,E)$, a weight vector $w \in \mathbb{R}^E$ associated with the edges of G and a subset of distinguished nodes $S \subseteq V$, called *terminals*, the *Steiner tree problem* (STP) is to find a minimum weight tree of G spanning S.

The STP arises in VLSI circuit layout design. It has seen a particular attention in the past two decades. It is NP-hard in general. It has been shown to be NP-hard even

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in special classes of graphs as bipartite graphs and grid graphs (see [12]). It can be solved in polynomial time if there is a fixed number of terminals (see [19]). Wald and Colbourn [29] and Cornuéjols et al. [6] show that the problem can be solved in linear time on series–parallel graphs.

In [15] Goemans and Bertsimas give, within the framework of a more general model, a heuristic for the STP with worst-case guarantee. Further heuristics are presented in [31–33,27,28]. In [16] Goemans and Myungt present some formulations for the STP. A survey of the algorithmic aspect of the problem can be found in Winter [30,18]. A recent study of the different formulations and techniques for the STP and its relaxations is presented in [21,22].

The polyhedral aspect of the STP has also been the subject of extensive research. In [4,5] Chopra and Rao discuss the dominant of the Steiner tree polytope in both the directed and undirected cases. In particular, they introduce two classes of facet defining inequalities called odd hole inequalities and Steiner partition inequalities. Moreover, they conjecture that these inequalities, together with the nonnegativity inequalities suffice to completely describe the dominant of the Steiner tree polytope if the underlying graph is a 2-tree.

In this paper we consider that polyhedron. We introduce a new class of valid inequalities that generalizes both classes the odd hole inequalities and the Steiner partition inequalities, and we give sufficient conditions for these inequalities to define facets. We describe some methods with which new facet defining inequalities can be constructed from known ones. Using this we give a counterexample to the conjecture of Chopra and Rao [5]. We also describe the dominant of the Steiner tree polytope and the closely related Steiner connected subgraph polytope on some special classes of graphs. In particular, we show that if the underlying graph is series–parallel and the terminals satisfy certain conditions, then both polyhedra are given by the trivial inequalities and the Steiner partition inequalities.

Related work can be found in [3,11,13,14,20,23,25]. In [23] Prodon et al. Gröflin characterize the dominant of the Steiner tree polytope in the directed case, when the underlying graph is series–parallel. Goemans [13] gives a complete description of the corresponding polytope on that class of graphs. In [14] he considers the vertex-weighted Steiner tree problem, that is when auxiliary variables are associated with the vertices. He completely describes the polytope associated with that problem when the underlying graph is series–parallel. Using projection he describes general classes of facets for the Steiner tree polytope. In [24] Prodon introduces a class of inequalities that can be shown to suffice to completely describe the dominant of the Steiner tree polytope. These inequalities, called *Prodon inequalities*, can also be derived by projection of the directed cut constraints and nonnegativity constraints, and can be separated in polynomial time (see [2,26]). In [20] Margot et al. and Liebling give an extended formulation for the Steiner tree problem and show that it is a complete linear description of the associated polytope when the graph is a 2-tree. Further algorithmic and polyhedral results on the vertex-weighted Steiner tree problem can be found in [25,11,3].

In the next section we introduce a new class of valid inequalities for the dominant of the Steiner tree polytope and we give sufficient conditions for these inequalities to define facets. In Sections 3 and 4 we describe some procedures for constructing facets from facets for the dominant of the Steiner tree polytope and the closely related Steiner connected subgraph polytope. In Section 5 we discuss these two polyhedra in series–parallel graphs. Concluding remarks are given in Section 6. The remainder of this section is devoted to more definitions and notations.

The graphs we consider are finite and undirected. We denote a graph by $G=(V,E)$ where V is the *node set* and E is the *edge set*. If e is an edge with endnodes u and v, then we write $e=(u,v)$.

Given a graph $G=(V,E)$ and a set of terminals $S\subseteq V$, a subgraph of G spanning S is called a *Steiner subgraph*. The vertices not in S are called *Steiner vertices*. A *tree* of G is a subgraph that is connected and acyclic. If $F\subseteq E$ is an edge set that induces a Steiner tree of G, then we also say that F is a Steiner tree. Note that a Steiner tree may be nonminimal.

Let $G=(V,E)$ be a graph. Let $x(e)$ be a variable associated with each edge e. For an edge subset $F\subseteq E$, the 0–1 vector $x^F\in\mathbb{R}^E$ with $x^F(e)=1$ if $e\in F$ and $x^F(e)=0$ if not is called the *incidence vector* of F.

The *Steiner tree polytope* of G, with respect to a set of terminals $S\subseteq V$, denoted by STP(G,S) is the convex hull of the incidence vectors of all the Steiner trees of G. The *dominant* of the STP(G,S) is the polyhedron $\mathrm{DSTP}(G,S)=\mathrm{STP}(G,S)+\mathbb{R}^E_+$.

If $w\geqslant 0$ then the STP is equivalent to solving the linear program $\min\{wx, x\in \mathrm{DSTP}(G,S)\}$.

It is clear that the DSTP(G,S) is full dimensional. Thus for every facet $\mathscr{F}$ of DSTP(G,S), there exists a unique (up to multiplication by a positive constant) valid inequality $a^Tx\geqslant\alpha$ such that $\mathscr{F}=\{x\in\mathrm{DSTP}(G,S)\mid a^Tx=\alpha\}$. Moreover $\mathscr{F}=\mathrm{conv}(X_{\mathscr{F}})+\mathrm{cone}(R_{\mathscr{F}})$ where $X_{\mathscr{F}}=\{x^T\in\mathscr{F}\mid T$ is a Steiner tree of $G\}$ and $R_{\mathscr{F}}=\{x^{\{e\}}\in\mathbb{R}^E\mid a(e)=0\}$. Consequently if $a(e)\neq 0$ for each $e\in E$ and $\mathrm{conv}(X_{\mathscr{F}})\neq\mathrm{STP}(G,S)$, then $\mathscr{F}=\mathrm{conv}(X_{\mathscr{F}})$ is a facet of STP(G,S).

Given a graph $G=(V,E)$ and a set of terminals, $S\subseteq V$, a *Steiner connected subgraph* of G is a subgraph of G spanning S, such that between every two nodes of S there is at least one path. The *Steiner connected subgraph polytope* denoted by SCSP(G,S) is the convex hull of the incidence vectors of all the edge sets of the Steiner connected subgraphs of G. This polytope is closely related to the DSTP(G,S). In fact if $w>0$, then $\min\{wx\mid x\in\mathrm{DSTP}(G,S)\}=\min\{wx\mid x\in\mathrm{SCSP}(G,S)\}$.

Given a graph $G=(V,E)$, an edge whose removal increases the number of connected components is called a *bridge*. A graph is said to be *2-edge connected* if it is connected and does not contain a bridge. Clearly, if G is not 2-edge connected, then the minimum weight Steiner tree (Steiner connected subgraph) of G can be determined by finding the minimum weight Steiner tree (Steiner connected subgraph) in each 2-edge connected component of G. Hence, throughout the paper, we consider 2-edge connected graphs. In consequence, SCSP(G,S) is full dimensional. And from Balas and Fischetti [1] we have the following.

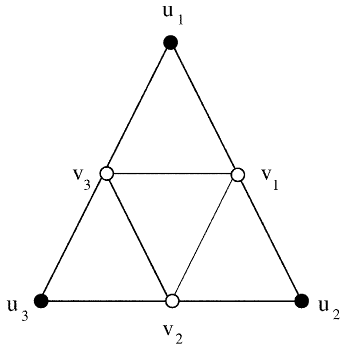

Fig. 1. The graph G_3.

Remark 1.1. A constraint different from $x(e)\geqslant 0$ and $x(e)\leqslant 1$ defines a facet of SCSP(G,S) if and only if it defines a facet of DSTP(G,S).

If $G=(V,E)$ is a graph and $F\subseteq E$, then $V(F)$ denotes the set of nodes of F and $G(F)$ the subgraph of G induced by F. If $W\subseteq V$ then $E(W)$ denotes the set of edges having both nodes in W.

2. Generalized Steiner partition inequalities

In [4] Chopra and Rao introduce a class of inequalities as follows.

Let m be an odd integer ($m\geqslant 3$) and $G_m=(V_m,E_m)$ a graph such that

$$V_m=\{u_1,\dots,u_m,v_1,\dots,v_m\},$$
$$E_m=\{(u_i,v_i),(u_i,v_{i-1}),(v_i,v_{i-1});\ i=1,\dots,m \text{ (modulo } m)\}.$$

Let $S_m=\{u_1,\dots,u_m\}$ be the set of terminals of G_m. The graph G_3 is shown in Fig. 1 where the terminals correspond to the black nodes.

Consider the inequality

$$x(E_m)\geqslant 2(m-1). \tag{2.1}$$

Chopra and Rao [4] show that inequality (2.1) defines a facet of the DSTP(G_m,S_m).

They also introduce lifting procedures that permit to construct facets of DSTP(G,S) from facets of DSTP(G',S') if G is contractible to G' (that is G' can be obtained from G by a sequence of deletions and contractions of edges) and S' is defined from S in an appropriate way. In particular they show the following:

Theorem 2.1 (Chopra and Rao [4]). *Let $G=(V,E)$ be a graph and $S\subseteq V$ a set of terminals. Let $G'=(V',E')$ be obtained from G by contracting an edge $\bar{e}=(u,v)$. Let $S'=(S\setminus\{u,v\})\cup\{w\}$ if $S\cap\{u,v\}\neq\emptyset$, and $S'=S$ if not, where w is the node obtained from the contraction of $\bar{e}$. If*

$$\sum_{e\in E'} a(e)x(e)\geqslant\alpha$$

defines a facet of DSTP(G', S') *then*

$$\sum_{e \in E} a(e)x(e) \geqslant \alpha$$

defines a facet of DSTP(G, S) *where* $a(\bar{e}) = 0$.

Chopra and Rao call inequalities of type (2.1) and those obtained from these inequalities by lifting, *odd hole inequalities*. A second class of facet defining inequalities introduced by Chopra and Rao [4] generalizes the so-called Steiner cut inequalities.

Let $G=(V, E)$ be a graph and $S \subseteq V$ a set of terminals. Let $(V_1, \ldots, V_p)$, $p \geqslant 2$, be a partition of V such that $V_i \cap S \neq \emptyset$ for $i = 1, \ldots, p$. Such a partition is called a *Steiner partition*. If (U, F) is a Steiner tree of G, then x^F, the incidence vector of F, satisfies the inequality

$$x(\delta(V_1, \ldots, V_p)) \geqslant p - 1, \tag{2.2}$$

where $\delta(V_1, \ldots, V_p)$ denotes the set of edges having nodes in different members of the partition. Thus inequality (2.2) is valid for the DSTP(G, S). Chopra and Rao [4] give sufficient conditions for such an inequality to be facet defining. Inequalities (2.2) are called *Steiner partition inequalities*.

Chopra and Rao introduce in [5] further classes of facet defining inequalities for the DSTP(G, S), namely the wheel, bipartite and anti-hole inequalities. In what follows we introduce a large class of valid inequalities for the DSTP(G, S) that generalizes these inequalities as well as the odd hole inequalities (2.1) and the Steiner partition inequalities. And we describe special cases in which these inequalities define facets. But first we state the following lemmas which are easy to prove. The first one can also be obtained as a consequence of a result of Balas and Fischetti [1].

Lemma 2.2. *Every facet defining inequality of* DSTP(G, S) *is of the form* $\sum_{e \in E} a(e) x(e) \geqslant \alpha$ *with* $a(e) \geqslant 0$ *for all* $e \in E$.

Lemma 2.3. *Let* $a^T x \geqslant \alpha$ *be a constraint where* $a(e) \geqslant 0$ *for all* $e \in E$. *If* $a^T x \geqslant \alpha$ *is valid for* STP(G, S), *then it is valid for* DSTP(G, S).

Let $G=(V, E)$ be a graph and $S \subseteq V$ a set of terminals. Let $\pi=(V_1, \ldots, V_p)$, $p \geqslant 2$, be a partition of V. Suppose that at least one of the sets V_i intersects S. Let $0 \leqslant r \leqslant p-1$ be the number of elements V_i such that $V_i \cap S = \emptyset$. Hence

$$V_i \cap S \neq \emptyset \quad \text{for } i = r+1, \ldots, p.$$

Let $G_\pi = (V_\pi, E_\pi)$ be the graph obtained by contracting $V_1, \ldots, V_p$. Let $w_1, \ldots, w_p$ be the nodes that arise from the contractions of $V_1, \ldots, V_p$, respectively. Let $S_\pi = \{w_i;\ i = r+1, \ldots, p\}$. Let

$$d = \max\{|U| \mid U \subseteq V_\pi \setminus S_\pi \text{ and } G_\pi \setminus U \text{ is Steiner connected}\},$$

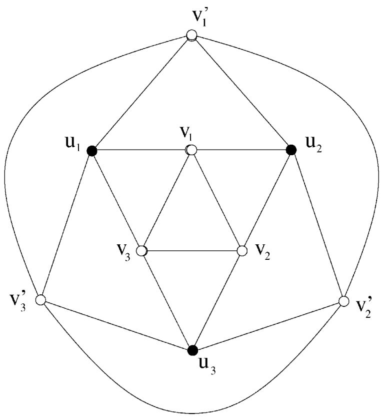

Fig. 2. The graph $\bar{G}_3$.

where $G_\pi \setminus U$ is the graph obtained from G_π by deleting the nodes of U and all the edges having nodes in U. Consider the inequality

$$x(\delta(V_1,\dots,V_p)) \geqslant p - d - 1. \tag{2.3}$$

Note that the right-hand side of (2.3) is nothing but the minimum cardinality of a Steiner tree in G_π. Hence, computing d is an NP-hard problem. Moreover we have the following.

Theorem 2.4. *Inequality* (2.3) *is valid for* DSTP(G,S).

We call inequalities of type (2.3) *generalized Steiner partition inequalities*. A Steiner partition inequality (2.2) corresponds to the case where $r = 0$ (and $d = 0$), and an odd hole inequality (2.1) corresponds to a generalized Steiner partition inequality with $p = 2m$ (and $d = 1$), that is when the elements of the partition correspond to the nodes of the graph. It can also be easily seen that the so-called wheel, bipartite and anti-hole inequalities introduced by Chopra and Rao [5] correspond to generalized Steiner partition inequalities.

In what follows, we give some sufficient conditions for generalized Steiner partition inequalities to define facets for DSTP(G,S).

Let m be an odd integer ($m \geqslant 3$) and $\bar{G}_m = (\bar{V}_m, \bar{E}_m)$ the graph such that

$$\bar{V}_m = \{u_1,\dots,u_m, v_1,\dots,v_m, v'_1,\dots,v'_m\},$$
$$\bar{E}_m = \{(u_i,v_i),(u_i,v_{i-1}),(v_i,v_{i-1}),(u_i,v'_i),(u_i,v'_{i-1}),(v'_i,v'_{i-1});\ i = 1,\dots,m \text{ (modulo } m)\}.$$

Let $\bar{S}_m = \{u_1,\dots,u_m\}$ be the set of terminals of $\bar{G}_m$. The graph $\bar{G}_3$ is shown in Fig. 2.

Consider the inequality

$$x(\bar{E}_m) \geqslant 2(m-1). \tag{2.4}$$

It is not hard to see that inequality (2.4) corresponds to the generalized Steiner partition inequality where $p=3m$ (and $d=m+1$). Moreover this inequality corresponds neither to an odd hole inequality nor to a Steiner partition inequality.

We have the following.

Theorem 2.5. *Inequality* (2.4) *is facet defining for* $\mathrm{DSTP}(\bar{G}_m, \bar{S}_m)$.

Proof. Let G_m^1 and G_m^2 be the subgraphs of $\bar{G}_m$ induced by $\{u_1,\dots,u_m,v_1,\dots,v_m\}$ and $\{u_1,\dots,u_m,v'_1,\dots,v'_m\}$, respectively. Note that G_m^1 and G_m^2 are copies of G_m. Also note that $\bar{G}_m$ is the graph obtained from G_m^1 and G_m^2 by identifying $u_1,\dots,u_m$.

As inequality (2.1) defines a facet of $\mathrm{DSTP}(G_m^1, \bar{S}_m)$ (resp. $\mathrm{DSTP}(G_m^2, \bar{S}_m)$) there are $3m$ affinely independent feasible points $x_1,\dots,x_{3m}$ of $\mathrm{DSTP}(G_m^1, \bar{S}_m)$ (resp. $y_1,\dots,y_{3m}$ of $\mathrm{DSTP}(G_m^2, \bar{S}_m)$) satisfying (2.1) with equality. The $6m$ points $(x_1,0),\dots,(x_{3m},0)$, $(0,y_1),\dots,(0,y_{3m})$ are easily seen to be affinely independent and satisfy (2.4) with equality. □

Let $G=(V,E)$ be a graph and let $S\subseteq V$ be a set of terminals. Let $\bar{G}=(\bar{V},\bar{E})$ be a graph obtained from G by contracting a set of edges F inducing a connected subgraph of G. Let $\bar{S}=S$ if $S\cap V(F)=\emptyset$ and $\bar{S}=S\cup\{w\}$ if not where w is the node that arises from the contraction of F.

By Theorems 2.1 and 2.5 it thus follows that a generalized Steiner partition inequality defines a facet of $\mathrm{DSTP}(G,S)$ if a graph of type $\bar{G}_m$ can be obtained from G by a sequence of contractions.

3. Construction of facets for DSTP(*G*, *S*)

In this section we give a procedure of construction of facets from known ones for $\mathrm{DSTP}(G,S)$. This procedure will be used to give a counterexample to the conjecture of Chopra and Rao [5] (see also [14]) concerning the $\mathrm{DSTP}(G,S)$ on 2-trees.

Theorem 3.1. *Let $G=(V,E)$ be a graph and $S\subseteq V$ a set of terminals. Let $E=E^*\cup\{f\}$ where $f=(v_1,v_2)$. Let*

$$\sum_{e\in E^*} a(e)x(e)+a(f)x(f)\geqslant\alpha \tag{3.1}$$

be a nontrivial facet defining inequality of $\mathrm{DSTP}(G,S)$.

Let $G'=(V',E')$ be the graph obtained from G by replacing the edge f by a path f_1, f_2 with $f_1=(v_0,v_1)$ and $f_2=(v_0,v_2)$ where v_0 is a new node (*see Fig.* 3). *Let $S'=S\cup\{v_0\}$. If $v_1\in S$ and $v_2\notin S$, then the inequality*

$$\sum_{e\in E^*} a(e)x(e)+(\alpha_0-\alpha)x(f_1)+a(f)x(f_2)\geqslant\alpha_0, \tag{3.2}$$

Fig. 3. Adding a node.

defines a facet of DSTP(G',S'), *where*

$$\alpha_0 = \min\left\{ \sum_{e\in E^*} a(e)x(e) + a(f)x(f_2) \,|\, x \in \mathrm{DSTP}(G' - f_1, S') \right\}.$$

Proof. It is not hard to see that (3.2) is valid for DSTP(G',S'). In what follows we show that it defines a facet.

Since (3.1) defines a facet of DSTP(G,S), there exist $m{=}|E|$ solutions of DSTP(G,S), $x_1,\ldots,x_m$ that satisfy (3.1) with equality and are linearly independent. Let $x'_1,\ldots,x'_m \in \mathbb{R}^{E'}$ be the solutions such that $x'_i = (x_i^*, 1, x_i(f))$, for $i = 1,\ldots,m$, where x_i^* is the restriction of x_i on E^*. And let x'_{m+1} be a solution of DSTP$(G' - f_1, S')$ that realizes the minimum defining α_0. Clearly, the solutions $x'_1,\ldots,x'_{m+1}$ all belong to DSTP(G',S'). Let A be the matrix whose columns are $x_1,\ldots,x_m$. Let B be the matrix whose columns are given by $x'_1,\ldots,x'_{m+1}$. Hence B can be written as

$$B = \begin{pmatrix} A & b \\ 1\ldots1 & 0 \end{pmatrix}.$$

where the last row of B corresponds to f_1 and $x'_{m+1} = (b, 0)$. If the last row of B depends of the m first ones, there must exist $y \in \mathbb{R}^m$ such that $y^{\mathrm{T}}A = (1,\ldots,1)$ and $y^{\mathrm{T}}b = 0$. As A is nonsingular, one should have $y(e) = a(e)/\alpha$ for all $e \in E^*$ and $y(f_2) = a(f)/\alpha$. But this implies that $y^{\mathrm{T}}b = \alpha_0 \neq 0$, a contradiction.

In consequence, B is nonsingular and thus $x'_1,\ldots,x'_{m+1}$ are linearly independent. Moreover they satisfy (3.2) with equality. Hence (3.2) defines a facet of DSTP(G',S'). □

Now using Theorem 3.1 we are going to give a counterexample to the conjecture of Chopra and Rao on the DSTP(G,S) in 2-trees. A 2-*tree* is a graph obtained recursively from a triangle using the following operation: add a new node w and two new edges (w, v_1) and (w, v_2) where (v_1, v_2) is an edge. In [5] Chopra and Rao have conjectured that if G is a 2-tree, then the polyhedron DSTP(G,S) is completely described by the nonnegativity inequalities, the Steiner partition inequalities and the odd hole inequalities. Unfortunately this conjecture does not hold. In fact consider the graph $G'_1 = (V'_1, E'_1)$ shown in Fig. 4 with $S'_1 = \{s_1, s_2, s_3\}$ its set of terminals.

Note that G'_1 is an odd hole with two pairs of parallel edges. The inequality

$$\sum_{e\in E'_1} x(e) \geqslant 4$$

is facet defining for DSTP(G'_1, S'_1) (see [4]).

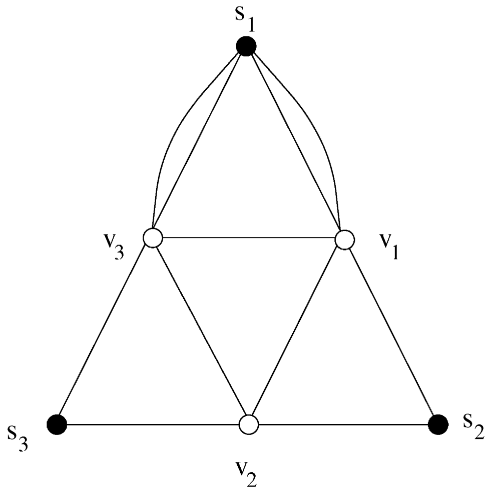

Fig. 4. The graph G_1'.

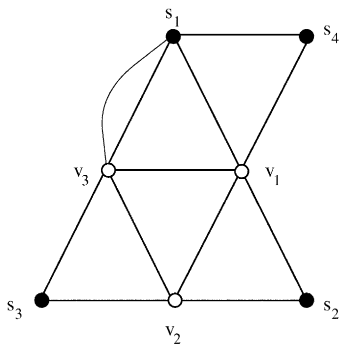

Fig. 5. The graph G_2'.

By adding a new terminal s_4 on one of the edges between s_1 and v_1 and applying the procedure described in Theorem 3.1, we obtain that the inequality

$$\sum_{e \in E_2'} x(e) \geqslant 5$$

defines a facet of DSTP(G_2', S_2') where G_2' is the graph of Fig. 5.

By adding a new terminal s_5 on one of the edges between s_1 and v_3, we obtain similary that the inequality

$$\sum_{e \in E_3'} x(e) \geqslant 6 \tag{3.3}$$

is facet defining for DSTP(G_3', S_3') where G_3' is the graph of Fig. 6.

Inequality (3.3) is a generalized Steiner partition inequality which is different from both a Steiner partition inequality and an odd hole inequality. This implies that the conjecture of Chopra and Rao [5] does not hold.

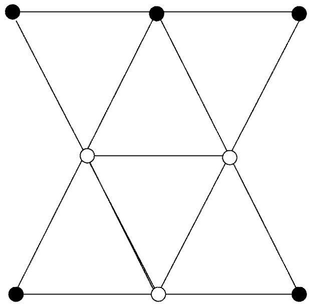

Fig. 6. The graph G_3'.

In the rest of the paper we will consider the polytope SCSP(G,S). We will describe three procedures of facet construction for the SCSP(G,S) and give a complete description of this polytope when the underlying graph is series–parallel and the terminals satisfy certain conditions. This will enable us to give a similar description for the DSTP(G,S).

4. Construction of facets for SCSP(*G, S*)

In this section we give three procedures that permit to construct facets from facets for the SCSP(G,S). The first one is similar to that of Theorem 3.1. A detailed proof is given in [9].

Theorem 4.1. *Let $G=(V,E)$ be a graph and $S\subseteq V$ a set of terminals. Let $E=E^*\cup\{f\}$ where $f=(v_1,v_2)$. Let*

$$\sum_{e\in E^*} a(e)x(e)+a(f)x(f)\geqslant\alpha \tag{4.1}$$

be a nontrivial facet defining inequality of SCSP(G,S). *Let $G'=(V',E')$ be the graph obtained from G by replacing the edge f by a path f_1, f_2 with $f_1=(v_0,v_1)$ and $f_2=(v_0,v_2)$ where v_0 is a new node (see Fig. 3). Suppose that $\{v_1,v_2\}\subseteq S$. Let $S'=S\cup\{v_0\}$. Then the inequality*

$$\sum_{e\in E^*} a(e)x(e)+a(f)x(f_1)+a(f)x(f_2)\geqslant\alpha+a(f) \tag{4.2}$$

defines a facet of SCSP(G',S').

Now before describing the two other operations, we first give a technical lemma.

Lemma 4.2. *Let $G=(V,E)$ be a graph and $S\subseteq V$ a set of terminals. Suppose there exists a node $v_0\in V$ such that $\delta(v_0)=\{f_1,f_2\}$ where $f_1=(v_0,v_1)$ and $f_2=(v_0,v_2)$.*

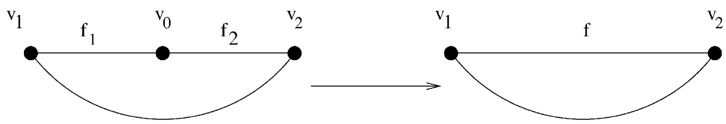

Fig. 7. Contracting an edge.

Let

$$\sum_{e\in E\setminus\{f_1,f_2\}} a(e)x(e)+a(f_1)x(f_1)+a(f_2)x(f_2)\geqslant\alpha \tag{4.3}$$

be a nontrivial facet defining inequality of SCSP(G,S).

(a) *If* $v_0\notin S$ *then* $a(f_1).a(f_2)=0$.

(b) *Suppose* $v_0\in S$.

(b.1) *If* $v_1\in S$ *and* $v_2\in S$, *then* $a(f_1)=a(f_2)$.

(b.2) *If* $v_1\in S$ *and* $v_2\notin S$, *then* $a(f_1)\geqslant a(f_2)$.

Proof. (a) Since (4.3) defines a facet of SCSP(G,S), there exists an edge set F inducing a Steiner connected subgraph of G such that $|F\cap\{f_1,f_2\}|=1$ and x^F satisfies (4.3) with equality. Indeed, if this is not the case, then (4.3) would be a multiple of the equation $x(f_1)-x(f_2)=0$. As the last equation contains negative coefficients, this contradicts Lemma 2.2. W.l.o.g. we may assume that $f_1\in F$ and $f_2\notin F$. Let $F'=F\setminus\{f_1\}$. Since $v_0\notin S$, F' induces a Steiner connected subgraph of G. This implies that $a(f_1)=0$.

(b) Suppose $v_0\in S$. We will show (b.1) (the proof of (b.2) is similar). (b.1) Suppose that, for instance, $a(f_1)>a(f_2)$. Since (4.3) is different from a nontrivial inequality, there exists an edge set $\tilde F\subseteq E$ such that $f_2\notin\tilde F$ and $x^{\tilde F}$ satisfies (4.3) with equality. Since $v_0\in S$, it follows that $f_1\in\tilde F$. Let $\tilde F'=(\tilde F\setminus\{f_1\})\cup\{f_2\}$. Clearly, $\tilde F'$ induces a Steiner connected subgraph of G. However $a^{\mathrm T}x^{\tilde F'}<\alpha$, which is a contradiction. □

Our second procedure for SCSP(G,S) consists of replacing a path of length two by an edge.

Theorem 4.3. *Let* $\bar G=(\bar V,\bar E)$ *be a graph and* $\bar S\subseteq\bar V$ *a set of terminals. Suppose* $\bar E=E^*\cup\{f_1,f_2\}$ *where* $f_1=(v_0,v_1)$, $f_2=(v_0,v_2)$ *and* $\delta(v_0)=\{f_1,f_2\}$ (*see* Fig. 7). *Let*

$$\sum_{e\in E^*} a(e)x(e)+a(f_1)x(f_1)+a(f_2)x(f_2)\geqslant\alpha \tag{4.4}$$

be a nontrivial facet defining inequality of SCSP$(\bar G,\bar S)$.

Let $G=(V,E)$ *be the graph obtained from* $\bar G$ *by removing the node* v_0 *and the edges* f_1 *and* f_2, *and adding a new edge* f *between* v_1 *and* v_2. *Let* $S=\bar S\setminus\{v_0\}$.

(a) *If* $S=\bar{S}$ *then the inequality*

$$\sum_{e\in E^*} a(e)x(e)+a(f)x(f)\geqslant \alpha \tag{4.5}$$

defines a facet of SCSP(G,S), *where* $a(f)=\max\{a(f_1),a(f_2)\}$.

(b) *Suppose* $S\neq\bar{S}$ *and* $v_1\in\bar{S}$. *Then the inequality*

$$\sum_{e\in E^*} a(e)x(e)+a(f_2)x(f)\geqslant \alpha-a(f_1) \tag{4.6}$$

defines a facet of SCSP(G,S).

Proof. We will show (a) (the proof for (b) is similar).

First we show the validity of (4.5). By Lemma 4.2(a) we have $a(f_1)a(f_2)=0$. W.l.o.g we may assume that $a(f_2)=0$. Hence $a(f)=a(f_1)$. Let $F\subseteq E$ be an edge subset inducing a Steiner connected subgraph of G. Let

$$\bar{F}=\begin{cases}(F\setminus\{f\})\cup\{f_1,f_2\} & \text{if } f\in F,\\ F & \text{otherwise.}\end{cases}$$

Since $v_0\notin S$, $\bar{F}$ induces a Steiner connected subgraph of $\bar{G}$. Moreover we have

$$\sum_{e\in E^*} a(e)x^F(e)+a(f)x^F(f)=\sum_{e\in E^*} a(e)x^{\bar{F}}(e)+a(f_1)x^{\bar{F}}(f_1)+a(f_2)x^{\bar{F}}(f_2)\geqslant\alpha,$$

which implies that the constraint (4.5) is valid for SCSP(G,S).

Let

$$\mathscr{F}=\left\{x\in \text{SCSP}(G,S)\;\middle|\;\sum_{e\in E^*} a(e)x(e)+a(f)x(f)=\alpha\right\}.$$

Notice that $\mathscr{F}$ is a proper face of SCSP(G,S), that is $\emptyset\neq\mathscr{F}\cap\text{SCSP}(G,S)\neq$ SCSP(G,S). Let $\bar{\mathscr{F}}$ be the facet of SCSP$(\bar{G},\bar{S})$ defined by (4.4). Suppose that $\mathscr{F}$ is not a facet of SCSP(G,S). Then there exists a facet $\mathscr{F}_1$ of SCSP(G,S) such that $\mathscr{F}\subset\mathscr{F}_1$. Suppose that $\mathscr{F}_1$ is defined by $\sum_{e\in E^*}a'(e)x(e)+a'(f)x(f)\geqslant\alpha'$. Since $S=\bar{S}$, by Theorem 2.1 it follows that the inequality

$$\sum_{e\in E^*} a'(e)x(e)+a'(f)x(f_1)\geqslant\alpha',$$

defines a facet $\bar{\mathscr{F}}_1$ of SCSP$(\bar{G},\bar{S})$. In what follows we are going to show that $\bar{\mathscr{F}}\subset\bar{\mathscr{F}}_1$. Let $\tilde{F}\subseteq\bar{E}$ be an edge set that induces a Steiner connected subgraph of $\bar{G}$ such that $x^{\tilde{F}}\in\bar{\mathscr{F}}$. Then

$$\sum_{e\in E^*} a(e)x^{\tilde{F}}(e)+a(f_1)x^{\tilde{F}}(f_1)=\alpha.$$

Let $F^*=(\tilde{F}\setminus\{f_1,f_2\})\cup\{f\}$ if $f_1\in\tilde{F}$ and $F^*=\tilde{F}\setminus\{f_2\}$ if not. The set F^* induces a Steiner connected subgraph of G such that $x^{F^*}\in\mathscr{F}$. As $\mathscr{F}\subset\mathscr{F}_1$, it follows that

$$\sum_{e\in E^*} a'(e)x^{F^*}(e)+a'(f)x^{F^*}(f)=\alpha'.$$

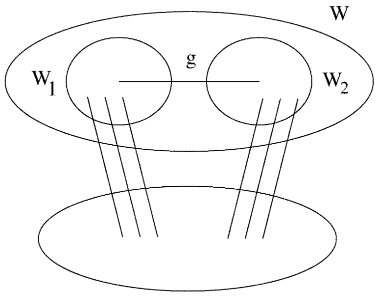

Fig. 8. Splitting a node.

Consequently, we obtain that

$$\sum_{e\in E^*} a'(e)x^{\tilde{F}}(e) + a'(f)x^{\tilde{F}}(f_1) = \alpha'.$$

Hence $x^{\tilde{F}} \in \bar{\mathscr{F}}_1$, and therefore $\bar{\mathscr{F}} \subseteq \bar{\mathscr{F}}_1$.

Since $\mathscr{F} \subset \mathscr{F}_1$, there exists an edge set $T \subseteq E$ such that $x^{\mathrm{T}} \in \mathscr{F}_1$ and $x^{\mathrm{T}} \notin \mathscr{F}$. Thus

$$\sum_{e\in E^*} a'(e)x^{\mathrm{T}}(e) + a'(f)x^{\mathrm{T}}(f) = \alpha' \quad \text{and} \quad \sum_{e\in E^*} a(e)x^{\mathrm{T}}(e) + a(f)x^{\mathrm{T}}(f) > \alpha.$$

Let $\bar{T} = (T \setminus \{f\}) \cup \{f_1, f_2\}$ if $f \in T$ and $\bar{T} = T \cup \{f_2\}$ if not. Obviously, $\bar{T}$ induces a Steiner connected subgraph of $\bar{G}$. Moreover $x^{\bar{T}} \in \bar{\mathscr{F}}_1 \setminus \bar{\mathscr{F}}$. Thus $\bar{\mathscr{F}} \subset \bar{\mathscr{F}}_1$. But this contradicts the fact that $\bar{\mathscr{F}}$ is a facet of SCSP($\bar{G}, \bar{S}$). □

Our third operation consists of splitting a node into two sets of nodes inducing 2-edge connected subgraphs.

Let $G = (V, E)$ be a graph and $S \subseteq V$ a set of terminals. Let $W \subset V$ be a node subset such that

$$W = W_1 \cup W_2,$$
$$W_1 \cap W_2 = \emptyset,$$
$$W_i \cap S \neq \emptyset, \quad i = 1, 2,$$
$$G[W_i] \text{ is 2-edge connected}, \quad i = 1, 2.$$

Suppose $[W_1, W_2] = \{g\}$ (see Fig. 8).

Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph obtained from G by contracting W. Denote by w_0 the node that arises from the contraction of W. Let $\bar{S} = (S \setminus W) \cup \{w_0\}$. We have the following result, for the detailed proof see [9].

Theorem 4.4. *Let* $\bar{a}^{\mathrm{T}}x \geqslant \bar{\alpha}$ *be a facet defining inequality of* SCSP($\bar{G}, \bar{S}$). *Set*

$$a(e) = \bar{a}(e) \quad if\ e \in \bar{E},$$
$$a(e) = 0 \quad if\ e \in E(W) \setminus \{g\},$$

$$a(g) = \alpha - \bar{\alpha},$$

$$\alpha = \min \left\{ \sum_{e \in \bar{E}} \bar{a}(e)x(e) \mid x \in \text{SCSP}(G - g, S) \right\}.$$

Then the inequality $a^{\mathrm{T}}x \geqslant \alpha$ *defines a facet of* SCSP(G,S).

5. SCSP(*G, S*) and DSTP(*G, S*) on series–parallel graphs

In this section we discuss SCSP(G,S) and DSTP(G,S) in series–parallel graphs. We give complete linear descriptions of SCSP(G,S) and DSTP(G,S) in that class of graphs when the terminals satisfy some conditions.

A connected graph is called *series–parallel* [10] if it can be obtained by a recursive application of the following operations starting from the graph consisting of two nodes joined by an edge:

(θ_1) duplicate an edge (i.e. add an edge joining the same endnodes),

(θ_2) subdivide an edge (i.e. replace an edge (u,v) by two edges (u,w) and (w,v), where w is a new node of degree 2).

Let $G = (V,E)$ be a 2-edge connected series–parallel graph and $S \subseteq V$ a set of terminals. We say that the set of terminals S verifies the *property P* (with respect to G), if G can be obtained by operations θ_1 and θ_2 in such a way that if a terminal is added between two nodes v_1 and v_2 by the operation θ_2, then at least one of nodes v_1 and v_2 is a terminal.

In what follows we are going to show that if a series–parallel graph is 2-edge connected and verifies property P, then the trivial inequalities together with the Steiner partition inequalities suffice to describe the SCSP(G,S). As a consequence, we obtain that in this case the DSTP(G,S) is given by the nonnegativity inequalities and the Steiner partition inequalities. For this, we first give a lemma which will be useful in the sequel. Its proof is omitted because it is similar to that of Theorem 4.3.

Lemma 5.1. *Let* $G=(V,E)$ *be a* 2*-node connected graph and* $S \subseteq V$ *a set of terminals. Let* f *and* g *be two multiple edges. Let*

$$\sum_{e \in E} x(e) \geqslant \alpha$$

be a facet defining inequality of SCSP(G,S). *Then the inequality*

$$\sum_{e \in E \setminus \{f\}} x(e) \geqslant \alpha \left(\text{resp.} \sum_{e \in E \setminus \{g\}} x(e) \geqslant \alpha \right),$$

defines a facet of SCSP$(G - f, S)$ (*resp.* SCSP$(G - g, S)$).

Theorem 5.2. *If $G=(V,E)$ is a 2-edge connected series–parallel graph and $S\subseteq V$ a set of terminals verifying the property P, then* SCSP(G,S) *is completely characterized by the trivial and the Steiner partition inequalities.*

Proof. The proof is by induction on $|E|$. It is easy to see that the theorem holds if G is a 2-edge connected series–parallel graph verifying the property P and having at most 4 nodes. So suppose that the theorem holds for every graph with no more than m edges and suppose that G has exactly $m+1$ edges. Let $a^{\mathrm{T}}x\geqslant\alpha$ be an inequality that defines a facet $\mathscr{F}$ of SCSP(G,S). Suppose that $a^{\mathrm{T}}x\geqslant\alpha$ is different from both a trivial inequality and a Steiner cut inequality. We will show that $a^{\mathrm{T}}x\geqslant\alpha$ is precisely a Steiner partition inequality.

Let us first examine the case where the last operation in the construction of G consists of adding a node v_0 (operation θ_2). Let $\delta(v_0)=\{f_1,f_2\}$ where $f_1=(v_0,v_1)$ and $f_2=(v_0,v_2)$. Let $G'=(V',E')$ be the graph obtained from G by removing the node v_0 and replacing the edges f_1 and f_2 by an edge f between v_1 and v_2. Let $S'=S\setminus\{v_0\}$. We consider three cases.

Case 1. $v_0\notin S$. By Theorem 4.3(a) the inequality

$$\sum_{e\in E'\setminus\{f\}} a(e)x(e)+a(f)x(f)\geqslant\alpha \tag{5.1}$$

defines a facet $\mathscr{F}'$ of SCSP(G',S') where $a(f)=\max\{a(f_1),a(f_2)\}$. By Lemma 4.2(a) we have $a(f_1).a(f_2)=0$. W.l.o.g. we may suppose that $a(f_2)=0$ and therefore $a(f)=a(f_1)$. Since $a^{\mathrm{T}}x\geqslant\alpha$ is different from both a trivial inequality and a Steiner cut inequality, (5.1) so is. Since $|E'|<|E|$, by the induction hypothesis, it follows that (5.1) is a Steiner partition inequality. Thus there exists a Steiner partition $V_1',\dots,V_p'$ of V' and a positive scalar β such that

$$\begin{cases} a(e)=\beta & \text{if } e\in\delta(V_1',\dots,V_p'),\\ a(e)=0 & \text{if } e\notin\delta(V_1',\dots,V_p'),\\ \alpha=\beta(p-1). \end{cases} \tag{5.2}$$

W.l.o.g. we may suppose that $v_2\in V_1'$. Let $V_1\dots,V_p$ be the Steiner partition of V given by

$$V_1=V_1'\cup\{v_0\},$$
$$V_i=V_i',\quad i=2,\dots,p.$$

If $x\in\mathscr{F}$ then by (5.2) we have that

$$\beta\sum_{e\in\delta(V_1,\dots,V_p)} x(e)=\beta(p-1),$$

and thus

$$\sum_{e\in\delta(V_1,\dots,V_p)} x(e)=p-1.$$

Since SCSP(G,S) is full dimensional, this implies that $a^{T}x\geqslant\alpha$ is a positive multiple of the Steiner partition inequality $\sum_{e\in\delta(V_1,\dots,V_p)}x(e)\geqslant p-1$.

Case 2. $v_0, v_1, v_2 \in S$. By Lemma 4.2(b.1) we have $a(f_1) = a(f_2) = \delta$, and by Theorem 4.3(b) the inequality

$$\sum_{e \in E' \setminus \{f\}} a(e)x(e) + \delta x(f) \geqslant \alpha - \delta$$

defines a facet $\mathscr{F}'$ of SCSP(G', S'). As we did for Case 1, we can show that the facet $\mathscr{F}'$ is defined by a Steiner partition constraint. Hence there exists a Steiner partition $V_1', \ldots, V_p'$ of V' and a positive scalar β such that

$$\begin{cases} a(e) = \beta & \text{if } e \in \delta(V_1', \ldots, V_p'), \\ a(e) = 0 & \text{if } e \notin \delta(V_1', \ldots, V_p'), \\ \alpha - \delta = \beta(p-1). \end{cases} \tag{5.3}$$

Suppose that $a(f) = \delta > 0$ (the case where $a(f) = \delta = 0$ is similar). Thus $f \in \delta(V_1', \ldots, V_p')$. Let $V_1, \ldots, V_{p+1}$ be the Steiner partition such that

$$V_i = V_i', \quad i = 1, \ldots, p,$$
$$V_{p+1} = \{v_0\}.$$

If $x \in \mathscr{F}$, from (5.3) it follows that

$$\beta \sum_{e \in \delta(V_1, \ldots, V_{p+1})} x(e) = \beta + \beta(p-1).$$

Thus

$$\sum_{e \in \delta(V_1, \ldots, V_{p+1})} x(e) = p.$$

Consequently $a^{\mathrm{T}}x \geqslant \alpha$ is a positive multiple of the inequality $\sum_{e \in \delta(V_1, \ldots, V_{p+1})} x(e) \geqslant p$.

Case 3. $v_0, v_1 \in S$, $v_2 \notin S$ (*the case* $v_0, v_2 \in S$, $v_1 \notin S$ *is similar*). By Lemma 4.2 (b.2) we have $a(f_1) \geqslant a(f_2)$ and by Theorem 4.3(b), the inequality

$$\sum_{e \in E' \setminus \{f\}} a(e)x(e) + a(f_2)x(f) \geqslant \alpha - a(f_1)$$

defines a facet $\tilde{\mathscr{F}}$ of SCSP(G', S'). As we did before, we can show that the facet $\tilde{\mathscr{F}}$ is defined by a Steiner partition constraint. Hence there exists a Steiner partition $V_1', \ldots, V_p'$ of V' and a positive scalar β such that

$$\begin{cases} a(e) = \beta & \text{if } e \in \delta(V_1', \ldots, V_p'), \\ a(e) = 0 & \text{if } e \notin \delta(V_1', \ldots, V_p'), \\ \alpha - a(f_1) = \beta(p-1). \end{cases} \tag{5.4}$$

We distinguish two cases.

Case 3.1. $v_1 \in V_i'$ *and* $v_2 \in V_j'$ $(i \neq j)$. This implies that $a(f_2) > 0$. Suppose, w.l.o.g., that $v_1 \in V_1'$ and $v_2 \in V_2'$. Let $V_1, \ldots, V_{p+1}$ be the Steiner partition of V such that

$$V_i = V_i', \quad i = 1, \ldots, p,$$
$$V_{p+1} = \{v_0\}.$$

Let $x \in \mathscr{F}$. Then by (5.4) it follows that

$$\beta \sum_{e \in \delta(V_1,\dots,V_{p+1})} x(e) = \beta + \beta(p-1).$$

Hence

$$\sum_{e \in \delta(V_1,\dots,V_{p+1})} x(e) = p.$$

This implies that $a^{\mathrm{T}}x \geqslant \alpha$ is a positive multiple of the Steiner partition inequality

$$\sum_{e \in \delta(V_1,\dots,V_{p+1})} x(e) \geqslant p.$$

Case 3.2. $v_1, v_2 \in V_1'$. Then $a(f_2) = 0$ (since $f \notin \delta(V_1',\dots,V_p')$). If $a(f_1) = 0$, then let $V_1,\dots,V_p$ be the partition of V such that

$$V_1 = V_1' \cup \{v_0\},$$
$$V_i = V_i', \quad i = 2,\dots,p.$$

If $x \in \mathscr{F}$, by (5.4) we have

$$\beta \sum_{e \in \delta(V_1,\dots,V_p)} x(e) = \beta(p-1) + a(f_1)$$
$$= \beta(p-1).$$

Hence $a^{\mathrm{T}}x \geqslant \alpha$ is a positive multiple of the Steiner partition inequality

$$\sum_{e \in \delta(V_1,\dots,V_p)} x(e) \geqslant (p-1).$$

Now suppose that $a(f_1) > 0$. We claim that there exists a partition (W_1, W_2) of V_1' such that $[W_1, W_2] = \{f\}$. In fact, suppose the contrary, and let C be a path in $G(V_1')$ between v_1 and v_2 such that f does not belong to C. As C is containend in the edge set induced by V_1', we have $a(e) = 0$ for all $e \in C$. Since $\mathscr{F}$ is a non-trivial facet, there exists an edge subset $F \subseteq E$ inducing a Steiner connected subgraph of G such that $x^F \in \mathscr{F}$ and $f_1 \in F$. Let $F' = (F \setminus \{f_1\}) \cup (C \cup \{f_2\})$. Since F' induces a Steiner connected subgraph of G, it follows that $a(f_1) = 0$, a contradiction.

Hence there exists a partition (W_1, W_2) of V_1' such that $[W_1, W_2] = \{f\}$. W.l.o.g. we may suppose that $v_1 \in W_1$ and $v_2 \in W_2$. Let $V_1,\dots,V_{p+1}$ be the Steiner partition of V such that

$$V_1 = W_1,$$
$$V_2 = W_2 \cup \{v_0\},$$
$$V_{i+1} = V_i', \quad i = 2,\dots,p.$$

We can show as in the previous cases that $a^{\mathrm{T}}x \geqslant \alpha$ is a positive multiple of the Steiner partition inequality $\sum_{e \in \delta(V_1,\dots,V_{p+1})} x(e) \geqslant p$.

Now let us suppose that the last operation in the construction of G consists of adding a parallel edge g between two nodes of V (operation θ_1). By Lemma 5.1, the inequality

$$\sum_{e\in E\setminus\{g\}} x(e)\geqslant\alpha$$

defines a facet of SCSP($G-g$,S). By the induction hypothesis, this facet is a Steiner partition facet. We can show along the same line as we did before that $\sum_{e\in E} a(e)x(e)\geqslant\alpha$ is a Steiner partition inequality, and the proof of our theorem is complete. □

By Remark 1.1 and Theorem 5.2, it follows that if G is a 2-edge connected series-parallel graph and S verifies the property P, then DSTP(G,S) is completely given by the nonnegativity and the Steiner partition inequalities.

6. Concluding remarks

We have studied the dominant of the Steiner tree polytope and have introduced a new class of valid inequalities, the generalized Steiner partition inequalities, that generalizes the so-called odd hole, Steiner partition, wheel, bipartite and anti-hole inequalities. We have described some methods that permit to construct facets from facets for the dominant of the Steiner tree polytope and the closely related Steiner connected subgraph polytope. These methods enabled us to give a counterexample to the conjecture of Chopra and Rao [5]. They have also been used to show that these two polyhedra are given by the trivial inequalities and the Steiner partition inequalities if the underlying graph is 2-edge connected series–parallel and satisfies the property P.

Let Ψ be the class of graphs $G_m=(V_m,E_m)$ introduced in Section 2, that is the graphs $G_m=(V_m,E_m)$ such that

$$V_m=\{u_1,\ldots,u_m,v_1,\ldots,v_m\},$$
$$E_m=\{(u_i,v_i),(u_i,v_{i-1}),(v_i,v_{i-1});i=1,\ldots,m \text{ (modulo } m)\}.$$

for $m\geqslant 3$ and odd. Let $S_m=\{u_1,\ldots,u_m\}$ be the set of terminals of G_m. We also have the following result. For a detailed proof see [9].

Theorem 6.1. *If* $G_m=(V_m,E_m)$ *is a graph of* Ψ and $S_m=\{u_1,\ldots,u_m\}$ *is the set of terminals, then* SCSP(G_m,S_m) *is given by the trivial inequalities, the Steiner partition inequalities and inequality* (2.1).

As inequalities (2.1) are a special case of the generalized Steiner partition inequalities, from Remark 1.1 and Theorem 6.1 it follows that if $G_m=(V_m,E_m)$ is a graph of Ψ and $S_m=\{u_1,\ldots,u_m\}$ is the set of terminals, then DSTP(G_m,S_m) is given by the nonnegativity and the generalized Steiner partition inequalities.

As we have seen, the polyhedron DSTP(G,S) may have generalized Steiner partition facets different from those defined by the odd hole inequalities and the Steiner partition inequalities, when the graph G is a 2-tree. We remark that the generalized Steiner partition inequalities that have been identified, when the graph is a 2-tree, are all produced by graphs contractible to graphs of Ψ. In the lights of this and Theorem 2.1 we give the following conjecture.

Conjecture 6.1. *Let $G=(V,E)$ be a 2-tree and $S \subseteq V$ a set of terminals. If G is noncontractible to a graph of Ψ, then* DSTP(G,S) *is completely described by the nonnegativity inequalities and the Steiner partition inequalities.*

To conclude this section, let us mention that, as the separation problem for the Steiner partition inequalities is NP-hard [17], it seems likely that the separation problem for the generalized Steiner partition inequalities is also NP-hard.

In [17] Grötschel et al. give a heuristic for the separation of the Steiner partition inequalities. In [7] Dahlhaus et al. devise a heuristic that permits to separate partition inequalities such that each element of the partition contains exactly one terminal (see also [8]). Their heuristic, given in connection with the k-cut problem, has a performance guarantee $2(k-1)/k$. That is, it is guaranteed to deliver a k-cut (Steiner partition with exactly one terminal in each element) of weight at most $2(k-1)/k$ times the minimum weight of a k-cut. Here k is the number of terminals. Now it would be interesting to extend these heuristics to the generalized Steiner partition inequalities and to use them in the framework of a cutting plane algorithm to test the utility of these inequalities in solving the Steiner tree problem.

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