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ON THE POLYTOPE OF THE (1,2)-SURVIVABLE NETWORK DESIGN PROBLEM*

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Abstract. This paper deals with the survivable network design problem where each node v has a connectivity type $r(v)$ equal to 1 or 2, and the survivability conditions require the existence of at least $\min\{r(s), r(t)\}$ edge-disjoint paths for all distinct nodes s and t. We consider the polytope given by the trivial and cut inequalities together with the partition inequalities. More precisely, we study some structural properties of this polytope which leads us to give some sufficient conditions for this polytope to be integer in the class of series-parallel graphs. With both separation problems for the cut and partition inequalities being polynomially solvable, we then obtain a polynomial time algorithm for the (1,2)-survivable network design problem in a subclass of series-parallel graphs including the outerplanar graph class. We also introduce a new class of facet-defining inequalities for the polytope associated to the (1,2)-survivable network design problem.

Key words. survivable network, cut and partition inequalities, polytope, facet, series-parallel graphs

AMS subject classifications. 68M10, 90C10, 90C57

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1. Introduction.

In order to protect telecommunication networks from equipment failures, one must maintain the survivability of networks when links are severed or nodes fail. As failures are not very common, robust networks are designed to withstand a single network equipment failure. Moreover, in practice a node usually fails completely because of major incidents (e.g., power outages), and thus it is more frequent to encounter link interface failures or severed links than node failures. Therefore, one of the main concerns when designing telecommunication networks is to devise network topologies that provide protection against single-link failures. The network topology problem is usually the first stage of the overall network design optimization process, and the second one involves traffic and routing issues.

Furthermore, some network nodes may be more important than others because of their specific functions. This fact thus leads to considering two kinds of nodes: the *specific nodes*, also called *terminals*, for which a "high" degree of survivability has to be guaranteed, and the *ordinary nodes*, which simply have to be connected to the network. The network topology problem then consists of selecting links such that the sum of their cost is minimized and the failure of any single link does not disconnect any two terminal nodes.

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More precisely, based on a model first introduced by Grötschel and Monma [8] (see also Stoer [19]), this problem can be stated as follows. Consider an undirected graph $G = (V, E)$ where V represents the node set and E represents the set of edges or potential links. The set V is partitioned into two subsets T and O corresponding, respectively, to the terminal and ordinary node sets. By associating to each node $u \in V$ a *connectivity type* $r(u)$ which is equal to 1 if u is an ordinary node and to 2 if u is a terminal, we have $O = \{u \in V : r(u) = 1\}$, $T = \{u \in V : r(u) = 2\}$, and $V = O \cup T$. A subgraph H of G fulfills the survivability conditions if there exist at least $\min\{r(s), r(t)\}$ edge-disjoint paths (i.e., paths sharing no edges) in H for any pair of nodes $s, t \in V$. Such a subgraph is then called *survivable.* Suppose that each edge $e \in E$ has a certain cost $c(e) \in \mathbb{R}_+$. Our network topology problem, called the (1,2)-*survivable network design problem* (SNDP), then consists of finding a survivable subgraph of G with minimum total cost. (The cost of a subgraph of G is equal to the sum of its edge costs.)

The optimization problem SNDP is NP-hard since it includes as a special case the 2-edge connected network problem (i.e., $r(u) = 2$ for all $u \in V$), which has been extensively studied in the past. Some heuristics have thus been devised such as the one of Monma and Shallcross [18] which was used to obtain near optimal solutions to both real-world and randomly generated problems. The SNDP has also been proved to be polynomially solvable in special cases. Particularly, if $T = \emptyset$ (i.e., $r(u) = 1$ for all $u \in V$, and r is then called a *unit connectivity type vector*), then the SNDP is nothing but the minimum-cost spanning tree problem which is well known to be polynomially solvable [15]. Furthermore, if the underlying graph G is series-parallel and $O = \emptyset$ (i.e., $r(u) = 2$ for all $u \in V$), then we have a linear time algorithm for the SNDP devised by Winter [20]. Many survivability problems related to the SNDP have received particular attention and complete surveys over the existing approaches can be found in Grötschel, Monma, and Stoer [11], Stoer [19], and Kerivin and Mahjoub [13].

Grötschel, Monma, and Stoer [9] studied the general model related to the SNDP from a polyhedral point of view. (They consider $r(u) \in \mathbb{Z}_+$ for all $u \in V$.) They introduced several families of valid inequalities for the polytope associated with this problem. They also derived some necessary and/or sufficient conditions under which these inequalities are facet-defining. Among all of the inequalities considered in [9], the so-called partition inequalities are of interest for solving the SNDP as pointed out in [10, 14]. Kerivin and Mahjoub [12] actually showed that the separation problem for the partition inequalities is polynomially solvable for the SNDP, even though this separation problem is NP-hard for general connectivity type vectors $r \in \mathbb{Z}_+^V$. Furthermore, Grötschel and Monma [8] showed that the partition inequalities together with the trivial lower-bound and upper-bound inequalities suffice to describe the polytope associated with the SNDP when $r(u) = 1$ for all $u \in V$. When the underlying graph G is series-parallel, Mahjoub [17] described the polytope associated with the 2-edge connected network problem by the trivial inequalities and the cut ones, the latter being a special case of the partition inequalities.

Let $\mathrm{SNDP}(G, r)$ be the convex hull of incidence vectors of all survivable subgraphs. This polytope is called the *survivable network polytope.* In this paper, we are interested in the polytope $\mathrm{CPP}(G, r)$ given by the trivial lower-bound and upper-bound inequalities and the so-called partition inequalities for connectivity type vectors $r \in \{1, 2\}^V$. This polytope is a strengthened linear relaxation of $\mathrm{SNDP}(G, r)$. Here, we give sufficient conditions for the $\mathrm{CPP}(G, r)$ to be an integer on series-parallel graphs. This study leads us to give a polynomial-time algorithm for solving the $(1,2)$-survivable network design problem on a class of graphs including the outerplanar ones.

This paper is organized as follows. In the next section, we formulate the problem SNDP as an integer linear program and introduce its associated polytope SNDP(G,r) as well as the polytope CPP(G,r). Section 3 is devoted to the study of some structural properties of the polytope CPP(G,r). In section 4, we consider the CPP(G,r) when G can be decomposed by one-node cutsets. We show in section 5 that the polytope CPP(G,r) may have noninteger extreme points even if the underlying graph G is series-parallel, and give sufficient conditions which make the CPP(G,r) integer on series-parallel graphs. In section 6, we prove this last result. A new class of facet-defining inequalities for the SNDP(G,r) is introduced in section 7. Finally, some concluding remarks are given in section 8.

The rest of this section is devoted to more definitions and notation. Throughout this paper, the graphs are undirected, finite, loopless, and connected. We consider a graph $G=(V,E)$ and denote by n the number of nodes of G, that is, $n=|V|$. For $W \subseteq V$, let $\overline{W}=V\setminus W$, and for $F\subseteq E$, let $\overline{F}=E\setminus F$. Given two distinct nodes u and v of V, an edge between u and v is denoted by uv. For a nonempty node subset $W \subsetneq V$, the set of edges having exactly one endnode in W is called a *cut* and is denoted by $\delta_G(W)$. If $W=\{u\}$, we then write $\delta_G(u)$ for $\delta_G(\{u\})$. A *partition* of V is a collection of disjoint subsets of V with union V. The elements of the partition are called its *classes*. Given a partition $\{V_1,\dots,V_p\}$ of the node set V, we denote by $\delta_G(V_1,\dots,V_p)$ the set of edges with endnodes in two different classes. Given a collection $(W_1,\dots,W_q)$ of node subsets, we write $[W_1,\dots,W_q]_G$ for the set of edges with endnodes in two different subsets. We notice that if $(W_1,\dots,W_q)$ is a partition of V, then we have $[W_1,\dots,W_q]_G=\delta_G(W_1,\dots,W_q)$. If u and v are two distinct nodes of V, we then write $[u,v]_G$ for $[\{u\},\{v\}]_G$. For all of our notation, we don't use the subscript G whenever the graph G can be deduced from the context. For $F\subseteq E$, we denote by $V(F)$ the set of nodes which are spanned by the edges in F. For $W\subseteq V$, we denote by $E(W)$ the set of edges with both endnodes in W, and $G(W)=(W,E(W))$ is called the *subgraph induced by* W. A maximal connected nonempty subgraph of G is called a *connected component*. (Here, "maximal" is taken with respect to inclusion.) A graph G is called 2-*node-connected* if for any node $u\in V$, the subgraph $G-u$ induced by $V\setminus\{u\}$ is connected. Given a ground set S, a set-function $f: 2^S \longrightarrow \mathbb{R}\cup\{\infty\}$ is called *fully submodular* if

$$f(A)+f(B)\geq f(A\cap B)+f(A\cup B) \tag{1.1}$$

for all $A,B\subseteq S$. A pair of subsets A and B of S is said to be *crossing* if none of $A\setminus B$, $B\setminus A$, $A\cap B$, $S\setminus(A\cup B)$ is empty. A set-function f is called *submodular on crossing pairs* if the inequality (1.1) is required only for crossing pairs. Moreover, if f satisfies the inequality (1.1) with equality for crossing pairs, then f is called *modular on crossing pairs*. For a vector $x\in\mathbb{R}^S$ and a subset $A\subseteq S$, we denote $\sum_{a\in A}x(a)$ by $x(A)$. For $F\subseteq S$, its *incidence vector* $x^F\in\mathbb{R}^S$ is defined by $x^F(e)=1$ if $e\in F$, and $x^F(e)=0$ if $e\in S\setminus F$. An *integer vector* is a vector with all entries integer. A polytope $P\subseteq\mathbb{R}^S$ is *integer* if and only if each extreme point of P is integer. An inequality $a^tx\geq\alpha$, where $a\in\mathbb{R}^S$ and $\alpha\in\mathbb{R}$, is *tight* for a point x^* if $a^Tx^*=\alpha$. If $a^Tx\geq\alpha$ is induced by a cut (respectively, a partition), we equivalently say that the cut (respectively, partition) is tight for x^*.

2. The (1,2)-survivable network design problem and some related polytopes. Let $G=(V,E)$ be an undirected graph and $r\in\{1,2\}^V$ be a connectivity type vector. Without loss of generality, we may assume throughout this paper that there exist at least two nodes having the largest connectivity type (i.e., $|T| \geq 2$). For a nonempty node subset $W \subsetneq V$, let $r(W) = \max\{r(u) : u \in W\}$ and $con(W) = \min\{r(W), r(V \setminus W)\}$. Given an edge subset $F \subseteq E$, if (V, F) is a survivable subgraph of G, then its incidence vector x^F satisfies

$$x(e) \geq 0 \qquad \text{for all } e \in E, \tag{2.1}$$

$$x(e) \leq 1 \qquad \text{for all } e \in E, \tag{2.2}$$

$$x(\delta(W)) \geq con(W) \qquad \text{for all } \emptyset \neq W \subsetneq V, \tag{2.3}$$

$$x(e) \in \{0, 1\} \qquad \text{for all } e \in E. \tag{2.4}$$

The inequalities (2.1) and (2.2) are, respectively, called *lower-bound* and *upper-bound trivial inequalities* (or more generally *trivial inequalities*), and inequalities (2.3) are called *cut inequalities*.

For a class of inequalities, the separation problem is as follows: given a vector y, find a violated inequality in the class or prove that none exists. An algorithm for the separation problem associated with a class of inequalities is a key ingredient for being able to use those inequalities within a branch-and-cut algorithm. The separation problem for the cut inequalities (2.3) is polynomially solvable using a polynomial-time maximum flow algorithm (e.g., preflow-push algorithm of Goldberg and Tarjan [6] running in $O(n^3)$ time).

In [9], Grötschel, Monma, and Stoer introduced a class of valid inequalities for the polytope SNDP(G, r) which can be stated as follows. Let $\{V_1, \ldots, V_p\}$, $p \geq 2$, be a partition of V. Let $I_2 = \{i : con(V_i) = 2,\ i = 1, \ldots, p\}$ be the set of subscripts whose corresponding classes of the partition contain at least one terminal. The *partition inequalities* induced by $\{V_1, \ldots, V_p\}$ is

$$x(\delta(V_1, \ldots, V_p)) \geq \begin{cases} p - 1 & \text{if } I_2 = \emptyset, \\ p & \text{otherwise.} \end{cases} \tag{2.5}$$

The inequalities (2.5) are a generalization of the cut inequalities (2.3). (The latter correspond to the case where $p = 2$.) Therefore, if we do not specify $p \geq 3$, then a partition inequality (2.5) may be a cut inequality (2.3). In the remainder of this paper, a partition of V with $I_2 = \emptyset$ (respectively, $I_2 \neq \emptyset$) will be called a *partition of type* 1 (respectively, *partition of type* 2), and the inequality (2.5) induced by it will be called a *partition inequality of type* 1 (respectively, a *partition inequality of type* 2). Grötschel, Monma, and Stoer [9] gave sufficient conditions and necessary conditions for the inequalities (2.5) to define facets of SNDP(G, r).

In [12], Kerivin and Mahjoub showed that the separation problem for the partition inequalities (2.5) reduces to minimizing a particular submodular function, and then is polynomially solvable. Later, Barahona and Kerivin [2] reduced the separation problem for those inequalities to a sequence of $O(n^4)$ minimum cut problem.

Let CPP(G, r) be the polytope given by the inequalities (2.1), (2.2), (2.3), and (2.5). This polytope, called the *cut and partition inequalities polytope*, is the linear relaxation of SNDP(G, r) strengthened by the partition inequalities (2.5). As we mentioned above, the separation problems for both cut and partition inequalities are polynomially solvable, and then this implies by the ellipsoid method [7] that the (1,2)-survivable network design problem can be solved in polynomial time on graphs for which SNDP$(G, r) =$ CPP(G, r).

The definition of the connectivity type of node subsets implies the following remarks.

Remark 2.1. The function $r : 2^V \longrightarrow \{1, 2\}$ is nondecreasing, that is, r satisfies $r(A) \le r(B)$ for all $A \subseteq B \subseteq V$.

Remark 2.2. Let $A \subseteq V$ such that $r(A) = 1$. Then, we have

(a) $r(A \cup B) = r(B)$ for any $B \subseteq V$, and

(b) $con(A \cup B) = con(B)$ for any $B \subseteq V$ such that $A \cup B \neq V$.

PROPOSITION 2.3. *The function $con : 2^V \longrightarrow \{1, 2\}$ is submodular on crossing pairs.*

Proof. Let $A, B \subseteq V$ such that $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$, $B \setminus A \neq \emptyset$, and $\overline{A \cup B} \neq \emptyset$. We must show that

$$con(A) + con(B) \ge con(A \cap B) + con(A \cup B). \tag{2.6}$$

If $con(A) + con(B) = 4$, then it is obvious that inequality (2.6) holds.

If $con(A) = con(B) = 1$, then $con(A \cap B) = con(A \cup B) = 1$ and hence, inequality (2.6) is satisfied.

Suppose now that $con(A) + con(B) = 3$. Without loss of generality, we may assume that $con(A) = 1$ and $con(B) = 2$. Consider first the case $r(A) = 1$. From Remark 2.1, we then have $r(A \cap B) = r(A)$. By the definition of the function con, we thus obtain $con(A \cap B) = con(A) = 1$. From Remark 2.2(b), we have $con(A \cup B) = con(B)$. Thus, inequality (2.6) holds. If $r(A) = 2$, then $r(A \cap B) = r(A \setminus B) = 2$ and $r(B \setminus A) = r(\overline{A \cup B}) = 1$ and hence, inequality (2.6) is satisfied. ∎

From the proof of Proposition 2.3, we deduce the remark below.

Remark 2.4. The function $con : 2^V \longrightarrow \{1, 2\}$ is modular on crossing pairs $A, B \subseteq V$ if one of the two following properties hold:

(a) $con(A) + con(B) \le 3$, or

(b) $con(A) = con(B) = 2$ and $r(A \cap B) = r(\overline{A \cup B}) = 2$.

To conclude this section, we define the connectivity type vectors associated with the subgraphs of G obtained by contractions or deletions of edges. For any subset F of E, *deleting* F gives rise to the graph $G - F = (V(\overline{F}), \overline{F})$. The connectivity type vector $r_F^d \in \{1, 2\}^{V(\overline{F})}$ is then obviously the restriction of $r \in \{1, 2\}^V$ on $V(\overline{F})$, that is,

$$r_F^d(u) = r(u) \qquad \text{for all } u \in V(\overline{F}). \tag{2.7}$$

Given an edge $e = uv \in E$, *contracting* e means deleting e and identifying u and v. If $F \subseteq E$ induces a connected subgraph of G, then G/F denotes the graph obtained from G by contracting F, that is, by contracting all of the edges in F. Let w be the node that arises from the contraction of F. The connectivity type vector $r_F^c \in \{1, 2\}^{V'}$ associated with the node set $V' = (V \setminus V(F)) \cup \{w\}$ is defined as follows:

$$r_F^c(u) = \begin{cases} r(u) & \text{if } u \in V' \setminus \{w\}, \\ con_G(V(F)) & \text{if } u = w. \end{cases} \tag{2.8}$$

A couple (H, r_H) is called a *minor* of (G, r) if H arises from G by a series of deletions and contractions of edges. The vector r_H is the connectivity type vector associated with H, and it is obtained from r by applying at each deletion/contraction the corresponding (2.7)/(2.8). This notion of minor will be important in section 5. In the next section, we study some structural properties of the CPP(G,r) which will be also useful later on.

3. Structural properties of the polytope CPP(G, r). One way to solve the (1,2)-survivable network design problem is to use a branch-and-cut framework. In such an approach, one may first consider a linear program whose constraints are given by CPP(G, r) which is a relaxation of SNDP(G, r). This linear program provides a lower bound for the SNDP, and in some special cases an optimal solution. This linear program has an exponential number of constraints, and one needs to use a cutting-plane algorithm to solve it. The knowledge of structural properties of the points (especially the extreme points) of CPP(G, r) may provide useful information which is important in order to determine violated inequalities more efficiently. The structural properties established in this section may also be used to characterize the polytope SNDP(G, r) on certain classes of graphs (see section 5).

Let us consider a point $\overline{x}$ of CPP(G, r).

PROPOSITION 3.1. *Let $F \subseteq E$ be an edge subset of E that induces a connected subgraph of G. Then $\overline{x}' \in \mathbb{R}^{E\setminus F}$, the restriction of $\overline{x}$ on $E \setminus F$, is a point of CPP($G/F, r_F^c$).*

Proof. The result comes directly from the fact that any inequality of CPP($G/F, r_F^c$) is also an inequality of CPP(G, r). □

Let us denote by π a partition $\{V_1, \ldots, V_p\}$ of V, $p \geq 2$, which is tight for $\overline{x}$.

PROPOSITION 3.2. *Suppose $p \geq 3$. Consider $i, j \in \{1, \ldots, p\}$ such that $i < j$. Let $\pi' = \{V_1', \ldots, V_{p-1}'\}$ be the partition defined below*

$$\begin{aligned} V_t' &= V_t, \qquad t = 1, \ldots, i-1, i+1, \ldots, j-1, \\ V_i' &= V_i \cup V_j, \\ V_t' &= V_{t+1}, \qquad t = j, \ldots, p-1. \end{aligned}$$

(a) *If π is of type 1, then $\overline{x}[V_i, V_j] \leq 1$.*
(b) *If π is of type 2 and π' is of type 1, then $\overline{x}[V_i, V_j] \leq 2$.*
(c) *If π and π' are of type 2, then $\overline{x}[V_i, V_j] \leq 1$.*
Moreover, if $\overline{x}[V_i, V_j] = 1$ (cases (a) *and* (c)*) or $\overline{x}[V_i, V_j] = 2$ (case* (b)*), then the partition π' is also tight for $\overline{x}$.*

Proof. We are going to prove (a). (The proofs of (b) and (c) are similar.) Since π is of type 1, then clearly π' is also of type 1. Hence, we have

$$\overline{x}(\delta(V_1', \ldots, V_{p-1}')) \geq p - 2.$$

We thus obtain

$$\begin{aligned} \overline{x}[V_i, V_j] &= \overline{x}(\delta(V_1, \ldots, V_p)) - \overline{x}(\delta(V_1', \ldots, V_{p-1}')) \\ &\leq p - 1 - (p - 2) \\ &= 1. \end{aligned}$$

Moreover, if $\overline{x}[V_i, V_j] = 1$, then the above inequalities are all satisfied as equalities, and we get $\overline{x}(\delta(V_1', \ldots, V_{p-1}')) = p - 2$. □

PROPOSITION 3.3. *Consider a partition $\{V_i^1, V_i^2\}$ of V_i for some $i \in \{1, \ldots, p\}$. Let $\pi' = \{V_1', \ldots, V_{p+1}'\}$ be the partition of V given by*

$$\begin{aligned} V_j' &= V_j, \qquad j = 1, \ldots, i-1, \\ V_i' &= V_i^1, \\ V_{i+1}' &= V_i^2, \\ V_j' &= V_{j-1}, \qquad j = i+2, \ldots, p+1. \end{aligned}$$

(a) *If π and π' are of type* 1*, then* $\overline{x}[V_i^1, V_i^2] \geq 1$.
(b) *If π is of type* 1 *and π' is of type* 2*, then* $\overline{x}[V_i^1, V_i^2] \geq 2$.
(c) *If π is of type* 2*, then* $\overline{x}[V_i^1, V_i^2] \geq 1$.

Moreover, if $\overline{x}[V_i^1, V_i^2] = 1$ *(cases* (a) *and* (c)*) or* $\overline{x}[V_i^1, V_i^2] = 2$ *(case* (b)*), then the partition π' is also tight for $\overline{x}$.*

Proof. The proof is omitted because of its similarity with the one of Proposition 3.2. ∎

An immediate consequence of Proposition 3.3 is the following.

Remark 3.4. The subgraphs $G(V_i)$ for $i = 1, \ldots, p$ are all connected.

We notice that the two previous propositions could be generalized as described in the remark below.

Remark 3.5. In Proposition 3.2, we may consider a subscript subset $I \subsetneq \{1, \ldots, p\}$ with $|I| \geq 2$ (instead of only two distinct subscripts i and j), and in Proposition 3.3, we may partition the node subset V_i into at least (rather than exactly) two subsets.

PROPOSITION 3.6. *Let $\delta(W_1)$ and $\delta(W_2)$ be two cuts tight for $\overline{x}$ such that W_1 and W_2 are two crossing subsets of V. Then, $\delta(W_1 \cap W_2)$ and $\delta(\overline{W_1 \cup W_2})$ are tight for $\overline{x}$, and $\overline{x}[W_1 \setminus W_2, W_2 \setminus W_1] = 0$ if one of the two following properties holds:*
(a) $con(W_1) + con(W_2) \leq 3$, *or*
(b) $con(W_1) = con(W_2) = 2$ *and* $r(W_1 \cap W_2) = r(\overline{W_i \cup W_2}) = 2$.

Proof. We have

$$\begin{aligned}\overline{x}(\delta(W_1)) + \overline{x}(\delta(W_2)) &= con(W_1) + con(W_2)\\ &= \overline{x}(\delta(W_1 \cap W_2)) + \overline{x}(\delta(\overline{W_1 \cup W_2})) + 2\overline{x}[W_1 \setminus W_2, W_2 \setminus W_1]\\ &\geq con(W_1 \cap W_2) + con(\overline{W_1 \cup W_2}),\end{aligned}$$

where the last inequality follows from $\overline{x} \in \mathrm{CPP}(G, r)$. Since one of the properties (a) and (b) holds and $con(\overline{W_1 \cup W_2}) = con(W_1 \cup W_2)$, by Remark 2.4 we have $con(W_1) + con(W_2) = con(W_1 \cap W_2) + con(\overline{W_1 \cup W_2})$. We then obtain the result. ∎

PROPOSITION 3.7. *Let $\pi_1 = \{V_1, \ldots, V_p\}$ and $\pi_2 = \{W_1, \ldots, W_q\}$, $p \geq 2$ and $q \geq 2$, be two partitions of V which are tight for $\overline{x}$. Consider two distinct subscripts $i,\ j \in \{1, \ldots, p\}$.*
(a) *If π_1 is of type* 1 *or π_1 is of type* 2 *with $p \geq 3$ and $r(\overline{V_i \cup V_j}) = 2$, then at most one class of π_2 only intersects both V_i and V_j.*
(b) *If π_1 is of type* 2 *with either $p = 2$ or $r(\overline{V_i \cup V_j}) = 1$, then at most two classes of π_2 only intersect both V_i and V_j.*

Proof. Assume that π_1 is a partition of type 1. (The proofs for the other cases are similar.) Let $K \subseteq \{1, \ldots, q\}$, $|K| > 1$, be the set of subscripts such that $W_k \cap V_i \neq \emptyset$, $W_k \cap V_j \neq \emptyset$, and $W_k = (W_k \cap V_i) \cup (W_k \cap V_j)$ for all $k \in K$. From Proposition 3.3, we have

$$\overline{x}[W_k \cap V_i, W_k \cap V_j] \geq 1 \qquad \text{for all } k \in K.$$

These last inequalities together with Proposition 3.2(a) lead to

$$\begin{aligned}1 \geq \overline{x}[V_i, V_j] &\geq \sum_{k \in K} \overline{x}[W_k \cap V_i, W_k \cap V_j]\\ &\geq |K|\\ &> 1,\end{aligned}$$

a contradiction. ∎

We remark that if we have $p = 2$ in Proposition 3.7, then at most $con(V_1)$ classes of the partition π_2 intersect both V_1 and V_2.

PROPOSITION 3.8. *Let $\pi_1 = \{V_1, \ldots, V_p\}$ and $\pi_2 = \{W_1, \ldots, W_q\}$, $p \geq 2$ and $q \geq 3$, be two partitions of V which are tight for $\overline{x}$. Given a subscript $i \in \{1, \ldots, p\}$, define $J = \{j_1, \ldots, j_k\} = \{j \in \{1, \ldots, q\} \mid V_i \cap W_j \neq \emptyset\}$. Assume $2 \leq k < q$. We then have*

(a) $\overline{x}[V_i \cap W_{j_1}, \ldots, V_i \cap W_{j_k}] = \overline{x}[W_{j_1}, \ldots, W_{j_k}] = k - 1$, *if π_1 is of type 1, and one of the following conditions hold:*

  (a.1) *π_2 is of type 1, or*

  (a.2) *π_2 is of type 2, $r(V_1) = 1$ and $r(V \setminus \bigcup_{j \in J} W_j) = 2$.*

(b) $\overline{x}[V_i \cap W_{j_1}, \ldots, V_i \cap W_{j_k}] = \overline{x}[W_{j_1}, \ldots, W_{j_k}] = k$, *if π_1 and π_2 are of type 2 and $r(V_i) = 2$.*

Proof. We are going to prove the case (a.1). (The proofs for the other cases use similar arguments.) Without loss of generality, we suppose $J = \{1, \ldots, k\}$. Since $[V_i \cap W_1, \ldots, V_i \cap W_k] \subseteq [W_1, \ldots, W_k]$ and $\overline{x} \geq 0$, we have

$$\overline{x}[V_i \cap W_1, \ldots, V_i \cap W_k] \leq \overline{x}[W_1, \ldots, W_k]. \tag{3.1}$$

From the definition of J and the fact that π_2 is a partition of V, $\{V_i \cap W_1, \ldots, V_i \cap W_k\}$ is a partition of V_i. By Proposition 3.3(a) and Remark 3.5, we have

$$\overline{x}[V_i \cap W_1, \ldots, V_i \cap W_k] \geq k - 1. \tag{3.2}$$

Furthermore, as π_2 is of type 1, the partition $\{\bigcup_{j \in J} W_j, W_{k+1}, \ldots, W_q\}$ is also of type 1. From Proposition 3.2(a) and Remark 3.5, we have

$$\overline{x}[W_1, \ldots, W_k] \leq k - 1.$$

This last inequality combined with the inequalities (3.1) and (3.2) gives $\overline{x}[V_i \cap W_1, \ldots, V_i \cap W_k] = \overline{x}[W_1, \ldots, W_k] = k - 1$. ∎

We remark that in Proposition 3.8, if we do not fulfill the conditions (a) and (b), we still have $k - 1 \leq \overline{x}[V_i \cap W_{j_1}, \ldots, V_i \cap W_{j_k}] \leq \overline{x}[W_{j_1}, \ldots, W_{j_k}] \leq k$.

PROPOSITION 3.9. *Let $\pi = \{V_1, \ldots, V_p\}$, $p \geq 2$, be a partition of V which is tight for $\overline{x}$. Consider two distinct subscripts $i, j \in \{1, \ldots, p\}$ such that $i < j$, $|V_i| \geq 2$, and $|V_j| \geq 2$. Given a nonempty node set $W \subsetneq V_i$ and a partition $\{V_j^1, \ldots, V_j^q\}$, $q \geq 2$, of V_j, let $\pi' = \{V'_1, \ldots, V'_{p'}\}$ be the following partition of V:*

$$\begin{aligned}
&V'_t = V_t, && t = 1, \ldots, i-1,\\
&V'_i = V_i \setminus W, && \\
&V'_t = V_t, && t = i+1, \ldots, j-1,\\
&V'_j = V_j^1 \cup W, && \\
&V'_{t+j} = V_j^{t+1}, && t = 1, \ldots, q-1,\\
&V'_{t+q-1} = V_t, && t = j+1, \ldots, p,
\end{aligned}$$

where $p' = p + q - 1$.

(a) *If π and π' are of the same type, then $\overline{x}[V_j^1, \ldots, V_j^q] \geq q - 1 + \overline{x}[W, V_j^1] - \overline{x}[W, V_i \setminus W]$.*

(b) *If π is of type 1 and π' is of type 2, then $\overline{x}[V_j^1, \ldots, V_j^q] \geq q + \overline{x}[W, V_j^1] - \overline{x}[W, V_i \setminus W]$.*

(c) *If* π *is of type* 2 *and* π' *is of type* 1*, then* $\overline{x}[V_j^1, \ldots, V_j^q] \geq q - 2 + \overline{x}[W, V_j^1] - \overline{x}[W, V_i \setminus W]$.

Moreover, if $\overline{x}[V_j^1, \ldots, V_j^q] = q-1+\overline{x}[W, V_j^1]-\overline{x}[W, V_i \setminus W]$ *(case* (a)*),* $\overline{x}[V_j^1, \ldots, V_j^q] = q+\overline{x}[W, V_j^1]-\overline{x}[W, V_i \setminus W]$ *(case* (b)*) or* $\overline{x}[V_j^1, \ldots, V_j^q] = q-2+\overline{x}[W, V_j^1]-\overline{x}[W, V_i \setminus W]$ *(case* (c)*), then the partition* π' *is also tight for* $\overline{x}$.

Proof. Using $\overline{x}(\delta(V_1', \ldots, V_{p'}')) = \overline{x}(\delta(V_1, \ldots, V_p)) + \overline{x}[V_j^1, \ldots, V_j^q] - \overline{x}[W, V_j^1] + \overline{x}[W, V_i \setminus W]$, the proof is similar to the one of Proposition 3.2, and thus it is omitted. ∎

In the remainder of this section, let us assume that $\overline{x}$ is an extreme point of $\text{CPP}(G, r)$. We denote by $E_0(\overline{x})$, $E_1(\overline{x})$ and $E_f(\overline{x})$ the set of edges $e \in E$ such that $\overline{x}(e) = 0$, $\overline{x}(e) = 1$ and $0 < \overline{x}(e) < 1$, respectively. Let $P_1(\overline{x})$ and $P_2(\overline{x})$ be the sets of partitions of type 1 and 2, respectively, which are tight for $\overline{x}$.

Since $\overline{x}$ is an extreme point of $\text{CPP}(G, r)$, there exist $P_1^*(\overline{x}) \subseteq P_1(\overline{x})$ and $P_2^*(\overline{x}) \subseteq P_2(\overline{x})$ such that $\overline{x}$ is the unique solution of the system

$$S(\overline{x}) \begin{cases} x(e) = 0 & \text{for all } e \in E_0(\overline{x}), \\ x(e) = 1 & \text{for all } e \in E_1(\overline{x}), \\ x(\delta(V_1, \ldots, V_p)) = p - 1 & \text{for all } \{V_1, \ldots, V_p\} \in P_1^*(\overline{x}), \\ x(\delta(W_1, \ldots, W_q)) = q & \text{for all } \{W_1, \ldots, W_q\} \in P_2^*(\overline{x}), \end{cases}$$

where $|E_0(\overline{x})|+|E_1(\overline{x})|+|P_1^*(\overline{x})|+|P_2^*(\overline{x})| = |E|$. Since the system $S(\overline{x})$ is not unique, we give the following remarks which will be useful later.

Remark 3.10. From any system of equations, induced by inequalities of $\text{CPP}(G, r)$ and whose unique solution is $\overline{x}$, we may extract a nonsingular subsystem having exactly $|E|$ equations.

Remark 3.11. For any $e \in E$, there exists at least one equation of $S(\overline{x})$ which contains $x(e)$ with a nonzero coefficient.

Proposition 3.12. *If* $\overline{x}$ *is fractional, then* $|E_f(\overline{x})| \geq 2$.

Proof. Every equation of the system $S(\overline{x})$ has coefficients equal to 0 or 1 and an integer right-hand side. Since $\overline{x}$ is fractional, Remark 3.11 then implies that $\overline{x}$ must contain at least two fractional components. ∎

Proposition 3.13. *For any edge pair of* $E_f(\overline{x})$*, there exists at least one equation of* $S(\overline{x})$ *which contains exactly one of the two edges.*

Proof. Suppose that there exist two edges $e_1,\ e_2 \in E_f(\overline{x})$ such that any equation of $S(\overline{x})$ contains either both or none of them. Let $x' \in \mathbb{R}^E$ be the point such that

$$x'(e) = \begin{cases} \overline{x}(e) + \epsilon & \text{if } e = e_1, \\ \overline{x}(e) - \epsilon & \text{if } e = e_2, \\ \overline{x}(e) & \text{if } e \in E \setminus \{e_1, e_2\}, \end{cases}$$

where $\epsilon \neq 0$. The point x' is also a solution of $S(\overline{x})$, which contradicts the extremality of $\overline{x}$. ∎

A direct consequence of the previous proposition is the following.

Remark 3.14. If u, v are two nodes of V, then $[u, v]$ contains at most one edge in $E_f(\overline{x})$.

Proposition 3.15. *Let* $W \subsetneq V$ *be a nonempty node subset such that* $\overline{x}(\delta(W)) = 1$*. We then have* $\overline{x}(e) \in \{0, 1\}$ *for all* $e \in \delta(W)$.

Proof. Suppose there exists $e \in \delta(W) \cap E_f(\overline{x})$. Since $\overline{x}(\delta(W)) = 1$, there must exist another edge $f \in \delta(W) \cap E_f(\overline{x})$. We are going to prove that the system $S(\overline{x})$ can be chosen such that any of its equations contains either e and f or none of them. Let $\{V_1, \ldots, V_p\}$ be a partition of V inducing an equation of the system $S(\overline{x})$. Assume that there exist two distinct subscripts $i,\ j \in \{1, \ldots, p\}$ such that $V_i \cap W \neq \emptyset \neq V_j \cap W$. From Proposition 3.3, we obtain $\overline{x}[V_i \cap W, V_i \setminus W] \geq 1$ and $\overline{x}[V_j \cap W, V_j \setminus W] \geq 1$ which give $\overline{x}(\delta(W)) \geq \overline{x}[V_i \cap W, V_i \setminus W] + \overline{x}[V_j \cap W, V_j \setminus W] \geq 2$, a contradiction. Therefore, we have $W \subseteq V_k$ for some $k \in \{1, \ldots, p\}$. If $W \neq V_k$, then from Proposition 3.3 and $\overline{x}(\delta(W)) = 1$ we must have $\delta(W) = [W, V_k \setminus W]$. We thus conclude that any equation of $S(\overline{x})$ contains either e and f or none of them. But this contradicts Proposition 3.13. $\square$

Proposition 3.16. *Let $W \subsetneq V$ be a nonempty node subset such that $con(W) = 2$. If $\overline{x}(\delta(W)) = 2$, then the system $S(\overline{x})$ can be chosen such that $P_1^*(\overline{x}) = \emptyset$.*

Proof. Suppose there exists a partition $\{V_1, \ldots, V_p\} \in P_1^*(\overline{x})$. Without loss of generality, we suppose that V_1 contains all the terminals. Since $con(W) = 2$, we obtain that $V_1 \cap W \neq \emptyset \neq V_1 \cap \overline{W}$. From Proposition 3.3(b), we have $\overline{x}[V_1 \cap W, V_1 \cap \overline{W}] \geq 2$ which implies $\delta(W) = [V_1 \cap W, V_1 \cap \overline{W}]$. Therefore by Proposition 3.2, we deduce that either W or $\overline{W}$ is a subset of V_1. Without loss of generality, we assume that $W \subset V_1$. The partition $\{W, V_1 \cap \overline{W}, V_2, \ldots, V_p\}$ is of type 2, then since $\overline{x}[W, V_1 \cap \overline{W}] = 2$ Proposition 3.3 gives $\{W, V_1 \cap \overline{W}, V_2, \ldots, V_p\} \in P_2^*(\overline{x})$.

We can thus replace in the system $S(\overline{x})$ the partition $\{V_1, \ldots, V_p\}$ by the partitions $\{W, \overline{W}\}$ and $\{W, V_1 \cap \overline{W}, V_2, \ldots, V_p\}$. The obtained system may have more than $|E|$ equations, yet from Remark 3.10 we can choose $|E|$ equations of this system whose unique solution is $\overline{x}$. $\square$

Proposition 3.17. *Let $W \subsetneq V$ be a nonempty node subset such that $con(W) = 2$. If $\overline{x}(\delta(W)) = 2$, then the system $S(\overline{x})$ can be chosen such that at most one class of any partition of $P_2^*(\overline{x})$ intersects both W and $\overline{W}$.*

Proof. From Proposition 3.7(b), any partition of $P_2^*(\overline{x})$ has at most two classes which intersect both W and $\overline{W}$. Suppose that there exists a partition $\{V_1, \ldots, V_p\}$ of $P_2^*(\overline{x})$ such that, without loss of generality, V_1 and V_2 intersect both W and $\overline{W}$. Proposition 3.3(c) then states that the partition $\{V_1 \cap W, V_1 \cap \overline{W}, V_2 \cap W, V_2 \cap \overline{W}, V_3, \ldots, V_p\}$ belongs to $P_2(\overline{x})$. We can thus replace in the system $S(\overline{x})$ the partition $\{V_1, \ldots, V_p\}$ by the partitions $\{W, \overline{W}\}$ and $\{V_1 \cap W, V_1 \cap \overline{W}, V_2 \cap W, V_2 \cap \overline{W}, V_3, \ldots, V_p\}$. The result thus follows from Remark 3.10. $\square$

Proposition 3.18. *Let $u,\ v$ be two distinct nodes such that $\overline{x}[u, v] \geq 1$. The system $S(\overline{x})$ can thus be chosen such that*

(a) *$\delta(V_1, \ldots, V_p) \cap [u, v] = \emptyset$, for all $\{V_1, \ldots, V_p\} \in P_1^*(\overline{x})$, and*

(b) *if $\{V_1, \ldots, V_p\} \in P_2^*(\overline{x})$ with $p \geq 3$ and $[u, v] \subseteq [V_i, V_j]$ for some distinct subscripts $i,\ j \in \{1, \ldots, p\}$, then $r(V \setminus (V_i \cup V_j)) = 1$.*

Proof. We will prove (a). (The proof of (b) uses similar arguments.) Let $\pi = \{V_1, \ldots, V_p\}$, $p \geq 2$, be a partition of type 1, tight for $\overline{x}$ and such that without loss of generality, $[u, v] \subseteq [V_1, V_2]$. If $p = 2$, then from Proposition 3.15, we have $\overline{x}(e) \in \{0, 1\}$ for all $e \in [u, v]$. The equation induced by π can thus be obtained from the equations $x(e) = 0$ for all $e \in [u, v] \cap E_0(\overline{x})$, and $x(e) = 1$ for all $e \in [u, v] \cap E_1(\overline{x})$.

Suppose now that $p \geq 3$. From Proposition 3.2, we have $\overline{x}[V_1, V_2] \leq 1$. Since $[u, v] \subseteq [V_1, V_2]$ and $\overline{x}[u, v] \geq 1$, we then obtain $\overline{x}[u, v] = 1$. Therefore, the partition $\pi' = \{V_1 \cup V_2, V_3, \ldots, V_p\}$ is also tight for $\overline{x}$. Moreover, by Remark 3.14, $\overline{x}(e) \in \{0, 1\}$ for all $e \in [u, v]$. The equation induced by π can thus be obtained from the one induced by π' together with the equations $x(e) = 0$ for all $e \in [u, v] \cap E_0(\overline{x})$ and $x(e) = 1$ for all $e \in [u, v] \cap E_1(\overline{x})$. $\square$

Proposition 3.19. *Let u be a node having exactly two neighbors, namely u_1*

and u_2. The system $S(\overline{x})$ can then be chosen such that any partition $\{V_1, \dots, V_p\} \in P_1^*(\overline{x}) \cup P_2^*(\overline{x})$ with $p \geq 3$ and $[u, u_1] \subset \delta(V_1, \dots, V_p)$ has the following property:

$$|V_i \cap \{u_1, u_2\}| \leq 1 \quad \textit{for all } i \in \{1, \dots, p\}.$$

Moreover, if $\overline{x}[u, u_2] < 1$, then we also have

$$|V_i \cap \{u, u_2\}| \leq 1 \quad \textit{for all } i \in \{1, \dots, p\}.$$

Proof. Let $\pi = \{V_1, \dots, V_p\} \in P_1^*(\overline{x}) \cup P_2^*(\overline{x})$ with $p \geq 3$ and $[u, u_1] \subset \delta(V_1, \dots, V_p)$. Without loss of generality, we assume $u \in V_1$ and $u_1 \in V_2$. Suppose $u_2 \in V_2$. By Remark 3.4, we have $V_1 = \{u\}$. If $\overline{x}(\delta(u)) > r(u)$, by considering the partition $\{V_1 \cup V_2, V_3, \dots, V_p\}$, we then contradict Proposition 3.2. Hence, we have $\overline{x}(\delta(u)) = r(u)$. Let π' be the partition $\{V_1 \cup V_2, V_3, \dots, V_p\}$. If $r(u) = 2$, then either $\overline{x}[u, u_1]$ or $\overline{x}[u, u_2]$ is greater than 1, and by Proposition 3.18, we have $\pi \in P_2^*(\overline{x})$, $r(V \setminus (V_1 \cup V_2)) = 1$, and then $\pi' \in P_1(\overline{x})$. If $r(u) = 1$, then we clearly have $\pi' \in P_1(\overline{x}) \cup P_2(\overline{x})$. We can thus replace in the system $S(\overline{x})$ the partition π by the partition π' and the cut $\delta(u)$. The obtained system may have more than $|E|$ equations, yet from Remark 3.10 we can choose $|E|$ equations of this system whose unique solution is $\overline{x}$. Therefore, the system $S(\overline{x})$ can be chosen such that no partition has one class containing u and a different one containing both u_1 and u_2.

We now suppose that $\overline{x}[u, u_2] < 1$. If $u_2 \in V_1$, we then have $\overline{x}[u, V_1 \setminus \{u\}] = \overline{x}[u, u_2] < 1$, a contradiction with Proposition 3.3. ∎

PROPOSITION 3.20. *Let u, v be two distinct nodes such that $\overline{x}[u, v] \geq 2$. We have the following:*

(a) *the system $S(\overline{x})$ can be chosen such that the variables $x(e)$, for all $e \in [u, v]$, only appear in equations induced by $E_0(\overline{x}) \cup E_1(\overline{x})$ with a nonzero coefficient, and*

(b) *$\overline{x}(e) \in \{0, 1\}$ for all $e \in [u, v]$.*

Proof. Using the same arguments as in the proof of Proposition 3.18, we get (a). Consequently, from Remark 3.11, we obtain $\overline{x}(e) \in \{0, 1\}$ for all $e \in [u, v]$. ∎

4. Composition of G by one-node cutset and the CPP(G, r). Given a graph $G = (V, E)$ and two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G, if $V = V_1 \cup V_2$ and $|V_1 \cap V_2| = 1$, then $G = (V, E_1 \cup E_2)$ is called the 1-*sum* of G_1 and G_2. In that case, the singleton $V_1 \cap V_2$ is called a *one-node cutset* of G.

LEMMA 4.1. *Let $G = (V, E)$ be a graph and $r \in \{1, 2\}^V$ be a connectivity type vector. Suppose that G is the 1-sum of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let u be the only node of $V_1 \cap V_2$. For $i = 1, 2$, let $r_i \in \{1, 2\}^{V_i}$ be the connectivity type vector such that $r_i(v) = r(v)$ if $v \in V_i \setminus \{u\}$ and $r_i(u) = \max\{r(u), r(V \setminus V_i)\}$. If $\overline{x}$ is an extreme point of CPP(G, r), then the restriction $\overline{x}_i$ of $\overline{x}$ on G_i is also an extreme point of CPP(G_i, r_i) for $i = 1, 2$.*

Proof. First, when we write $\overline{x}_i$, we refer to one of the two restrictions $\overline{x}_1$ and $\overline{x}_2$, and then the subscript i may be either 1 or 2 in the proof. From Proposition 3.1, $\overline{x}_i \in$ CPP(G_i, r_i). (We remark that the couples (G_1, r_1) and (G_2, r_2) are nothing but the couples $(G/E_2, r^c_{E_2})$ and $(G/E_1, r^c_{E_1})$, respectively.) To prove that $\overline{x}_i$ is an extreme point of CPP(G_i, r_i), it suffices to show that we can choose the system $S(\overline{x})$ such that for any pair of edges $e_1 \in E_1$ and $e_2 \in E_2$, none of its equations involves both $x(e_1)$ and $x(e_2)$ with nonzero coefficients.

Let $\pi = \{W_1, \dots, W_p\}$ be a partition which is tight for $\overline{x}$. Suppose $u \in W_q$ for some $q \in \{1, \dots, p\}$. Since $\{u\}$ is a one-node cutset, by Remark 3.4 W_q is the only class which may intersect both V_1 and V_2. Without loss of generality, we can assume that $W_j \subseteq V_1$ for all $j \in \{1, \ldots, q-1\}$ and $W_k \subseteq V_2$ for all $k \in \{q+1, \ldots, p\}$.

Consider the two partitions $\pi_1 = \{W_1, \ldots, W_{q-1}, W_q^1\}$ and $\pi_2 = \{W_q^2, W_{q+1}, \ldots, W_p\}$ of V, where

$$W_q^1 = \bigcup_{j \geq q} W_j \quad \text{and} \quad W_q^2 = \bigcup_{k \leq q} W_k.$$

We clearly have

$$\overline{x}(\delta(W_1, \ldots, W_p)) = \overline{x}(\delta(W_1, \ldots, W_{q-1}, W_q^1)) + \overline{x}(\delta(W_q^2, W_{q+1}, \ldots, W_p)). \tag{4.1}$$

We remark that we cannot have both π_1 and π_2 of type 2. The partition π would be of type 2 otherwise, and then (4.1) would give

$$\begin{aligned} p &= \overline{x}(\delta(W_1, \ldots, W_p)) \\ &\geq q + (p - q + 1) \\ &= p + 1, \end{aligned}$$

which is impossible.

Without loss of generality, we now suppose that π_1 is of type 1. We notice that, in this case, π and π_2 have the same type. Assume that π_2 is of type 2. (The case where π_2 is of type 1 is similar.) The inequality (4.1) then gives

$$\begin{aligned} p &= \overline{x}(\delta(W_1, \ldots, W_p)) \\ &\geq (q - 1) + (p - q + 1) \\ &= p. \end{aligned}$$

We thus have that both partitions π_1 and π_2 are tight for $\overline{x}$. Hence, we can replace in the system $S(\overline{x})$ the partition π by the two partitions π_1 and π_2, and we then get a new system $S'(\overline{x})$. By Remark 3.10, there is a nonsingular subsystem of $S'(\overline{x})$ with exactly $|E|$ equations and whose solution is $\overline{x}$. Since $\delta(W_1, \ldots, W_{q-1}, W_q^1) \subseteq E_1$ and $\delta(W_q^2, W_{q+1}, \ldots, W_p) \subseteq E_2$, any equation of this subsystem only contains variables induced by edges from either E_1 or E_2. The proof is thus complete. $\square$

Using the same notations as those introduced in Lemma 4.1, the following is an immediate consequence of this lemma.

COROLLARY 4.2. *If CPP(G_1, r_1) and CPP(G_2, r_2) are integer, then so is CPP(G, r).*

Suppose that G is decomposable by one-node custsets into $G_1, G_2, \ldots, G_t$ with $t \geq 2$. Let r_i, $i = 1, \ldots, t$, be the connectivity type vector associated to G_i, defined as in Lemma 4.1. A direct consequence of Corollary 4.2 is that if CPP(G_i, r_i) is integer for all $i \in \{1, \ldots, t\}$, then so is CPP(G, r).

5. The polytope CPP(G, r) on series-parallel graphs. In this section, we are interested in the cut and partition inequalities polytope CPP(G, r) on series-parallel graphs. A graph is called *series-parallel* if and only if it does not contain K_4 (i.e., the complete graph with 4 nodes) as a minor [4]. We are going to give sufficient conditions for this polytope to be an integer on this class of graphs, that is, sufficient conditions for the polytope SNDP(G, r) to be completely described by the trivial inequalities (2.1) and (2.2) together with the partition inequalities (2.5). (We recall that the cut inequalities (2.3) are partition inequalities induced by partitions having exactly two classes.)

We can remark that if the graph G contains at most one terminal (i.e., $|T| \leq 1$), then the SNDP is nothing but the spanning tree problem. In this case and for general graphs G, Fulkerson [5] completely described the polytope SNDP(G, r) by the trivial inequalities and the inequalities (2.5) induced by partitions of type 1. For general graphs G, we then have the following.

THEOREM 5.1 (see [5]). *If $|T| \leq 1$, then CPP(G, r) is integer.*

On the other hand, Mahjoub [17] showed that the polytope SNDP(G, r) is completely described by the trivial inequalities (2.1) and (2.2), and the cut inequalities (2.3) on series-parallel graphs when $r(u) = 2$ for all $u \in V$. We notice that for such connectivity type vectors, the partition inequalities (2.5) are dominated by the cut inequalities (2.3). Therefore we have the following theorem.

THEOREM 5.2 (see [17]). *If G is series-parallel and $T = V$, then CPP(G, r) is integer.*

In the remainder of this paper, we are going to restrict our attention to graphs G having at least one ordinary node and two terminals. We remark that according to the definition of the survivability conditions, the case with exactly one terminal can be trivially reduced to the one with $T = \emptyset$.

In view of Theorems 5.1 and 5.2, one would have expected the integrality of the polytope CPP(G, r) if G is series-parallel and $r \in \{1, 2\}^V$. Unfortunately, it turns out that this result does not hold. In fact, let us consider the two graphs $G_p^1 = (V_p^1, E_p^1)$ and $G_p^2 = (V_p^2, E_p^2)$ of Figure 5.1, where the terminals are represented by black circles. (G_p^1 has three terminals $\{u_1, u_2, u_3\}$ and two ordinary nodes $\{v_1, v_2\}$, while G_p^2 has four terminals $\{u_1, u_2, u_3, u_4\}$ and one ordinary node $\{v_1\}$.) These two graphs are clearly series-parallel. Moreover, the fractional solutions of $\frac{1}{2}$ for all of the dashed edges and 1 for all of the solid ones are extreme points of the corresponding polytopes CPP(G_p^1, r_p^1) and CPP(G_p^2, r_p^2), where $r_p^1 \in \{1, 2\}^{V_p^1}$ and $r_p^2 \in \{1, 2\}^{V_p^2}$ are the connectivity type vectors associated to G_p^1 and G_p^2, respectively. This implies that the partition inequalities (2.5) together with the trivial ones (2.1) and (2.2) do not suffice to completely describe the survivable network design polytope SNDP(G, r) on series-parallel graphs.

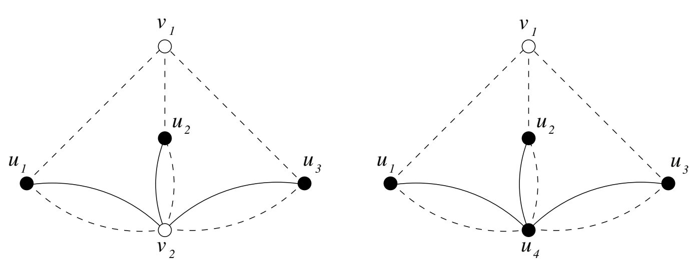

FIG. 5.1. *Fractional extreme points of CPP(G_p^1, r_p^1) and CPP(G_p^2, r_p^2).*

Let us now consider the following inequalities:

$$x(E_p^i) + x(\delta(v_1)) \geq 8 \qquad \text{for } i = 1, 2. \tag{5.1}$$

It is not hard to see that the inequality (5.1) for $i = 1$ (respectively, $i = 2$) should be satisfied by any point of the polytope SNDP(G_p^1, r_p^1) (respectively, SNDP(G_p^2, r_p^2)). Moreover, this inequality cuts off the fractional extreme point of CPP(G_p^1, r_p^1) (respectively, CPP(G_p^2, r_p^2)) given above. We will see in section 7 that inequalities (5.1) are actually special cases of a more general class of facet-defining inequalities of SNDP(G, r).

In the next theorem, we give sufficient conditions based on both graphs (G_p^1, r_p^1) and (G_p^2, r_p^2) for the cut and partition inequalities polytope $\mathrm{CPP}(G, r)$ to be an integer on series-parallel graphs.

THEOREM 5.3. *Let $G = (V, E)$ be a series-parallel graph and $r \in \{1, 2\}^V$ be its associated connectivity type vector. If (G, r) does not have either (G_p^1, r_p^1) or (G_p^2, r_p^2) as a minor, then CPP(G, r) is integer.*

Before giving the proof of the theorem, let us first note that its converse does not hold as shown by the following example. (The black circles still depict the terminal nodes.)

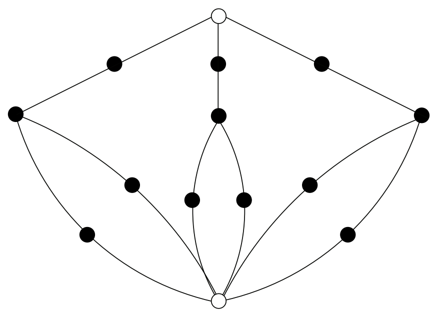

FIG. 5.2. *Counterexample for the converse of Theorem* 5.3.

Let $G_0 = (V_0, E_0)$ be the graph given in Figure 5.2 and $r_0 \in \{1, 2\}^{V_0}$ its associated connectivity type vector. The polytope $\mathrm{CPP}(G_0, r_0)$ is integer. In fact it is reduced to the point with all components equal to 1. The couple (G_0, r_0) clearly has (G_p^1, r_p^1) as a minor, proving that the converse of Theorem 5.3 is not true.

Proof of Theorem 5.3. The proof is by induction on the number of edges. It is not hard to see that the statement holds for any graph with no more than two edges. Suppose that for any series-parallel graph G having no more than m edges and any connectivity type vector $r \in \{1, 2\}^V$ such that (G, r) has neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor, we have that $\mathrm{CPP}(G, r)$ is integer. Let us consider a series-parallel graph $G = (V, E)$ and a connectivity type vector $r \in \{1, 2\}^V$ such that G has $m+1$ edges, (G, r) has neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor and $\mathrm{CPP}(G, r)$ is not integer. There thus exists a fractional extreme point $\overline{x}$ of $\mathrm{CPP}(G, r)$. We can choose $\overline{x}$ among all the fractional extreme points of $\mathrm{CPP}(G, r)$ such that $|E_0(\overline{x}) \cup E_1(\overline{x})|$ is maximum. (Associated with $\overline{x}$, we consider the system $S(\overline{x})$ as well as the sets $E_0(\overline{x})$, $E_1(\overline{x})$ and $E_f(\overline{x})$ as defined in section 3.)

If $r(v) = 2$ for all $v \in V$, then from Theorem 5.2 the polytope $\mathrm{CPP}(G, r)$ is integer. Therefore, without loss of generality, we can also suppose that the cardinality of the terminal set T is maximum. It means that for any series-parallel graph $G' = (V', E')$ and any connectivity type vector $r' \in \{1, 2\}^{V'}$ such that $|E'| = m + 1$, (G', r') has neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor and $|\{v \in V' : r'(v) = 2\}| > |T|$, the polytope $\mathrm{CPP}(G', r')$ is integer.

From the induction hypothesis, it follows that

$$\overline{x}(e) > 0 \qquad \text{for all } e \in E, \tag{5.2}$$

that is, $E_0(\overline{x}) = \emptyset$.

CLAIM 1. *Any variable $x(e)$, $e \in E$, has a nonzero coefficient in at least two equations of $S(\overline{x})$.*

Proof. Let us consider an edge $f = uv$ of E, and denote $F = [u, v]$. From Remark 3.11, there exists at least one equation of $S(\overline{x})$ which contains $x(f)$ with a nonzero coefficient. Assume that there is exactly one such equation. Let $S'(\overline{x})$ be the system obtained from $S(\overline{x})$ by deleting all of the equations involving $x(e)$ for $e \in F$. We notice that the system $S'(\overline{x})$ contains exactly $|E| - |F|$ equations. Let x' be the restriction of $\overline{x}$ on $E \setminus F$. By Proposition 3.1, $x' \in \text{CPP}(G/F, r_F)$. Furthermore, x' is a solution of the system $S'(\overline{x})$. Since $S'(\overline{x})$ is nonsingular and its equations come from constraints of $\text{CPP}(G/F, r_F)$, this implies that x' is an extreme point of $\text{CPP}(G/F, r_F)$. By Proposition 3.12 and Remark 3.14, the point x' is fractional. As $(G/F, r_F)$ is a series-parallel graph having neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor, this contradicts the induction hypothesis. $\square$

CLAIM 2. *The graph G is 2-node connected.*

Proof. Suppose the graph G is not 2-node connected. There then exists a node $u \in V$ defining a one-node cutset of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two subgraphs of G such that $V_1 \cap V_2 = \{u\}$ and G is the 1-sum of G_1 and G_2. For $i = 1, 2$ we associate with G_i the connectivity type vector $r_i \in \{1, 2\}^{V_i}$ such that $r_i(v) = r(v)$ if $v \in V_i \setminus \{u\}$, and $r_i(u) = con_G(V \setminus V_i)$. Let $\overline{x}_1$ and $\overline{x}_2$ be the two restrictions of $\overline{x}$ on E_1 and E_2, respectively. From Lemma 4.1, both $\overline{x}_1$ and $\overline{x}_2$ are extreme points of $\text{CPP}(G_1, r_1)$ and $\text{CPP}(G_2, r_2)$, respectively. Since $E_f(\overline{x}) \neq \emptyset$, at least one among both restrictions is clearly fractional. Without loss of generality, we assume that $\overline{x}_1$ is fractional. We point out that (G_1, r_1) is nothing but the minor of (G, r) obtained by the contraction of E_2. Therefore, since the couple (G, r) has neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor, then neither does (G_1, r_1). The latter combined with $|E_1| < |E|$ and $\overline{x}_1$ fractional contradicts the induction hypothesis. $\square$

The proof now proceeds by successively establishing a sequence of claims which build on each other. Therefore, for the sake of clarity, we only mention the highlights of our argument, and the detailed sequences and proofs are deferred to section 6.

Let $u \in V$ be a node having exactly two neighbors, say u_1 and u_2. Since G is series-parallel, such a node u must exist [4]. Let us denote by F_1 and F_2 the set of edges between u and u_1, and u and u_2, respectively. Without loss of generality, we suppose

$$\overline{x}(F_1) \geq \overline{x}(F_2). \tag{5.3}$$

Suppose that $\overline{x}(F_1) < 1$. From the inequality (5.3), we also have $\overline{x}(F_2) < 1$, and then, by Remark 3.14, we obtain $F_1 = \{f_1 = uu_1\}$ and $F_2 = \{f_2 = uu_2\}$. From Proposition 3.13, there exists a partition $\{V_1, \ldots, V_p\}$ of $P_1^*(\overline{x}) \cup P_2^*(\overline{x})$ such that $|\delta(V_1, \ldots, V_p) \cap \{f_1, f_2\}| = 1$. Without loss of generality, assume that $f_1 \in \delta(V_1, \ldots, V_p)$, and then $\{u, u_2\} \subseteq V_i$ for some $i \in \{1, \ldots, p\}$. Hence, we obtain $\overline{x}[u, V_i \setminus \{u\}] = \overline{x}(f_2) < 1$, which contradicts Proposition 3.3. Therefore, $\overline{x}(F_1) \geq 1$.

CLAIM 3. $r(u) = 2$.

CLAIM 4. $r(u_2) = 1$.

CLAIM 5. *There does not exist a node of degree* 2.

CLAIM 6. $1 < \overline{x}(F_1) < 2$.

Let $F_1 = \{e_1, f_1\}$ with $\overline{x}(e_1) = 1$ and $0 < \overline{x}(f_1) < 1$. From Claims 3, 4, 5, 6, we can make the following remarks.

Remark 5.4. If v is a terminal adjacent to exactly two nodes, then between v and one of its neighbors there are exactly two edges.

Remark 5.5. Given two nodes v_1 and v_2, then any path P between v_1 and v_2, whose internal nodes have exactly two neighbors in G, satisfies the following.

(a) All of the internal nodes of P are terminals.

(b) P has at most two internal nodes.

Since $G = (V, E)$ is series-parallel and 2-connected, there is a 2-node cutset $\{v_1, v_2\}$ such that G decomposes with respect to $\{v_1, v_2\}$ into two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $\{v_1, v_2\} = V_1 \cap V_2$ and G_1 is a cycle, possibly with parallel edges (see Figure 5.3). Let L_1 and L_2 be the two walks of G_1 having only v_1 and v_2 in common. (In our context, a walk is a path between v_1 and v_2, possibly with parallel edges.) We remark that if $|V| \geq 3$, then there exists such a decomposition with $|V_1| \geq 3$.

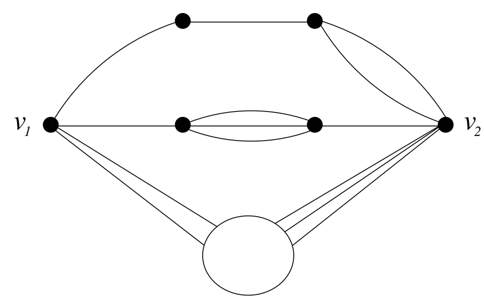

FIG. 5.3. *Decomposition of G by the 2-node cutset $\{v_1, v_2\}$.*

By Remark 5.5, all of the internal nodes of L_1 and L_2 are terminals and both L_1 and L_2 have each at most two internal nodes. Throughout the rest of the proof, we only consider the case where both L_1 and L_2 have each at most one internal node. The case of two internal nodes in L_1 or L_2 can be handled using the same arguments.

Without loss of generality, we can consider that L_1 has an internal node, say v. Under the induction assumptions, we remark that the claims 3, 4, and 6 are the results of the fact that u has exactly two neighbors. From now on, we consider $u = v$, $u_1 = v_1$, and $u_2 = v_2$.

CLAIM 7. $|F_2| = 1$

Let $F_2 = \{f_2 = uu_2\}$.

CLAIM 8. $\overline{x}(f_2) < 1$

CLAIM 9. *L_2 contains an internal node.*

Let $W = V \setminus \{u, u_1, u_2, u'\}$, where u' is an internal node in L_2. We can suppose, without loss of generality, that $|\delta(u)| \leq |\delta(u')|$. From Remark 5.5(a), we have $r(u') = 2$. To finish the proof of our theorem, we give the following three claims.

CLAIM 10. $|[u', u_1]| > |[u', u_2]|$.

From Claims 6 and 7, we obtain $|\delta(u')| = |\delta(u)| = 3$. Using similar arguments for u' as those used for u, we have $F_1' = \{e_1', f_1'\}$ and $F_2' = \{f_2'\}$ with $\overline{x}(e_1') = 1$, $0 < \overline{x}(f_1') < 1$, and $0 < \overline{x}(f_2') < 1$.

CLAIM 11. $\overline{x}(f_1) = \overline{x}(f_2) = \overline{x}(f_1') = \overline{x}(f_2') = \frac{1}{2}$.

CLAIM 12. *The couple (G, r) has the following properties:*

(a) *$W \cap T \neq \emptyset$. (We recall that $T = \{u \in V : r(u) = 2\}$.)*

(b) *$|[S, u_1]| \geq 2$ and $|[S, u_2]| \geq 1$ for every $S \subseteq W$ such that $G(S)$ is a connected component of $G(W)$ and $S \cap T \neq \emptyset$.*

Among the series-parallel graphs, the class of outerplanar graphs has received particular attention (see, for instance, [3, 20]). A graph is called *outerplanar* if it can be embedded in the plane such that all nodes lie on the boundary of its exterior region. In order to give a complete linear description of the survivable network polytope on outerplanar graphs, we give this second characterization devised from Kuratowski's theorem [16]. A graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor. (We recall that $K_{2,3}$ is the complete bipartite graph having its node set decomposed into V_1 and V_2 with $|V_1| = 2$, $|V_2| = 3$, and $E(V_1) = E(V_2) = \emptyset$.)

First of all, we remark that the graph G_p of Figure 5.1 contains $K_{2,3}$ as a minor. In fact, $K_{2,3}$ can be obtained from G_p by deleting the three solid edges. Therefore, from the second definition of outerplanar graphs and Theorems 5.1, 5.2, and 5.3, we can deduce the following result.

THEOREM 5.6. *Let $G = (V, E)$ be an outerplanar graph and $r \in \{1, 2\}^V$ be an associated connectivity type vector. The survivable network polytope SNDP(G, r) is then completely described by the trivial inequalities* (2.1) *and* (2.2) *together with the partition inequalities* (2.5).

We notice that there exist some series-parallel graphs which are not outerplanar and for which the polytope CPP(G, r) is integer. For instance, from Theorem 5.3, we know that the cut and partition inequalities polytope is an integer for $K_{2,3}$ with exactly three terminals, and clearly, $K_{2,3}$ is not outerplanar. Therefore, the class of outerplanar graphs is strictly included in the subclass of series-parallel graphs implying an integer cut and partition inequalities polytope.

Since the separation problem for the partition inequalities (2.5) is polynomially solvable, a direct consequence of Theorem 5.6 is the following corollary.

COROLLARY 5.7. *The* $(1, 2)$*-survivable network design problem can be solved in polynomial time on outerplanar graphs.*

6. Proof of Theorem 5.3. In order to allow a better understanding of the proof of Theorem 5.3, we have just presented its main ideas. This section is thus devoted to give the details of the proof.

Proof of Claim 3. Suppose that $r(u) = 1$ and the system $S(\overline{x})$ satisfies Proposition 3.18. Consider a partition $\pi = \{V_1, \ldots, V_p\}$ of $P_1^*(\overline{x}) \cup P_2^*(\overline{x})$ such that $F_1 \subseteq \delta(V_1, \ldots, V_p)$. From Claim 1, we know that such a partition exists. Since $\overline{x}(F_1) \geq 1$, by Proposition 3.18(a), we have $\pi \in P_2^*(\overline{x})$. Without loss of generality, assume $u \in V_1$ and $u_1 \in V_2$. From Proposition 3.18(b) and the assumption $r(u) = 1$, we obtain $r(V_1 \setminus \{u\}) = 2$. We can then deduce from Remark 3.4 that $u_2 \in V_1$. With π, we associate the partition $\pi_a = \{V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p\}$ which clearly is of type 2. Since $\overline{x}(\delta(V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p)) \geq p$, we then obtain $\overline{x}(F_2) \geq \overline{x}(F_1)$. This last inequality combined with inequality (5.3) gives $\overline{x}(F_2) = \overline{x}(F_1)$, and thus, $\pi_a \in P_2(\overline{x})$.

Let $S_{F_1}(\overline{x})$ be the system arisen from $S(\overline{x})$ when we delete every equation induced by $e \in E_1(\overline{x}) \cap F_1$ and the ones induced by the partitions π containing F_1, and we add the equations induced by the associated partitions π_a. (We remark that π_a might already belong to $P_2^*(\overline{x})$.) From Proposition 3.1, the restriction $\overline{x}_{F_1}$ of $\overline{x}$ on G/F_1 belongs to CPP$(G/F_1, r_{F_1})$. Moreover, the couple $(G/F_1, r_{F_1})$ clearly has neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor. Since G/F_1 has less edges than G, by the induction hypothesis, the polytope CPP$(G/F_1, r_{F_1})$ is an integer. Therefore $\overline{x}_{F_1}$, which is clearly fractional, is not an extreme point of CPP$(G/F_1, r_{F_1})$. There must thus exist an extreme point $y \in$ CPP$(G/F_1, r_{F_1})$ which is also a solution of $S_{F_1}(\overline{x})$. Let $\overline{y} \in \mathbb{R}^E$ be the unique point such that $\overline{y}(e) = y(e)$ if $e \in E \setminus F_1$, $\overline{y}(e) = 1$ if $e \in E_1(\overline{x}) \cap F_1$, and $\overline{y}(F_1) = y(F_2)$. We clearly have $\overline{y} \neq \overline{x}$. Moreover, it is obvious that $\overline{y}$ is also a solution of the system $S(\overline{x})$. This contradicts the fact that the system $S(\overline{x})$ is nonsingular. ∎

Proof of Claim 4. Assume on the contrary that $r(u_2) = 2$. Let us consider a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ of $P_1^*(\overline{x}) \cup P_2^*(\overline{x})$ such that $u \in V_1$ and $u_1 \in V_2$.

From Claim 1, such a partition must exist and from Proposition 3.18, $\pi \in P_2^*(\overline{x})$. We claim that this partition exists providing $\overline{x}(F_1) = \overline{x}(F_2)$. In fact, suppose that $\overline{x}(F_1) > \overline{x}(F_2)$. Since $\overline{x}(\delta(u)) = \overline{x}(F_1) + \overline{x}(F_2) \geq 2$, we clearly get $\overline{x}(F_1) > 1$. If $u_2 \notin V_1 \cup V_2$, Remark 3.4 implies that the class V_1 is then reduced to the single node u. The partition $\pi_1 = \{V_1 \cup V_2, V_3, \ldots, V_p\}$ is of type 2 because of $u \in V_1 \cup V_2$, $r(u) = 2$, $u_2 \notin V_1 \cup V_2$, and $r(u_2) = 2$. We then have

$$\begin{aligned}\overline{x}(\delta(V_1 \cup V_2, V_3, \ldots, V_p)) &= \overline{x}(\delta(V_1, V_2, \ldots, V_p)) - \overline{x}(F_1)\\ &= p - \overline{x}(F_1)\\ &< p - 1,\end{aligned}$$

which contradicts $\overline{x} \in \text{CPP}(G, r)$.

If $u_2 \in V_1$, the partition $\pi_2 = \{V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p\}$ is obviously of type 2. We then get

$$\begin{aligned}\overline{x}(\delta(V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p)) &= \overline{x}(\delta(V_1, V_2, \ldots, V_p)) - \overline{x}(F_1) + \overline{x}(F_2)\\ &= p - \overline{x}(F_1) + \overline{x}(F_2)\\ &< p,\end{aligned}$$

which contradicts $\overline{x} \in \text{CPP}(G, r)$. We therefore deduce that the node u_2 belongs to V_2, and thus $V_1 = \{u\}$. From Proposition 3.19, we could have chosen the system $S(\overline{x})$ such that $p = 2$, and thus partition π is nothing but the cut $\delta(u)$. Since $\overline{x}(F_1) > 1$, $\overline{x}(F_2) > 0$, and $\overline{x}(\delta(u)) = 2$, we clearly deduce that there exists $f_1 \in F_1 \cap E_f(\overline{x})$. This edge belongs to exactly one equation of $S(\overline{x})$, contradicting Claim 1. Therefore $\overline{x}(F_1) = \overline{x}(F_2)$.

We still consider the partition π introduced at the beginning of this proof. If $p = 2$, then we have $\overline{x}(F_1) = \overline{x}(F_2) = 1$ which implies that the equation induced by π is redundant with respect to $\overline{x}(e_1) = \overline{x}(e_2) = 1$, where $F_1 = \{e_1\}$ and $F_2 = \{e_2\}$. Hence, we can consider $p \geq 3$. From Proposition 3.19, we have either $u_2 \notin V_1 \cup V_2$ or $u_2 \in V_1$. If $u_2 \notin V_1 \cup V_2$ (respectively, $u_2 \in V_1$), then by considering the partition π_1 (respectively, π_2) previously defined, we obtain $\overline{x}(F_1) = 1$ and $\pi_1 \in P_2(\overline{x})$ (respectively, $\pi_2 \in P_2(\overline{x})$). Using the same arguments as the ones of the proof of Claim 3, we can exhibit a vector $\overline{y} \neq \overline{x}$ which is also a solution of $S(\overline{x})$. This contradiction completes our proof. ◻

Proof of Claim 5. Suppose that the statement does not hold. Let v be a node of degree 2 adjacent to exactly two different nodes, say v_1 and v_2. (Remark that the 2-node connectivity of G implies the existence of v_1 and v_2.) Without loss of generality, we may suppose that $\overline{x}(vv_1) \geq \overline{x}(vv_2)$. By Claims 3 and 4, it then follows that $r(v) = 2$ and $r(v_2) = 1$. Let $r^* \in \mathbb{R}^V$ be the connectivity type vector such that $r^*(w) = r(w)$ if $w \neq v_2$, and $r^*(w) = 2$ if $w = v_2$. We claim that (G, r^*) contains neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor. In fact, suppose, on the contrary, that (G, r^*) contains one of these minors, and let us denote that (H, r_H) contains one of these minors. Let $\pi = (S_1, \ldots, S_5)$ be the partition of V that induces H, where S_1 corresponds to the ordinary node of H of degree 3, S_2, S_3, S_4 to the terminal nodes of H of degree 3, and S_5 to the node of H of degree 6. As $r^*(v_2) = 2$, v_2 belongs to a class S_i of connectivity type 2, that is, $i \neq 1$. If S_i contains a node $w \neq v_2$ with $r^*(w) = 2$, then (H, r_H) would also be a minor of (G, r), a contradiction. Therefore $S_i \setminus \{v_2\}$ only contains nodes of connectivity type 1 with respect to r^*. In consequence, v belongs to S_j with $j \neq i$ and $j \neq 1$. Note that $v_1 \in S_j$. Otherwise as $G(S_j)$ is connected, one would have $S_j = \{v\}$, and thus, H would contain a node of degree 2. Since neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) has a node of degree 2, this is impossible.

If $i = 5$, then (H, r_H) is nothing but (G_p^2, r_p^2). Since $r^*(S_i) = 2$ and $r^*(S_i \setminus \{v_2\}) = 1$, we have $r(S_i) = 1$, and (G_p^1, r_p^1) is a minor of (G, r). This contradicts our hypothesis on (G, r). Therefore $2 \le i \le 4$.

If $j = 5$, let us consider the partition $\pi' = (S'_1, \ldots, S'_5)$ such that

$$\begin{aligned} S'_t &= S_t && t \in \{1, \ldots, 4\} \setminus \{i\}, \\ S'_i &= S_i \cup \{v\}, \\ S'_5 &= S_5 \setminus \{v\}. \end{aligned}$$

We can easily see that the partition π' induces either (G_p^1, r_p^1) or (G_p^2, r_p^2) with respect to r. This yields to a contradiction, and then $2 \le j \le 4$.

Since $i \neq j$, $v \in S_j$, and $v_2 \in S_i$, we have $[S_i, S_j] \neq \emptyset$. Therefore, G is not series-parallel.

Consequently, (G, r^*) contains neither (G_p^1, r_p^1) nor (G_p^2, r_p^2) as a minor. Let us now prove that $\overline{x}$ is an extreme point of CPP(G, r^*). We first show that $\overline{x}$ belongs to CPP(G, r^*). Let $\pi = (V_1, V_2, \ldots, V_p)$ be a partition of V. It is obvious that if v and v_2 are in the same class of π or if π is of type 2 with respect to r, then the type of π doesn't change by considering r^* instead of r. The inequality induced by π with respect to r^* is then satisfied by $\overline{x}$. Therefore, we only have to focus on the case where π is a partition of type 1 with respect to r and $e_2 \in \delta(V_1, V_2, \ldots, V_p)$. Without loss of generality, suppose that $v \in V_1$ and $v_2 \in V_2$. Since π is of type 1 with respect to r, all of the nodes u with $r(u) = 2$ belong to V_1. (We recall that (G, r) has at least two terminals which implies that $|V_1| \ge 2$.) The partition $(\{v\}, V_1 \setminus \{v\}, V_2, \ldots, V_p)$ is of type 2 with respect to r, and hence, we have

$$\begin{aligned} \overline{x}(\delta(\{v\}, V_1 \setminus \{v\}, V_2, \ldots, V_p)) &= \overline{x}(\delta(V_1, V_2, \ldots, V_p)) + \overline{x}[\{v\}, V_1 \setminus \{v\}] \\ &\ge p + 1. \end{aligned}$$

Since $\overline{x}[\{v\}, V_1 \setminus \{v\}] \le 1$, we obtain $\overline{x}(\delta(V_1, V_2, \ldots, V_p)) \ge p$. Therefore, $\overline{x}$ belongs to CPP(G, r^*). Using similar arguments, we can prove that $S(\overline{x})$ is a system of tight inequalities of CPP(G, r^*). Thus, $\overline{x}$ is an extreme point of CPP(G, r^*). As (G, r^*) has more terminals than (G, r), this contradicts the maximality of T. □

Proof of Claim 6. From Claim 5, we have $\overline{x}(F_1) > 1$. If $\overline{x}(F_1) \ge 2$, then by Proposition 3.20(a), the variable $x(e)$ belongs to exactly one equation of the system $S(\overline{x})$ with a nonzero coefficient, for all $e \in F_1$. Yet, this contradicts Claim 1. □

Since by Claim 6, $1 < \overline{x}(F_1) < 2$, it follows from Remark 3.14 that there exist two edges e_1, $f_1 \in E$ such that $F_1 = \{e_1, f_1\}$, $e_1 \in E_1(\overline{x})$, and $f_1 \in E_f(\overline{x})$, that is, $\overline{x}(e_1) = 1$ and $0 < \overline{x}(f_1) < 1$.

Proof of Claim 7. Suppose that $|F_2| = 2$. We then have $F_2 = \{e_2, f_2\}$ with $\overline{x}(e_2) = 1$ and $0 < \overline{x}(f_2) < 1$. A consequence of Proposition 3.18(b) is that F_1 and F_2 cannot belong to a same partition inducing an equation of $S(\overline{x})$. Let $F_0 = [u_1, u_2]$. We are going to consider two cases. We first consider $F_0 \neq \emptyset$. Using Proposition 3.2(b) and the previous remark about F_1 and F_2, we obtain $\overline{x}(F_0) < 1$. We then have $F_0 = \{f_0 = u_1u_2\}$. Let $y \in \mathbb{R}^E$ be the point defined as follows:

$$y(e) = \begin{cases} \overline{x}(e) & \text{if } e \in E \setminus \{f_0, f_1, f_2\}, \\ \overline{x}(f_0) - \epsilon & \text{if } e = f_0, \\ \overline{x}(f_1) + \epsilon & \text{if } e = f_1, \\ \overline{x}(f_2) + \epsilon & \text{if } e = f_2, \end{cases}$$

where ϵ is any arbitrary scalar. The point y is also a solution of $S(\overline{x})$. In fact, any partition inducing an equation of $S(\overline{x})$ contains either none of the edges in $\{f_0, f_1, f_2\}$, or f_0 and exactly one among f_1 and f_2. Since $y \neq \overline{x}$, this contradicts the extremality of $\overline{x}$.

We now suppose that $F_0 = \emptyset$. Let us denote by u^* the internal node in L_2. Without loss of generality, we can suppose that $|\delta(u)| \leq |\delta(u^*)|$, which makes us deduce that $|[u^*, u_1]| = |[u^*, u_2]| = 2$. Proposition 3.2(b) then implies that there is no partition of $P_1(\overline{x}) \cup P_2(\overline{x})$ containing an edge in $\delta(u)$. This is a contradiction with Claim 1. ∎

Proof of Claim 8. Let us suppose that $\overline{x}(f_2) = 1$. From Proposition 3.2(b) and $\overline{x}(F_1) > 1$, the system $S(\overline{x})$ can be chosen such that any of its equations containing $x(f_1)$ with a nonzero coefficient does not contain $x(f_2)$. In fact, suppose that there is a partition $\pi = \{V_1, \ldots, V_p\}$ of $S(\overline{x})$ such that $\delta(u) \subset \delta(V_1, \ldots, V_p)$. Thus, π is of type 2. Since $\overline{x}(\delta(u)) > 2$, we have $p \geq 3$. Without loss of generality, suppose that $V_1 = \{u\}$. If u_1 and u_2 are both in the same class of π, say V_2, then $\overline{x}(\delta(V_1 \cup V2, V_3, \ldots, V_p)) = p - \overline{x}(\delta(u)) < p-2$, which is a contradiction. Now suppose, without loss of generality, that $u_1 \in V_2$ and $u_2 \in V_3$. Since $\overline{x}[V_1, V_2] > 1$, then by Proposition 3.2 , the partition $\{V_1 \cup V_2, V_3, \ldots, V_p\}$ is of type 1. Thus, the partition $\pi' = \{V_1 \cup V_3, V_4, \ldots, V_p\}$ is of type 1. Moreover, π' is tight for $\overline{x}$. We can thus replace in the system $S(\overline{x})$ the partition π by the partition π' and the equation $x(f_2) = 1$.

Let $\pi^1 = \{V_1, V_2, \ldots, V_p\}$ be a partition of $S(\overline{x})$ such that $F_1 \subseteq \delta(V_1, \ldots, V_p)$. We thus have $f_2 \notin \delta(V_1, V_2, \ldots, V_p)$. By Proposition 3.18, we have $\pi^1 \in P_2^*(\overline{x})$. Without loss of generality, we suppose $\{u, u_2\} \subseteq V_1$ and $u_1 \in V_2$.

From Claim 1, there also exists a partition $\pi^2 = \{W_1, \ldots, W_q\}$ inducing an equation of $S(\overline{x})$ such that $f_2 \in \delta(W_1, \ldots, W_q)$. We are going to prove that the system $S(\overline{x})$ can be chosen such that $q \geq 3$. From above, we clearly have $F_1 \cap \delta(W_1, \ldots, W_q) = \emptyset$. Suppose that $q = 2$. Without loss of generality, we assume that $u_2 \in W_1$. We then have $\{u, u_1\} \subseteq W_2$. From $\overline{x}(f_2) = 1$, it is obvious that $con(W_1) = 2$. Proposition 3.17 then implies that at most one class of the partition π^1 intersects both W_1 and W_2. From the definition of π^1, we have $V_1 \cap W_1 \neq \emptyset$ and $V_1 \cap W_2 \neq \emptyset$. We then obtain $V_2 \subseteq W_2$. Since $r(V \setminus (V_1 \cup V_2)) = 1$ and $con(W_1) = 2$, we get $r(V_1 \setminus \{u\}) = 2$. Thus, the partition $\{V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p\}$ is clearly of type 2, and from Claim 7, we have

$$\begin{aligned}\overline{x}(\delta(V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p)) &= \overline{x}(\delta(V_1, \ldots, V_p)) - \overline{x}(F_1) + \overline{x}(f_2)\\ &= p - \overline{x}(F_1) + \overline{x}(f_2)\\ &< p,\end{aligned}$$

a contradiction.

Without loss of generality, we suppose $\{u, u_1\} \subseteq W_1$ and $u_2 \in W_2$. Since $\overline{x}(f_2) = 1$, Proposition 3.18 implies that $\pi^2 \in P_2^*(\overline{x})$ and $r(V \setminus (W_1 \cup W_2)) = 1$. Without loss of generality, let us assume that $W_i \cap V_2 \neq \emptyset$ for $i = 1, \ldots, k$, and $W_i \cap V_2 = \emptyset$ for $i = k+1, \ldots, q$. In fact, we have $u_1 \in W_1$ and $u_1 \in V_2$. Moreover, since $\overline{x}(f_2) < \overline{x}(F_1)$ by Claim 7, the partition $\{V_1 \setminus \{u\}, V_2 \cup \{u\}, V_3, \ldots, V_p\}$ must be of type 1, and then $r(V_1 \setminus \{u\}) = 1$. As $\pi^2 \in P_2^*(\overline{x})$ and $r(V \setminus (W_1 \cup W_2)) = 1$, we have $r(W_2) = 2$, which, combined with $r(V \setminus (V_1 \cup V_2)) = 1$ and $u \notin W_2$, implies that $W_2 \cap V_2 \neq \emptyset$. Let $\{U_1, \ldots, U_k\}$ be the partition of V_2 such that $U_i = W_i \cap V_2$ for $i = 1, \ldots, k$. It comes directly from the previous argument that $r(U_1) = r(U_2) = 2$. Therefore, by Proposition 3.9(a), we have $\overline{x}[U_1, \ldots, U_k] \geq k - 1 + \overline{x}(F_1) - \overline{x}(f_2)$. Since

$\overline{x}(F_1) = \overline{x}(e_1) + \overline{x}(f_1) = 1 + \overline{x}(f_1)$, we obtain

$$\overline{x}[U_1, \ldots, U_k] \geq k + \overline{x}(f_1) - \overline{x}(f_2). \tag{6.1}$$

Furthermore, we notice that $F_2 \cap [U_1, \ldots, U_k] = \emptyset$. Hence, from (5.2), we have

$$\overline{x}[U_1, \ldots, U_k] \leq \overline{x}(\delta(W_1, \ldots, W_q)) - \overline{x}(f_2). \tag{6.2}$$

If $k = q$, by the inequalities (6.1) and (6.2), we then obtain $k + \overline{x}(f_1) - \overline{x}(f_2) \leq k - \overline{x}(f_2)$. This implies that $\overline{x}(f_1) \leq 0$ which contradicts (5.2). Suppose now that $k < q$. Since $r(V \setminus (W_1 \cup W_2)) = 1$ and $k \geq 2$, it is straightforward that the partition $\{Z_1, \ldots, Z_{q-k+1}\}$ defined as

$$\begin{aligned} &Z_1 = \bigcup_{i=1}^{k} W_i, \\ &Z_i = W_{i+k-1}, \qquad i = 2, \ldots, q-k+1, \end{aligned}$$

is of type 1. We then have

$$\begin{aligned} \overline{x}(\delta(W_1, \ldots, W_q)) - \overline{x}[U_1, \ldots, U_k] - \overline{x}(f_2) &\geq \overline{x}(\delta(Z_1, \ldots, Z_{q-k+1})) \\ &\geq q - k + 1 - 1 \\ &= q - k. \end{aligned}$$

Thus, we obtain $\overline{x}[U_1, \ldots, U_k] + \overline{x}(f_2) \leq k$. From this last inequality combined with (6.1), we then get $\overline{x}(f_1) \leq 0$. This contradicts (5.2). Consequently, we obtain $\overline{x}(f_2) < 1$. □

Proof of Claim 9. Suppose that $F_0 = [u_1, u_2] \neq \emptyset$. Suppose that $\overline{x}(F_0) \geq 1$. Let be $\pi = \{V_1, \ldots, V_p\} \in S(\overline{x})$ such that $F_1 \in \delta(V_1, \ldots, V_p)$. Since $\overline{x}(F_1) = \overline{x}[u, u_1] > 1$, by Proposition 3.18(a), π is of type 2. Since $\overline{x}(F_0) \geq 1$, by Proposition 3.18(b), u and u_2 are in the same class of π, say V_1. Suppose, without loss of generality, that $u_1 \in V_2$. Since $F_0 \cup F_1 \in \delta(V_1, \ldots, V_p)$ and $\overline{x}(F_0 \cup F_1) > 2$, we then have $p \geq 3$ and $\overline{x}(\delta(V_1 \cup V2, V_3, \ldots, V_p)) = p - \overline{x}(\delta(u)) < p - 2$, a contradiction. We therefore obtain $\overline{x}(F_0) < 1$. Let $F_0 = \{f_0 = u_1u_2\}$. Let $y \in \mathbb{R}^E$ be the point defined as follows:

$$y(e) = \begin{cases} \overline{x}(e) & \text{if } e \in E \setminus \{f_0, f_2\}, \\ \overline{x}(f_0) - \epsilon & \text{if } e = f_0, \\ \overline{x}(f_2) + \epsilon & \text{if } e = f_2, \end{cases}$$

where $\epsilon = \min\{\overline{x}(f_0), 1 - \overline{x}(f_2)\}$. We first remark that any partition of $P_1(\overline{x}) \cup P_2(\overline{x})$, different from $\{\{u\}, V \setminus \{u\}\}$, contains either both f_0 and f_2 or none of them. If the partition $\{\{u\}, V \setminus \{u\}\}$ doesn't belong to $P_2(\overline{x})$, then the point y is a solution of $S(\overline{x})$ which is different to $\overline{x}$. This is a contradiction with the extremality of $\overline{x}$.

We suppose now that $\{\{u\}, V \setminus \{u\}\}$ belongs to $P_2(\overline{x})$. We first show that y is a point of CPP(G, r). To obtain that, we only need to prove that for any partition $\pi = \{V_1, V_2, \ldots, V_p\}$ with $f_0 \in \delta(V_1, V_2, \ldots, V_p)$ and $f_2 \notin \delta(V_1, V_2, \ldots, V_p)$, we have $y(\delta(V_1, V_2, \ldots, V_p)) \geq t$, where $t = p-1$ if π is of type 1 and $t = p$ otherwise. Without loss of generality, we suppose that $u \in V_1$. Let us consider the partition $\{\{u\}, V_1 \setminus \{u\}, V_2, \ldots, V_p\}$. Note that this partition is of the same type as $\{V_1, V_2, \ldots, V_p\}$.

Hence we have

$$\begin{aligned}\overline{x}(\delta(\{u\}, V_1 \setminus \{u\}, V_2, \dots, V_p)) &= y(\delta(\{u\}, V_1 \setminus \{u\}, V_2, \dots, V_p)) \\ &= y(\delta(V_1, V_2, \dots, V_p)) + y(f_2) \\ &= y(\delta(V_1, V_2, \dots, V_p)) + \overline{x}(f_2) + \epsilon \\ &\geq t + 1.\end{aligned}$$

This implies that

$$\begin{aligned}y(\delta(V_1, V_2, \dots, V_p)) &\geq t + 1 - (\overline{x}(f_2) + \epsilon) \\ &\geq t + 1 - 1 = t.\end{aligned}$$

From the definition of ϵ, we clearly have $0 \leq y(f_0) \leq 1$ and $0 \leq y(f_2) \leq 1$. Therefore, y belongs to $\text{CPP}(G, r)$.

We remark that $|E_f(y)| < |E_f(\overline{x})|$ since at least one variable among $y(f_0)$ and $y(f_2)$ is an integer. Moreover, as $0 < y(f_1) < 1$, y is fractional. By the induction hypothesis, y isn't an extreme point of $\text{CPP}(G, r)$. Hence, there exist $t \geq 2$ extreme points $y^1, \dots, y^t$ of $\text{CPP}(G, r)$ and t scalars $0 < \alpha_i < 1$, $i = 1, \dots, t$, such that

$$y = \sum_{i=1}^{t} \alpha_i y^i \quad \text{and} \quad \sum_{i=1}^{t} \alpha_i = 1.$$

It is clear that $|E_f(y^i)| < |E_f(\overline{x})|$ for $i = 1, \dots, t$. From the extremality of the y^i and the induction hypothesis on $\overline{x}$, we obtain that the points y^i are integer. All of the constraints that are tight for y are also tight for y^i. Moreover, since $y(\delta(u)) = 2+\epsilon < 3$, there exists $i_0 \in \{1, \dots, t\}$ such that $y^{i_0}(\delta(u)) < 3$. The integrality of y^{i_0} then implies that $y^{i_0}(\delta(u)) = 2$. Therefore, the point y^{i_0} is also a solution of $S(\overline{x})$ which contradicts the extremality of $\overline{x}$. We then conclude that $F_0 = \emptyset$. □

Proof of Claim 10. Denote $F'_1 = [u', u_1]$ and $F'_2 = [u', u_2]$. Suppose that $|F'_2| \geq 2$. From (5.2) and Remark 3.14, we have $\overline{x}(F'_2) > 1$. Let $\pi = \{V_1, \dots, V_p\}$ be a partition of $P^*_1(\overline{x}) \cup P^*_2(\overline{x})$ different from $\{\{u\}, V \setminus \{u\}\}$. By Claim 1, such a partition exists. Moreover, Claim 8 together with Proposition 3.3 implies that $p \geq 3$. From Proposition 3.19, u, u_1, and u_2 belong to three different classes of π. Without loss of generality, suppose that $u \in V_1$, $u_1 \in V_2$, and $u_2 \in V_3$. We remark that $V_1 = \{u\}$. Using Proposition 3.18, we obtain that $u' \in V_3$. Therefore, π is a partition of type 2. The partition $\pi' = \{V_1 \cup V_2, V_3, \dots, V_p\}$ is of type 2, and is such that

$$\begin{aligned}\overline{x}(\delta(V_1 \cup V_2, V_3, \dots, V_p)) &= \overline{x}(\delta(V_1, \dots, V_p)) - \overline{x}(F_1) \\ &< p - 1.\end{aligned}$$

The last inequality comes from $\overline{x}(F_1) > 1$. We then get a contradiction. We conclude that $|F'_2| = 1$. By Claims 5 and 6, we have $|F'_1| = 2$. □

To make the proofs of the next two claims clearer and shorter, we introduce additional notation. Given an edge subset $F \subseteq E$, we denote by $P(\overline{x}, F)$ the subset of partitions $\pi = \{V_1, \dots, V_p\} \in P^*_1(\overline{x}) \cup P^*_2(\overline{x})$ such that $p \geq 3$ and $F \subseteq \delta(V_1, \dots, V_p)$.

Proof of Claim 11. In order to prove the result, it is enough to prove that $\overline{x}(F_1) = \overline{x}(F'_1)$, $\overline{x}(\delta(u)) = \overline{x}(\delta(u')) = 2$, and $\overline{x}(f_2) + \overline{x}(f'_2) = 1$.

We first prove that $\overline{x}(F_1) = \overline{x}(F'_1)$. Without loss of generality, we suppose $\overline{x}(F_1) \geq \overline{x}(F'_1)$. Since $0 < \overline{x}(f_1) < 1$, by Claim 1, there must exist a partition $\pi = \{V_1, \dots, V_p\}$ such that $\pi \in P(\overline{x}, F_1)$. By Proposition 3.18(a), $\pi \in P^*_2(\overline{x})$. Without loss of generality, assume $u \in V_1$ and $u_1 \in V_2$. From $\overline{x}(F_2) < 1$ and Proposition 3.3(c), we obtain $V_1 = \{u\}$. Since $\overline{x}[u, u_1] > 1$, by Proposition 3.18(b), we have $r(V \setminus (V_1 \cup V_2)) = 1$, and then $u' \in V_2$. Consider then the partition $\{\{u'\}, (V_2 \setminus \{u'\}) \cup \{u\}, V_3, \ldots, V_p\}$. It clearly is a partition of type 2, and then we have

$$\begin{aligned}\overline{x}(\delta(\{u'\}, (V_2 \setminus \{u'\}) \cup \{u\}, V_3, \ldots, V_p)) &= \overline{x}(\delta(V_1, \ldots, V_p)) + \overline{x}(F_1') - \overline{x}(F_1)\\ &= p + \overline{x}(F_1') - \overline{x}(F_1)\\ &\geq p.\end{aligned}$$

We get $\overline{x}(F_1) \leq \overline{x}(F_1')$, and consequently $\overline{x}(F_1) = \overline{x}(F_1')$.

Now we are going to prove that $\overline{x}(\delta(u)) = \overline{x}(\delta(u')) = 2$. Since $0 < \overline{x}(f_1) < 1$, as we have shown in the proof of Claim 10, the system $S(\overline{x})$ can be chosen such that there exists a partition $\pi^0 = \{V_1^0, \ldots, V_{p_0}^0\} \in P(\overline{x}, F_1)$ such that u, u_1, and u_2 belong to three different classes of π_0. Without loss of generality, suppose that $V_1^0 = \{u\}$, $u_1 \in V_2^0$, and $u_2 \in V_3^0$. Since $r(u') = 2$, Proposition 3.18(b) implies $u' \in V_2^0$. Let $\pi_0' = \{V_1', V_2', V_3^0, \ldots, V_{p_0}^0\}$, where $V_1' = \{u'\}$ and $V_2' = (V_2^0 \setminus \{u'\}) \cup \{u\}$. This partition clearly is of type 2, and since $\overline{x}(F_1) = \overline{x}(F_1')$, we get

$$\begin{aligned}\overline{x}(\delta(V_1', V_2', V_3^0, \ldots, V_{p_0}^0)) &= \overline{x}(\delta(V_1^0, \ldots, V_{p_0}^0)) + \overline{x}(F_1') - \overline{x}(F_1)\\ &= p_0 + \overline{x}(F_1') - \overline{x}(F_1)\\ &= p_0.\end{aligned}$$

Let π be any partition of $P(\overline{x}, F_1) \setminus \{\pi_0\}$. The partition π' obtained from π by switching u and u', as π_0' was obtained above from π_0, belongs to $P_2(\overline{x})$. Consider the system $S'(\overline{x})$ obtained from $S(\overline{x})$ by adding the equation induced by π_0' and replacing those induced by $\pi \in P(\overline{x}, F_1) \setminus \{\pi_0\}$ by the ones induced by π'. This system is nonsingular, and $\overline{x}$ is its unique solution. The system $S(\overline{x})$ can then be chosen such that $P(\overline{x}, F_1) = \{\pi_0\}$. Since $0 < \overline{x}(f_1) < 1$, Claim 1 implies $\overline{x}(\delta(u)) = 2$, and thus, $\overline{x}(\delta(u')) = 2$. Moreover, we have $\overline{x}(F_1) = \overline{x}(F_1')$, and we then get $\overline{x}(f_2) = \overline{x}(f_2')$.

Finally, we prove that $\overline{x}(f_2) + \overline{x}(f_2') = 1$. Suppose that $\overline{x}(f_2) + \overline{x}(f_2') > 1$. (The case $\overline{x}(f_2) + \overline{x}(f_2') < 1$ is similar.) Let $S_{F_1}(\overline{x})$ be the system obtained from $S(\overline{x})$ by removing the equations $x(e_1) = 1$, $x(\delta(u)) = 2$, and the one induced by π_0. Let $\overline{x}_1$ be the restriction of $\overline{x}$ on G/F_1. By the induction hypothesis, the polytope $\mathrm{CPP}(G/F_1, r_{F_1})$ is an integer. Since $\overline{x}_1$ is fractional, it is not an extreme point of $\mathrm{CPP}(G/F_1, r_{F_1})$. Hence, there exist $k \geq 2$ extreme points $y^1, \ldots, y^k$ of $\mathrm{CPP}(G/F_1, r_{F_1})$ and k scalars $0 < \alpha_i < 1$, $i = 1, \ldots, k$, such that

$$\overline{x}_1 = \sum_{i=1}^{k} \alpha_i y^i \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 1.$$

By the integrality of $\mathrm{CPP}(G/F_1, r_{F_1})$, the points y^i, $i = 1, \ldots, k$, are integer. Since $\overline{x}$ is a solution of $S_{F_1}(\overline{x}_1)$ and is a convex combination of the points y^i, $i = 1, \ldots, k$, it is clear that y^i is also a solution of $S_{F_1}(\overline{x}_1)$ for all $i \in \{1, \ldots, k\}$. There must then exist some $j \in \{1, \ldots, k\}$ such that $y^j(f_2) + y^j(f_2') > 1$. Since y^j is integer, we have $y^j(f_2) + y^j(f_2') \geq 2$, which gives $y^j(f_2) = y^j(f_2') = 1$. Let $\overline{y} \in \mathbb{R}^E$ be the point defined below

$$\overline{y}(e) = \begin{cases} y^j(e) & \text{if } e \in E \setminus F_1,\\ 1 & \text{if } e = e_1,\\ 0 & \text{if } e = f_1.\end{cases}$$

This point is also a solution of $S(\overline{x})$. In fact, additionally, to be a solution of $S_{F_1}(\overline{x})$, y^j clearly satisfies the equations $x(e_1) = 1$, $x(\delta(u)) = 2$, and $x(\delta(V_1^0, \ldots, V_{p_0}^0)) = p_0$, which are the only three equations removed from $S(\overline{x})$ to get $S_{F_1}(\overline{x})$. Since $\overline{y} \neq \overline{x}$, this contradicts the fact that $\overline{x}$ is an extreme point of $\mathrm{CPP}(G, r)$. $\square$

Proof of Claim 12. We first prove (a), that is, $W \cap T \neq \emptyset$. From Claim 4 we know that $u_2 \notin T$. Suppose now that $T \subseteq \{u, u', u_1\}$. Let $G^* = (V^*, E^*)$ be the graph obtained from G by contracting F_1 and F_1', and deleting f_2 as well. Let $w \in V^*$ be the node arising from the contraction, that is, $V^* \setminus V = \{w\}$. We have $r(v) = 1$ for all $v \in V \setminus \{u, u', u_1\}$ and therefore, the connectivity type vector r^* associated with V^* can be defined such that $r^*(v) = 1$ for all $v \in V^*$. Let x^* be the restriction of $\overline{x}$ on E^*.

We are going to show that $x^* \in \mathrm{CPP}(G^*, r^*)$. Consider a partition $\pi^* = \{V_1^*, \ldots, V_p^*\}$, $p \geq 2$, of V^*. (We remark that any partition of V^* is obviously of type 1 with respect to r^*.) Without loss of generality, we suppose $w \in V_1^*$. Let $\{U_1, \ldots, U_p\}$ be the partition of V such that $U_1 = (V_1^* \setminus \{w\}) \cup \{u, u', u_1\}$ and $U_i = V_i^*$ for all $i = 2, \ldots, p$. This partition clearly is of type 1 with respect to r. If $f_2' \notin \delta(V_1^*, \ldots, V_p^*)$, then we get

$$\begin{aligned} x^*(\delta(V_1^*, \ldots, V_p^*)) &= \overline{x}(\delta(U_1, \ldots, U_p)) \\ &\geq p - 1. \end{aligned}$$

If $f_2' \in \delta(V_1^*, \ldots, V_p^*)$, we suppose, without loss of generality, that $u_2 \in V_2^*$. By considering the partition $\{\{u\}, U_1 \setminus \{u\}, U_2, \ldots, U_p\}$ of V which is of type 2 with respect to r, we then have

$$\begin{aligned} x^*(\delta(V_1^*, \ldots, V_p^*)) &= \overline{x}(\delta(\{u\}, U_1 \setminus \{u\}, U_2, \ldots, U_p)) - \overline{x}(\delta(u)) \\ &\geq p + 1 - 2 \\ &= p - 1. \end{aligned}$$

Therefore, we conclude that $x^* \in \mathrm{CPP}(G^*, r^*)$.

Proposition 3.16 and $\overline{x}(\delta(u)) = 2$ imply that the system $S(\overline{x})$ can be chosen such that $P_1^*(\overline{x}) = \emptyset$. Let $\pi_0^* = \{U_1^0, \ldots, U_{p_0-1}^0\}$ be the partition of V^* such that $U_1^0 = (V_2^0 \setminus \{u', u_1\}) \cup \{w\}$ and $U_i^0 = V_{i+1}^0$ for all $i = 2, \ldots, p_0 - 1$, where $\pi^0 = \{V_1^0, \ldots, V_{p_0}^0\}$ is the unique partition in $P(\overline{x}, F_1)$. (See the proof of Claim 11 for the definition of π^0.) We then get

$$\begin{aligned} x^*(\delta(U_1^0, \ldots, U_{p_0-1}^0)) &= \overline{x}(\delta(V_1^0, \ldots, V_{p_0}^0)) - \overline{x}(\delta(u)) \\ &= p_0 - 2. \end{aligned}$$

Let $\pi_1 = \{V_1, \ldots, V_p\}$ be a partition in $P(\overline{x}, F_1')$ and then, in $P_2^*(\overline{x})$ too. We suppose, without loss of generality, that $\{u, u_1\} \subseteq V_1$ and $V_2 = \{u'\}$. Let $\pi_1^* = \{V_1', \ldots, V_{p-1}'\}$ be the partition of V^* such that $V_1' = (V_1 \setminus \{u, u_1\}) \cup \{w\}$ and $V_i' = V_{i+1}$ for all $i = 2, \ldots, p - 1$. This partition clearly is of type 1 with respect to r^*, and then

$$\begin{aligned} x^*(\delta(V_1', \ldots, V_{p-1}')) &= \overline{x}(\delta(V_1, \ldots, V_p)) - (\overline{x}(f_2) + \overline{x}(F_1')) \\ &= p - \left(\frac{1}{2} + \frac{3}{2}\right) \\ &= p - 2. \end{aligned}$$

Let $Q^*(F_1') = \{\pi_1^* : \pi_1 \in P(\overline{x}, F_1')\}$ be the set of partitions of V^* obtained from the partitions in $P(\overline{x}, F_1')$ as described above. The point x^* is a solution of the following system:

$$S(x^*)\begin{cases} x(e) = 1 & \text{for all } e \in E_1(\overline{x}) \setminus \{e_1,\ e_1'\}, \\ x(\delta(U_1^0, \dots, U_{p_0-1}^0)) = p_0 - 2, & \\ x(\delta(V_1^*, \dots, V_p^*)) = p - 2 & \text{for all } \{V_1^*, \dots, V_p^*\} \in Q^*(F_1'). \end{cases}$$

Since x^* is fractional, from Theorem 5.1, it cannot be an extreme point of $\text{CPP}(G^*, r^*)$. There must then exist an extreme point y^* of $\text{CPP}(G^*, r^*)$ such that y^* is a solution of $S(x^*)$ and $y^*(f_2^*) > 0$. By Theorem 5.1, y^* is an integer, and thus, $y^*(f_2') = 1$. Let $\overline{y} \in \mathbb{R}^E$ be the point defined as

$$\overline{y}(e) = \begin{cases} y^*(e) & \text{if } e \notin \delta(u) \cup F_1', \\ 1 & \text{if } e \in \{e_1, e_1', f_2\}, \\ 0 & \text{if } e \in \{f_1, f_1'\}. \end{cases}$$

The point $\overline{y}$ clearly satisfies the two equations $x(\delta(u)) = 2$ and $x(\delta(u')) = 2$. Therefore, since $P_2^*(\overline{x}) = P(\overline{x}, F_1) \cup P(\overline{x}, F_1')$ and y^* is a solution of $S(x^*)$, the point $\overline{y}$ is also a solution of $S(\overline{x})$. This leads to a contradiction, and we conclude that $W \cap T \neq \emptyset$.

To prove (b), we consider a node subset $S \subseteq W$ such that $G(S)$ is a connected component of $G(W)$ and $S \cap T \neq \emptyset$. Suppose that $[S, u_1] = \emptyset$. (The proof for the case $[S, u_2] = \emptyset$ is similar.) The node u_2 then defines a one-node cutset of G which contradicts Claim 2. Therefore $|[S, u_1]| \geq 1$.

Suppose that $|[S, u_1]| = 1$. From Proposition 3.18(b), we have $r(V \setminus (V_1^0 \cup V_2^0)) = 1$, where $\pi_0 = \{V_1^0, \dots, V_{p_0}^0\}$ is the only partition in $P(\overline{x}, F_1)$ as previously defined. We then have $(W \cap T) \subset V_2^0$. Thus, $S \cap T \subset V_2^0$. Let $S^0 = S \cap V_2^0$. We obviously have $r(S^0) = 2$. Since $|[S, u_1]| = 1$, we get $\overline{x}[S^0, u_1] \leq 1$. By considering the partition $\{S^0, (V_2^0 \setminus S^0) \cup \{u\}, V_3^0, \dots, V_{p_0}^0\}$ which is of type 2, we have

$$\begin{aligned} \overline{x}(\delta(S^0, (V_2^0 \setminus S^0) \cup \{u\}, V_3^0, \dots, V_{p_0}^0)) &= \overline{x}(\delta(V_1^0, \dots, V_{p_0}^0)) - \overline{x}(F_1) + \overline{x}[S^0, u_1] \\ &\leq p_0 - \frac{3}{2} + 1 \\ &= p_0 - \frac{1}{2}. \end{aligned}$$

This leads to a contradiction with $\overline{x} \in \text{CPP}(G, r)$. Consequently, we have $|[S, u_1]| \geq 2$. ∎

7. New facet-defining inequalities for the polytope SNDP(G, r). In this section, we introduce a new family of facet-defining inequalities of the polytope $\text{SNDP}(G, r)$. Given a graph $G = (V, E)$, not necessarily series-parallel, and a connectivity type vector $r \in \{1, 2\}^V$, let $\{V_0, V_1, \dots, V_t, V_{t+1}\}$ be a partition of V such that $t \geq 2$ and

(a) $r(V_0) = 1$,

(b) $r(V_i) = 2$ for $i = 1, 2, \dots, t$,

(c) $[V_i, V_j] = \emptyset$ for $i = 1, 2, \dots, t-1$ and $j = i+1, \dots, t$.

Consider the inequality

$$x(\delta(V_0, V_1, \dots, V_t, V_{t+1})) + x(\delta(V_0)) \geq 2t + 2 \qquad (7.1)$$

THEOREM 7.1. *Inequality* (7.1) *is valid for the polytope SNDP(G, r).*

Proof. Let $H = (V, F)$ be a survivable subgraph of G. Let $F' = F \cap \delta(V_0, V_1, \ldots, V_t, V_{t+1})$. It suffices to prove that $|F'| + |F' \cap \delta(V_0)| \geq 2t + 2$. Since $r(V_0) = 1$, we have $|F' \cap \delta(V_0)| \geq 1$. If $|F' \cap \delta(V_0)| = 1$, let $i_0 \in \{1, \ldots, t\}$ such that $F' \cap \delta(V_0) = [V_0, V_{i_0}]$. Since H is survivable, we have $|F' \cap \delta(V_0 \cup V_{i_0})| \geq 2$ and $|F' \cap \delta(V_i)| \geq 2$ for all $i \in \{1, \ldots, t\} \setminus \{i_0\}$. We then obtain that $|F' \setminus \delta(V_0)| \geq 2t$. From $|F' \setminus \delta(V_0)| + 2|F' \cap \delta(V_0)| = |F'| + |F' \cap \delta(V_0)|$, we get $|F'| + |F' \cap \delta(V_0)| \geq 2t + 2$. If $|F' \cap \delta(V_0)| \geq 2$, by summing up the $t + 1$ inequalities $|F' \cap \delta(V_i)| \geq 2$ for all $i \in \{0, 1, \ldots, t\}$, we obtain $|F'| + |F' \cap \delta(V_0)| \geq 2t + 2$. □

We call inequalities of type (7.1) *spinning-top inequalities*. In the next theorem, we give necessary and sufficient conditions for spinning-top inequalities to define facets of SNDP(G, r) when $G(V_{t+1})$ is 2-edge connected. We denote by $\mathcal{F}$ the face of SNDP(G, r) induced by a spinning-top inequality. From the proof of Theorem 7.1, we give the following remark which is useful to prove the theorem.

Remark 7.2. Let (V, F) be a survivable subgraph of G. If $|F \cap \delta(V_0)| \geq 3$ or $|F \cap \delta(V_0)| \geq 4$ for some $i \in \{1, \ldots, t\}$, then $x^F \notin \mathcal{F}$.

THEOREM 7.3. *Suppose that $G(V_{t+1})$ is 2-edge connected. Inequality* (7.1) *then defines a facet of the polytope SNDP(G,r) if and only if the following holds:*

(a) *$G(V_i)$ is 2-edge connected for $i = 0, 1, \ldots, t$,*

(b) *$|[V_0, V_i]| \geq 1$ for $i = 1, \ldots, t$, and*

(c) *$|[V_i, V_{t+1}]| \geq 2$ for $i = 1, \ldots, t$.*

Proof. The proof uses Remark 7.2 and standard polyhedral techniques. □

A direct consequence of Theorem 7.3 is that for a general couple (G, r), inequality (7.1) must be considered to obtain a complete linear description of the polytope SNDP(G, r). The previous statement remains true even when G is a series-parallel graph, as we showed in section 5.

8. Final remarks. In this paper, we studied the polytope CPP(G, r) given by the trivial inequalities and the partition ones. We first gave some structural properties of the extreme points of CPP(G, r). Using these, we proved that the polytope CPP(G, r) is an integer on a nontrivial subclass of series-parallel graphs, which includes the outerplanar graph class. This result leads to a polynomial-time algorithm, based on the ellipsoid method, for solving the $(1, 2)$-survivable network design problem in that class of graphs. To the best of our knowledge, the complete linear description of SNDP(G, r) given in this paper is the first one which combines even and odd connectivity types for an important class of graphs. We also introduced a new family of facet-defining inequalities for the polytope SNDP(G, r), called the *spinning-top inequalities*. This class of inequalities must be considered in linear descriptions of the survivable network polytope SNDP(G, r).

From Theorem 5.3, we can deduce that CPP(G, r) is an integer when G is a series-parallel graph and $|T| \leq 2$. (We remind that T is the set of terminal nodes, that is, nodes u such that $r(u) = 2$.) For a general graph G and $|T| \leq 1$, we know that SNDP$(G, r) =$ CPP(G, r) [5]. Moreover, for a general graph G, a positive cost function, and $|T| = 2$, the survivable network design problem can be solved in polynomial time, since Arkin and Hassin [1] have shown that this special case of the SNDP can be reduced to the matroid intersection problem. In view of the previous discussion, we give the following conjecture.

CONJECTURE 8.1. *Let $G = (V, E)$ be a graph and $r \in \{1, 2\}^V$ its connectivity type vector such that $|\{u \in V : r(u) = 2\}| = 2$. The polytope SNDP$(G,r)$ is then completely described by both the trivial and the partition inequalities (i.e., CPP(G,r) is an integer).*

An important problem which deserves to be addressed is to study the separation problem of the spinning-top inequalities. We think that this problem is polynomial on series-parallel graphs and NP-hard in general.

Our work has started with the objective of giving a linear description of the polytope SNDP(G, r) on series-parallel graphs. Our study then leads us to give the following conjecture.

CONJECTURE 8.2. *Let $G = (V, E)$ be a series-parallel graph and $r \in \{1, 2\}^V$ its connectivity type vector. The polytope SNDP(G, r) is then completely described by the trivial, partition, and spinning-top inequalities.*

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