Journal of Combinatorial Optimization 4, 131–144 (2000)

Steiner *k*-Edge Connected Subgraph Polyhedra

M. DIDI BIHA biha@crt.umontreal.ca
GERAD and Département de Mathématiques et Génie Industriel, École Polytechnique, CP 6079, succursale Centre-Ville, Montreal (QC) H3C 3A7, Canada

A.R. MAHJOUB mahjoub@ucfma.univ-bpclermont.fr
LIMOS, Université de Clermont II, Complexe Scientifique des Cézeaux, 63177 Aubière Cedex, France

Received October 20, 1997; Revised October 5, 1999

Abstract. In this paper we consider the Steiner k-edge survivable network problem. We discuss the polytope associated with the solutions to that problem. We show that when the graph is series-parallel and k is even, the polytope is completely described by the trivial constraints and the so called Steiner-cut constraints. This generalizes recent work of Baïou and Mahjoub, *SIAM J. Discrete Mathematics,* vol. 10, pp. 505–514, 1997 for the case $k = 2$. As a consequence, we obtain in this case a linear description of the polyhedron associated with the problem when multiple copies of an edge are allowed.

Keywords: k-edge connected subgraph, polytope, series-parallel graph, cut, facet

1. Introduction

With the trend in communication networks to the use of fiber optic technology, it has become important to design networks with lower cost which are survivable. Survivable networks must satisfy certain connectivity requirements. A typical survivability condition is that between every pair of nodes of the network there are at least k edge- (node-) disjoint paths. In practice, there may exist specific nodes for which the survivability condition has to be satisfied. In this paper we discuss this problem. The problem of designing general communication survivable networks has been studied by Grötschel and Monma (1990) and Grötschel et al. (1991, 1992a, b). Related work and applications can also be found in Bienstock et al. (1990), Christofides and Whitlock (1981), Erickson et al. (1987), Monma et al. (1990), Steiglitz et al. (1969), Voss (1990) and Winter (1985, 1986, 1987).

A graph $G = (V, E)$ is called *k-edge connected* (where k is a positive integer) if for any pair of nodes $i, j \in V$, there are at least k edge-disjoint paths from i to j. Let $G = (V, E)$ be a graph and ω a weight function on E that associates with an edge $e \in E$, the weight $\omega(e) \in \mathbb{R}$. Given a subset of distinguished nodes $S \subseteq V$, called *terminals*, the *Steiner k-edge survivable network problem* (SkESNP) is the problem of finding a minimum weight subgraph of G spanning S such that between every two nodes $i, j \in S$, there are at least k edge-disjoint paths.

Polyhedral combinatorics has been successfully applied to prove the polynomiality and obtain efficient cutting plane algorithms for combinatorial optimization problems. In particular, if a polyhedral description of a combinatorial optimization problem is known and the associated separation problem (the problem that consists to determine whether or not a given point x satisfies all the inequalities describing the polyhedron and if not to find an inequality which is violated by x) is solvable in polynomial time, then the problem can be solved in polynomial time. In this paper we discuss the polytope associated with the solutions to the SkESNP. We show that when the graph is series-parallel and k is even, the polytope is completely described by the trivial constraints and the so-called Steiner-cut constraints. This generalizes recent work of Baïou and Mahjoub (1997) for the case $k = 2$. As a consequence, we obtain in this case a linear description of the polyhedron associated with the problem when multiple copies of an edge are allowed. Both descriptions yield polynomial time algorithms for solving these problems on series-parallel graphs.

The SkESNP is NP-hard in general. Winter devised a linear time algorithm to solve the S2ESNP in Halin graphs (Winter, 1985) and series-parallel graphs (Winter, 1986). He mentioned in Winter (1987) that for Halin graphs he also found a linear algorithm to solve the S3ESNP. The SkESNP has been studied by Grötschel and Monma (1990) and Grötschel et al. (1991, 1992a, b) within the framework of a more general model. In particular, Grötschel and Monma (1990) described several basic facets of the polytope associated with that model and Grötschel et al. (1991, 1992a, b) studied further facets and polyhedral aspects of that model, and devised cutting plane algorithms. They also presented experimental results for both the low and high connectivity cases. A complete survey of that model can be found in Stoer (1992).

Given a graph $G = (V, E)$ and a node subset $W \subseteq V$ of G, the set of edges having one endnode in W and the other in $V \backslash W$ is called a *cut* and denoted by $\delta(W)$. If $W = \{v\}$ for some $v \in V$, then we write $\delta(v)$ for $\delta(W)$. If a cut contains r edges, it is also called *r-edge cutset*.

Let $G = (V, E)$ be a graph. Let $x(e)$ be a variable associated with each edge e. For an edge subset $F \subseteq E$, the 0-1 vector $x^F \in \mathbb{R}^E$ with $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ if not, is called the *incidence vector* of F. For any subset of edges $T \subseteq E$, we define $x(T) = \sum_{e \in T} x(e)$. If $W \subseteq V$, then we denote by $E(W)$, the set of edges having both endnodes in W.

The SkESNP can be formulated as the following integer linear program

$$
\begin{array}{rlll}
\text{Min} & \omega x & & \\
\text{Subject to} & x(e) \geq 0, & \text{for all } e \in E, & (1.1)\\
& x(e) \leq 1, & \text{for all } e \in E, & (1.2)\\
& x(\delta(W)) \geq k & \text{for all } W \subset V,\ S \neq W \cap S \neq \emptyset, & (1.3)\\
& x(e) \in \{0, 1\}, & \text{for all } e \in E. & (1.4)
\end{array}
$$

Inequalities (1.1), (1.2) are called *trivial constraints* and inequalities (1.3) are called *Steiner-cut constraints*.

Let SkESNP$(G, S) = \text{conv}\{x \in \mathbb{R}^E \mid x$ satisfies (1.1)–(1.4)$\}$ be the polytope associated with the SkESNP.

The SkESNP(G, S) has been extensively investigated for $S = V$ and $k \leq 2$. It has been described for $k = 1$ for general graphs and for $k = 2$ for some classes of graphs. Using Edmonds' characterization of matroid polytopes (Edmonds, 1970, 1971), Grötschel and

Monma (1990) (see also Cornuéjols et al., 1985) showed that the so-called partition inequalities together with the trivial inequalities suffice to describe the S1ESNP(G, S) for $S = V$. In Mahjoub (1994), Mahjoub gives a complete description of the S2ESNP(G, S) when the graph is series-parallel and $S = V$.

In Chopra (1994), Chopra considers a relaxation of the SkESNP(G, S), namely when multiple copies of an edge are allowed. The problem here consists of determining an integer vector $x \in \mathbb{N}^E$ such that i) the graph $H = (V, E(x))$ is Steiner k-edge connected and ii) $\sum_{e\in E} w(e)x(e)$ is minimum. Here $E(x)$ is the set of edges obtained by replacing each edge $e = ij$ of E by $x(e)$ parallel edges between i and j. This relaxation of the SkESNP is important because it may provide a lower cost solution than the case where at most one copy of an edge may be used. He studies the polyhedron $P_k(G, S)$ associated with the solutions to that problem, that is

$$P_k(G, S) = \text{conv}\{x \in \mathbb{N}^E \mid (V, E(x)) \text{ is } k\text{-edge connected}\}.$$

He gives a complete description of $P_k(G, S)$ when G is outerplanar, $S = V$ and k is odd. (A graph is outerplanar if it is planar and it can be embedded on the plane so that all nodes lie on the outermost face). The polyhedron $P_k(G, S)$ has been previously studied by Cornuéjols et al. (1985). They showed that when the graph is series-parallel, $S = V$ and k is even, $P_k(G, S)$ is completely described by the nonnegativity and the cut inequalities.

In Didi Biha and Mahjoub (1996), Didi Biha and Mahjoub discuss the SkESNP when $S = V$. They describe, in this case, the SkESNP(G, S) for all k when G is series-parallel. In Barahona and Mahjoub (1995), Barahona and Mahjoub describe the S2ESNP(G, S) for Halin graphs when $S = V$. In Baïou and Mahjoub (1997), Baïou and Mahjoub discuss the S2ESNP and show that when the graph is series-parallel, S2ESNP(G, S) is given by the trivial and the Steiner-cut constraints. The purpose of this paper is to extend this to even k.

Related work can also be found in Coullard et al. (1991a, b), Fonlupt and Naddef (1992), Fonlupt and Mahjoub (1999), Margot et al. (1994) and Steiglitz et al. (1969). In Fonlupt and Naddef (1992), Fonlupt and Naddef characterize the class of graphs for which the system given by the nonnegativity constraints and the cut constraints, when $S = V$, defines the convex hull of the incidence vectors of the tours of G. (A tour is a cycle going at least once through each node). In Coullard et al. (1991a, b), Coullard et al. discuss the Steiner 2-node connected subgraph polytope. In Coullard et al. (1991a), they describe that polytope for series-parallel graphs, and in Coullard et al. (1991b), they describe the dominant of that polytope for the graphs noncontractible to W_4 (the wheel on 5 nodes).

The problem S1ESNP is closely related to the well known Steiner tree problem in graphs. In Chopra and Rao (1994), Chopra and Rao describe several classes of the Steiner tree polytope in both the directed and undirected cases. In Margot et al. (1994), Margot et al. (see also Goemans, 1994) give an extended formulation for the Steiner tree problem and show that it is a complete linear description of the associated polytope when the graph is a 2-tree (a maximal series-parallel graph). In Goemans (1994), Goemans discusses an extended formulation of the Steiner tree problem and describes the associated polytope when the underlying graph is series-parallel. He also describes some classes of facets for the Steiner tree polytope.

In the next section, we give a complete description of the polytope SkESNP(G, S) when the graph is series-parallel and k is even. In Section 3 we describe the polyhedron $P_k(G, S)$ in that class of graphs when k is even. In Section 4 we give some concluding remarks.

The remainder of this section is devoted to more definitions and notations. The graphs we consider are finite, undirected, connected and may have multiple edges. If e is an edge between two nodes i and j, then we write $e = ij$. If $G = (V, E)$ is a graph and $e \in E$, then $G - e$ will denote the graph obtained from G by removing e. For $W \subseteq V$, we let $G(W)$ denote the induced subgraph of G on W. Given W_1, W_2 two disjoint subsets of V, $[W_1, W_2]$ will denote the set of edges of G having one node in W_1 and the other one in W_2. If $W \subseteq V$, then $\bar{W}$ denotes $V \backslash W$. Given a constraint $ax \geq \alpha$, $a \in \mathbb{R}^E$, and a solution x^*, we will say that $ax \geq \alpha$ is *tight* for x^* if $ax^* = \alpha$.

2. The SkESNP(G, S) of a series-parallel graph

A *homeomorph* of K_4 (the complete graph on 4 nodes) is a graph obtained from K_4 when its edges are subdivided into paths by inserting new nodes of degree two. A graph is called *series-parallel* if it contains no homeomorph of K_4 as a subgraph. Connected series-parallel graphs have the following property.

Lemma 2.1. *If $G = (V, E)$ is a connected series-parallel graph with $|V| \geq 3$, then G contains a node that is adjacent to exactly two nodes.*

Let $G = (V, E)$ be a graph and $S \subseteq V$ a set of terminals. We will suppose that $|S| \geq 2$ (if $|S| = 1$, then the polytope SkESNP(G, S) would be given by the trivial constraints). Let $Q_k(G, S)$ be the polytope given by inequalities (1.1)–(1.3). (Recall that k is a fixed positive integer). In what follows we shall show that if G is series-parallel and k is even, then $Q_k(G, S) = $ SkESNP(G, S). Since the minimum cut problem can be solved in polynomial time (see Nagamochi and Ibaraki, 1992; Stoer and Wagner, 1994), a consequence of our result is that the SkESNP is solvable in polynomial time in these graphs using the ellipsoid method (Grötschel et al., 1981).

We have the following lemma, its proof is omitted because it is similar to that of a similar result in Baïou and Mahjoub (1997).

Lemma 2.2. *Let x be a solution of $Q_k(G, S)$. If $\delta(W_1)$ and $\delta(W_2)$ are two Steiner cuts tight for x with $(W_1 \cap W_2) \cap S \neq \emptyset$ and $(\overline{W_1 \cup W_2}) \cap S \neq \emptyset$ then $\delta(W_1 \cap W_2)$ and $\delta(\overline{W_1 \cup W_2})$ are two Steiner-cuts tight for x, and $x([W_1 \backslash W_2, W_2 \backslash W_1]) = 0$.*

If x is an extreme point of $Q_k(G, S)$, then there exist two edge subsets $E_0, E_1 \in E$ and a family of Steiner-cuts $\{\delta(W_i), i = 1, \ldots, r\}$ such that x is the unique solution of the system

$$\begin{cases} x(e) = 0, & \text{for all } e \in E_0, \\ x(e) = 1, & \text{for all } e \in E_1, \\ x(\delta(W_i)) = k, & \text{for } i = 1, \ldots, r, \end{cases} \tag{2.1}$$

where $|E_0| + |E_1| + r = |E|$.

Lemma 2.3. *Let $\delta(W_i)$ be a Steiner-cut of system* (2.1). *Then system* (2.1) *can be chosen so that either $W_j \subseteq W_i$ or $W_j \subseteq \bar{W}_i$ for all $j \in \{1, \ldots, r\}\backslash\{i\}$.*

Proof: The proof uses some ideas developed by Cornuéjols et al. (1985) for a similar result. W.l.o.g. we may suppose that $i = 1$. Suppose for instance that $W_1 \cap W_2 \neq \emptyset$, $W_1 \not\subset W_2$, $W_2 \not\subset W_1$ and $W_1 \cup W_2 \neq V$. W.l.o.g. we may suppose that $(W_1 \cap W_2) \cap S \neq \emptyset$. Since $\bar{W}_2 \cap S \neq \emptyset$, at least one of the sets $W_1 \backslash W_2$ and $\overline{W_1 \cup W_2}$ intersects S.

Case 1. $(\overline{W_1 \cup W_2}) \cap S \neq \emptyset$. As $(W_1 \cap W_2) \cap S \neq \emptyset$, it follows that both $\delta(W_1 \cap W_2)$ and $\delta(\overline{W_1 \cup W_2})$ are Steiner cuts. By Lemma 2.2 we then have

$$\begin{cases} x(\delta(W_1 \cap W_2)) = x(\delta(\overline{W_1 \cup W_2})) = k, \\ x([W_1 \backslash W_2, W_1 \backslash W_2]) = 0. \end{cases}$$

This together with $x(\delta(W_1)) = k$ implies that $x(\delta(W_2)) = k$. Hence $x(\delta(W_2)) = k$ can be replaced in the system (2.1) by $x(\delta(W_1 \cap W_2)) = k$ and $x(\delta(\overline{W_1 \cup W_2})) = k$, the new system still has x as a unique solution.

Case 2. $(\overline{W_1 \cup W_2}) \cap S = \emptyset$. As $\bar{W}_1 \cap S \neq \emptyset \neq \bar{W}_2 \cap S$, it follows that $(W_2 \backslash W_1) \cap S \neq \emptyset$ and $(W_1 \backslash W_2) \cap S \neq \emptyset$. Hence by considering $\delta(\bar{W}_1)$ instead of $\delta(W_1)$, by Lemma 2.2 we obtain that $\delta(W_1 \backslash W_2)$ and $\delta(W_2 \backslash W_1)$ are two Steiner cuts tight for x and $x([W_1 \cap W_2, \overline{W_1 \cup W_2}]) = 0$. Moreover, as we did in Case 1, if we replace $x(\delta(W_2)) = k$ by $x(\delta(W_1 \backslash W_2)) = k$ and $x(\delta(W_2 \backslash W_1)) = k$ we obtain a system still having x as a unique solution.

So any equation $x(\delta(W_j)) = k$, $j \in \{2, \ldots, r\}$, can be replaced by equations of the form $x(\delta(W)) = k$ where $\delta(W)$ is a Steiner cut with either $W \subseteq W_1$ or $W_1 \subseteq W$ ($\delta(W)$ and $\delta(W_j)$ may be the same). Let L be the system thus obtained. And let M be the system given by the trivial inequalities of system (2.1) and $x(\delta(W_1)) = k$. Note that the constraints of M belong to L. As x is the unique solution of L and M is a nonsingular system, there must exist $|E| - (|E_0| + |E_1| + 1)$ equations of L different from those of M that form with M a nonsingular system having x as a unique solution. This new system is as required. $\square$

Now we can state our main result.

Theorem 2.4. *Let $G = (V, E)$ be a series-parallel graph and $S \subset V$ a set of terminals. If k is even, then SkESNP$(G, S) = Q_k(G, S)$.*

Proof: The proof is by induction on $|E| + |V|$. The statement is trivially true if G consists of two nodes (terminals) joined by k edges. So suppose that it is true for any series-parallel graph with no more than m edges and suppose that G contains $m + 1$ edges. Also suppose that, under this hypothesis, $|S|$ is maximum. That is, for any series-parallel graph $G' = (V', E')$ with $|E'| = m + 1$ and a set of terminals S' such that $|S'| > |S|$, we have SkESNP$(G', S') = Q_k(G', S')$. Note that such assumption can be made since, as shown in Didi Biha and Mahjoub (1996), SkESNP$(G', V) = Q_k(G', V)$ holds if G is series-parallel.

Now suppose that, on the contrary, SkESNP$(G, S) \neq Q_k(G, S)$ and let x be a fractional extreme point of $Q_k(G, S)$. We assume that x is the unique solution of system (2.1). □

Claim 1. $x(e) > 0$ *for all* $e \in E$.

Proof: If f is an edge such that $x(f) = 0$, then let $x^* \in \mathbb{R}^{E^*}$ be the solution given by $x^*(e) = x(e)$ for all $e \in E^*$ where $E^* = E \backslash \{f\}$. Obviously, $x' \in Q_k(G - f, S)$. Moreover x^* is an extreme point of $Q_k(G - f, S)$. Since x^* is fractional, this contradicts the induction hypothesis.

Claim 2. *Each variable* $x(e)$ *has a nonzero coefficient in at least two equations of the system* (2.1) *defining* x.

Proof: It is clear that $x(e)$ must have a nonzero coefficient in at least one of the equations of (2.1). Otherwise, $x(e)$ would be fractional and the solution $\bar{x} \in \mathbb{R}^E$ such that $\bar{x}(e) = x(e) + \epsilon$ and $\bar{x}(e') = x(e')$ if $e' \in E \backslash \{e\}$ where $\epsilon \in \mathbb{R}$, would be a solution of system (2.1). Since $\bar{x} \neq x$ this is impossible. Now suppose that for an edge $f = uv \in E$, $x(f)$ has a nonzero coefficient in exactly one equation of (2.1). And let $(2.1)'$ be the system obtained from (2.1) by deleting this equation. Obviously, $(2.1)'$ is a nonsingular system. Let $G' = (V', E')$ be the graph obtained by contracting f. Let $S' = (S \backslash \{u, v\}) \cup \{w\}$ if $S \cap \{u, v\} \neq \emptyset$ and $S' = S$ if not, where w is the node arising from the contraction of f. Let x' be the restriction of x on E'. Clearly, $x' \in Q_k(G', S')$. Also note that the equations of system $(2.1)'$ all correspond to constraints of $Q_k(G', S')$. This implies that x' is an extreme point of $Q_k(G', S')$. Since G' is series-parallel and $|E'| < |E|$, this contradicts the induction hypothesis. □

From Claims 1 and 2 we have the following.

Claim 3. *Each variable* $x(e)$ *has a nonzero coefficient in at least one Steiner-cut constraint of system* (2.1).

Since G is series-parallel, by Lemma 2.1, there exists a node v which is adjacent to exactly two nodes v_1, v_2. Let F_1 (F_2) be the set of edges between v and v_1 (v_2). W.l.o.g. we may suppose that $|F_1| \geq |F_2|$ and if $|F_1| = |F_2|$, $x(F_1) \geq x(F_2)$.

Claim 4. *The set* F_1 (F_2) *contains at most one edge* e *with* $0 < x(e) < 1$.

Proof: Suppose that there are two edges $e_1, e_2 \in F_1$ with $0 < x(e_1) \leq x(e_2) < 1$. Let $x' \in \mathbb{R}^E$ be the solution such that

$$x'(e) = \begin{cases} x(e) + \epsilon & \text{if } e = e_1, \\ x(e) - \epsilon & \text{if } e = e_2, \\ x(e) & \text{if } e \in E \backslash \{e_1, e_2\}. \end{cases}$$

Since every cut $\delta(W_i)$ either contains both edges e_1 and e_2 or does not contain any one of these edges, x' is a solution of system (2.1). As $x' \neq x$, this is a contradiction. □

Claim 5. *System* (2.1) *can be chosen so that if* $|W_i| \geq 2$, *then* $(W_i \backslash \{v\}) \cap S \neq \emptyset$, *for* $i \in \{1, \ldots, r\}$.

Proof: The claim is trivially true if either $v \notin W_i \cap S$ or $|W_i \cap S| \geq 2$. Now, suppose that for some $i \in \{1, \ldots, r\}$, $W_i \cap S = \{v\}$ and $|W_i| \geq 2$. By Lemma 2.3, we may suppose that system (2.1) is such that $W_j \subseteq W_i$ or $W_j \subset \bar{W}_i$ for $j \in \{1, \ldots, r\} \backslash \{i\}$. W.l.o.g. we may suppose that $|W_i|$ is minimum with respect to this assumption. We have that $x(\delta(W)) \geq k$ for every $W \subset V$ with $v_1 \in W$ and $\bar{W} \cap S \neq \emptyset$. In fact, this is clear if $W \cap S \neq \emptyset$. So suppose that $W \cap S = \emptyset$. As $v \in S$, we have that $v \in \bar{W}$ and, consequently, $F_1 \subseteq \delta(W)$. Furthermore $\delta(W \cup \{v\})$ is a Steiner-cut. If $v_2 \in W$ then $x(\delta(W)) \geq x(\delta(v)) \geq k$. If $v_2 \in \bar{W}$, then $\delta(W \cup \{v\}) = (\delta(W) \backslash F_1) \cup F_2$. As $x(F_1) \geq x(F_2)$ and $x(\delta(W \cup \{v\})) \geq k$, it follows that $x(\delta(W)) \geq k$. Now, we claim that $v_1 \in S$. Indeed, if this is not the case, then let $S' = S \cup \{v_1\}$. As $x(\delta(W)) \geq k$ for every cut $\delta(W)$ with $v_1 \in W$ and $\bar{W} \cap S \neq \emptyset$, it follows that x is at the same time an extreme point of the polytope $Q_k(G, S')$. As $|S'| > |S|$ and x is fractional, this contradicts the maximality of $|S|$. So, as $W_i \cap S = \{v\}$, it follows that $v_1 \in \bar{W}_i$ and $v_2 \in W_i$. By Claim 3, the edges of F_2 must belong to at least one Steiner-cut $\delta(T)$ of the system (2.1). As $T \subseteq W_i$ and $x(e) > 0$, it follows that $T = \{v\}$ and, by the minimality of W_i, $\delta(T)$ is the only tight cut of the system (2.1) where $T \subseteq W_i$. By Claim 2 it follows that $x(e) = 1$ for all $e \in F_2$. As $x(\delta(v)) = k$ and by Claim 4, F_1 can contain at most one edge with fractional value, this implies that $x(e) = 1$ for all $e \in F_1$. Moreover, we have $F_2 = E(W_i)$. If for instance there is $f \in E(W_i) \backslash F_2$, then by the minimality of W_i, f cannot belong to any tight cut of system (2.1), contradicting Claim 3. Thus we have

$$\begin{cases} W_i = \{v, v_2\}, \\ x(\delta(v)) = k, \\ x(e) = 1, \qquad \text{for all } e \in \delta(v). \end{cases}$$

Let $G' = (V', E')$ be the graph obtained from G by contracting F_2. Let x' be the restriction of x on E'. Let $S' = (S \backslash \{v\}) \cup \{w\}$ where w is the node that arises from the contraction of F_2. Clearly, $x' \in Q_k(G', S')$. Furthermore, it is not hard to see that x' is an extreme point of $Q_k(G', S')$. As $|E'| < |E|$ and x' is fractional, this contradicts the induction hypothesis. □

For the rest of the proof we suppose that $v \in W_i$ for $i = 1, \ldots, r$. Now, by Claim 2, there must exist a cut W_i, $i \in \{1, \ldots, r\}$, such that $F_1 \subset \delta(W_i)$. Let us suppose that $|W_i|$ is maximum. We claim that $|W_i| \geq 2$. Indeed, suppose that $W_i = \{v\}$. Thus by the maximality of $|W_i|$, $\delta(W_i)$ is the only Steiner-cut containing F_1. By Claim 2 it follows that $x(e) = 1$ for all $e \in F_1$. In consequence we have

$$x(e) = 1 \quad \text{for all } e \in \delta(W_i),$$

implying that $x(\delta(v)) = x(\delta(W_i)) = k$ is redundant in system (2.1), a contradiction. Consequently, $|W_i| \geq 2$ and $v, v_2 \in W_i$. Also by Claim 4, we may assume that $(W_i \backslash \{v\}) \cap$

$S \neq \emptyset$ (and $(\bar{W}_i \cup \{v\}) \cap S \neq \emptyset$). Thus

$$\begin{aligned} k \leq x(\delta(W_i \backslash \{v\})) &= x(\delta(W_i)) + x(F_2) - x(F_1) \\ &= k + x(F_2) - x(F_1) \\ &\leq k. \end{aligned}$$

This implies that the inequalities above are all satisfied with equality. Hence,

$$\begin{cases} x(\delta(W_i \backslash \{v\})) = k, \\ x(F_2) = x(F_1), \\ |F_1| = |F_1|. \end{cases} \tag{2.2}$$

The last equation is obtained using Claim 4 and the fact that $x(F_2) = x(F_1)$. Now, suppose that $\delta(v)$ is tight for x. If there is an edge $e_1 \in F_1$ with $0 < x(e_1) < 1$, then by (2.2) together with Claim 4 there must exist an edge e_2 of F_2 such that $x(e_2) = x(e_1)$. Let $l = |F_1| = |F_2|$. We have

$$\begin{aligned} x(\delta(v)) &= x(F_1) + x(F_2) \\ &= (l-1) + x(e_1) + (l-1) + x(e_2) \\ &= 2(l-1) + 2x(e_1) \\ &= k. \end{aligned}$$

However the last equation cannot hold since k is even and $0 < x(e_1) < 1$. Thus $x(e) = 1$ for all $e \in F_1$. Similary we have $x(e) = 1$ for all $e \in F_2$. Hence $x(\delta(v)) = k$ is redundant with respect to system (2.1). And in consequence, one may assume that $x(\delta(v)) = k$ does not belong to system (2.1). Let

$$F_i^1 = \{e \in F_i \mid x(e) = 1\}, \quad i \in \{1, 2\}.$$

Let J be the set of indices $j \in \{1, \ldots, r\}$ such that $F_1 \subset \delta(W_j)$. Let $W_j' = W_j \backslash \{v\}$ for $j \in J$. Clearly, by Claim 5, $\delta(W_j')$ is a Steiner cut for $j \in J$. As $x(\delta(v)) = k$ and $x(F_1) = x(F_2)$ we have $x(\delta(W_j') = k$ for $j \in J$.

Now, consider the system $(2.1)'$ obtained from system (2.1) by replacing the equations $x(\delta(W_j)) = k$ by $x(\delta(W_j')) = k$ for $j \in J$, and deleting the equations $x(e) = 1$, for $e \in F_1^1$. Let $G' = (V', E')$ be the graph obtained by contracting F_1. Let $S' = (S \backslash \{v, v_1\}) \cup \{v_0\}$ if $\{v, v_1\} \cap S \neq \emptyset$ and $S' = S$ if not, where v_0 is the node arising from the contraction of F_1. Let x' be the restriction of x on E'. Clearly, $x' \in Q_k(G', S')$ and x' satisfies system $(2.1)'$. As $|E'| < |E|$, by the induction hypothesis $Q_k(G', S')$ is integral. And, since x' is fractional, there must exist an integral extreme point y' of $Q_k(G', S')$ which is a solution of $(2.1)'$. We distinguish two cases.

Case 1. $F_1^1 \subset F_1$. Thus $F_2^1 \subset F_2$. And there are two edges $e_1 \in F_1$ and $e_2 \in F_2$ with $0 < x(e_1) = x(e_2) < 1$. Let $y \in \mathbb{R}^E$ be the solution given by

$$y(e) = \begin{cases} y'(e) & \text{if } e \in E \backslash F_1, \\ 1 & \text{if } e \in F_1^1, \\ y'(e_2) & \text{if } e = e_1. \end{cases}$$

It is not hard to see that y is a solution of system (2.1). As y is integral and hence $y \neq x$, this contradicts the extremality of x.

Case 2. $F_1^1 = F_1$. Thus $F_2^1 = F_2$. Let $y \in \mathbb{R}^E$ be defined as

$$y(e) = \begin{cases} y'(e) & \text{if } e \in E \backslash F_1, \\ 1 & \text{if } e \in F_1. \end{cases}$$

As in Case 1, it is easy to see that y is a solution of system (2.1). But this is a contradiction, which finishes the proof of our theorem. □

An immediate consequence of Theorem 2.4 is the following.

Corollary 2.5. *The SkESNP is solvable in polynomial time on series-parallel graphs when k is even.*

3. The polyhedron $P_k(G, S)$ of a series-parallel graph

In this section we shall discuss $P_k(G, S)$, the polyhedron associated with the SkESNP when multiple copies of an edge are allowed. Using Theorem 2.4 we will show that inequalities (1.1) and (1.3) are sufficient to describe $P_k(G, S)$ when G is series-parallel and k is even.

Theorem 3.1. *Let $G = (V, E)$ be a series-parallel graph and $S \subseteq V$ a set of terminals. If k is even then $P_k(G, S)$ is completely described by inequalities* (1.1) *and* (1.3).

Proof: Let $P_k^*(G, S)$ be the polyhedron described by inequalities (1.1) and (1.3). It is clear that inequalities (1.1) and (1.3) are valid for $P_k(G, S)$. Thus $P_k(G, S) \subseteq P_k^*(G, S)$. To show that $P_k^*(G, S) \subseteq P_k(G, S)$ it suffices to show that the extreme points of $P_k^*(G, S)$ are integral. Suppose, on the contrary, that there exists a fractional extreme point $x \in \mathbb{R}^E$ of $P_k^*(G, S)$. Let $G' = (V, E')$ be the graph obtained from G by replacing each edge $e = ij$ of E such that $x(e) > 0$ by $\lceil x(e) \rceil$ edges $e_1, \ldots, e_{\lceil x(e) \rceil}$ between i and j. Let $x' \in \mathbb{R}^{E'}$ be the solution given by

$$\begin{cases} \left.\begin{array}{ll} x'(e_i) = 1 & \text{for } i = 1, \ldots, \lceil x(e) \rceil - 1, \\ x'(e_i) = x(e) - \lceil x(e) - 1 \rceil & \text{for } i = \lceil x(e) \rceil, \end{array}\right\} \text{if } x(e) \neq 0, \\ x'(e) = 0 \quad \text{if not.} \end{cases}$$

It is easily seen that x' satisfies inequalities (1.1) and (1.3). Moreover, x' is an extreme point of $Q_k(G', S)$. In fact, it is clear that x' satisfies inequalities (1.2). Now if the statement does not hold, as, by Theorem 2.4, $Q_k(G', S)$ is integral, there must exist t integer solutions $(t \geq 2)$ $y'_1, \ldots, y'_t$ of $Q_k(G', S)$ and $\lambda_1, \ldots, \lambda_t \in \mathbb{R}^*$ such that $x' = \sum_{j=1}^{t} \lambda_j y'_j$ and $\sum_{j=1}^{t} \lambda_j = 1$. Now let $y_1, \ldots, y_t \in \mathbb{R}^E$ be the solutions such that

$$y_i(e) = \sum_{j=1}^{\lceil x(e) \rceil} y'_i(e_j),$$

for $e \in E$ and $i = 1, \ldots, t$. It is clear that $y_1, \ldots, y_t \in P_k^*(G, S)$. Moreover we have that $x = \sum_{j=1}^{t} \lambda_j y_j$. But this contradicts the fact that x is an extreme point of $P_k^*(G, S)$. Consequently, x' is an extreme point of $Q_k(G', S)$. Since x' is fractional and G' is series-parallel, this contradicts Theorem 2.4. $\square$

4. Concluding remarks

We have studied the Steiner k-edge survivable network problem and have given a complete linear description of the associated polytope when the underlying graph is series-parallel and k is even. We have shown that in this case, the trivial and the Steiner-cut inequalities suffice to describe the polytope. As a consequence we obtained that the nonnegativity inequalities together with the Steiner-cut inequalities characterize the polyhedron in this case, when multiple copies of an edge are allowed. Both characterizations yield polynomial time algorithms for the corresponding optimization problems on series-parallel graphs.

The trivial inequalities and the Steiner-cut inequalities do not suffice to describe the polytope SkNSP(G, S) on series-parallel graphs when k is odd. In fact, as shown by Didi Biha and Mahjoub (1996), for $S = V$ and k odd, a further class of constraints called *series-parallel partition inequalities* is needed to have a complete description when the graph is series-parallel. This class generalizes the cut inequalities, and may be extended to the case when $S \subset V$ as follows:

Let $G = (V, E)$ be a series-parallel graph and S be a set of terminals. Let $V_1, \ldots, V_p$ be a partition of V such that

i) $G(V_i)$ is connected for $i = 1, \ldots, p$, and
ii) $V_i \cap S \neq \emptyset$ for $i = 1, \ldots, p$.

Then the inequality

$$x(\delta(V_1, \ldots, V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1$$

is valid for SkESNP(G, S). Here $\delta(V_1, \ldots, V_p)$ denotes the set of edges between the members of the partition. The trivial inequalities and the series-parallel partition inequalities do not, unfortunately, suffice to describe the polytope SkENSP(G, S) on series-parallel graphs even for $k = 1$ (see Chopra and Rao, 1994; Didi Biha et al., to appear; Goemans, 1994).

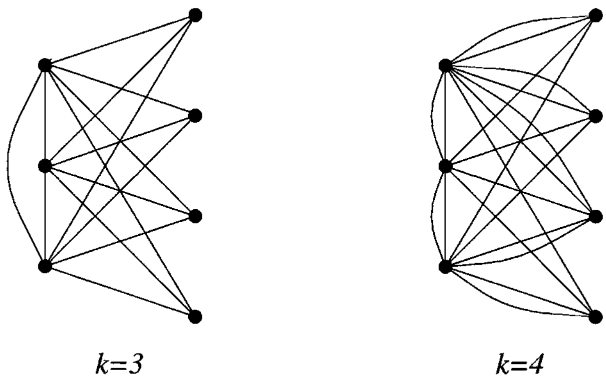

Figure 1. Graphs of Γ.

It would be interesting to characterize the class of graphs G such that $Q_k(G, S) =$ SkESNP(G, S). The problem SkESNP can be solved in polynomial time in that class of graphs. Theorem 2.4 shows that that class contains for instance the class of series-parallel graphs, if k is even. In what follows we describe further classes for which $Q_k(G, S) =$ SkESNP(G, S) when $S = V$, for the proofs see Didi Biha (1998).

Let $\mathbf{\Gamma}$ be the class of graphs $G = (V, E)$ such that

1) $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$,
2) $|V_1| = 3$ and $E(V_2) = \emptyset$,
3) $|V_2| \geq 3$ and if $|V_2| = 3$, then $E(V_1) = \emptyset$,
4) $|[v_1, v_2]| \leq \lfloor \frac{k}{2} \rfloor$ for all nodes v_1, v_2 such that $v_1 \in V_1$ and $v_2 \in V_2$.

Figure 1 shows some graphs of $\mathbf{\Gamma}$. Note that graphs of $\mathbf{\Gamma}$ can be recognized in polynomial time and may be non series-parallel.

The following theorem generalizes a result of Mahjoub (1997) for $k = 2$.

Theorem 4.1. *If G is a graph of $\mathbf{\Gamma}$, then $Q_k(G, V) = SkESNP(G, V)$.*

In Didi Biha (1998) it is also shown that if $G = (V_1 \cup V_2, E)$ is a bipartite graph without parallel edges such that either $|V_1 \cup V_2| \leq 4k - 1$ or $|V_1| \leq k + 1$, then $Q_k(G, V) =$ SkESNP(G, V).

In Fonlupt and Mahjoub (1999), Fonlupt and Mahjoub characterize the graphs for which $Q_2(G, V) =$ S2ESNP(G, V). In Didi Biha (1998), Didi Biha gives sufficient conditions under which $Q_k(G, V) =$ SkESNP(G, V).

In Mahjoub (1994), Mahjoub introduces a class of facet defining inequalities for the S2ESNP(G, S) when $S = V$, called *odd wheel inequalities*. Wheels with $2n + 2$ nodes (where the exterior cycle of the wheel contains $2n + 1$ nodes, and $n \geq 1$) are examples of graphs producing odd wheel inequalities.

A *Halin* graph $G = (V, T \cup C)$ consists of a tree T that has no degree-two nodes, together with a simple cycle C whose nodes are pendant nodes of T, the graph should be embeddable in the plane with C as the exterior face. These are examples of minimally 3-connected graphs given by Halin (1971). In Barahona and Mahjoub (1995), Barahona and Mahjoub show that if G is a Halin graph then S2ESNP(G, S) with $S = V$, is given by the trivial, cut and odd wheel inequalities. In Didi Biha (1998) Didi Biha gives a generalization of that result as follows. Let $\mathcal{H}_k$ be the class of graphs G such that G can be obtained from a Halin graph by replacing each edge ij of T by $k-1$ parallel edges between i and j. Note that graphs in $\mathcal{H}_k$ are minimally $k+1$-connected.

In Didi Biha (1998), Didi Biha describes a class of inequalities that generalises the odd wheel inequalities for graphs of $\mathcal{H}_k$ and arbitrary k, and shows that these inequalities together with the trivial and the cut inequalities describe the SkESNP(G, S) for $S = V$ when G is a graph of $\mathcal{H}_k$.

The SkESNP can be seen as a relaxation of the following problem called the *Steiner k-edge connected subgraph problem* (SkECSP) introduced by Monma et al. (1990). Given a graph $G = (V, E)$ with weights on its edges and a set of terminals $S \subseteq V$, the problem is to find a minimum k-edge connected subgraph of G, spanning S. Note that if the weights are positive, the two problems are equivalent. This problem has been studied by Baïou and Mahjoub (1997) and by Baïou (1997) for $k = 2$. In Baïou and Mahjoub (1997) it is shown that the associated polytope SkECSP(G, S) on series-parallel graphs is given by the trivial inequalities, Steiner-cut inequalities and the inequalities

$$x(\delta(W)) - 2x(e) \geq 0 \quad \text{for all } W \subset V, S \subseteq W, e \notin E(W).$$

A natural question that may arise here is whether or not this result can be extended to the case where k is even. Our study of that question motivates us to give the following conjecture.

Conjecture 4.2. *If G is series-parallel with a set of terminals S, and k is even, then SkECSP(G, S) is given by the trivial inequalities, the Steiner-cut inequalities and the inequalities*

$$x(\delta(W)) - kx(e) \geq 0 \quad \textit{for all } W \subset V, S \subseteq W, e \notin E(W).$$

In Baïou (to appear), Baïou gives a complete description of the dominant of the SkECSP(G, S) in the class of graphs for which SkECSP(G, S) coincides with its linear relaxation. This class contains series-parallel graphs as a subclass.

Acknowledgments

We are grateful to the referees for their helpful comments.

References

M. Baïou and A.R. Mahjoub, "Steiner 2-edge connected subgraph polytope on series-parallel graphs," *SIAM J. Discrete Mathematics*, vol. 10, pp. 505–514, 1997.

M. Baïou, "On the dominant of the Steiner 2-edge connected subgraph polytope," *Discrete Applied Mathematics*, to appear.

F. Barahona and A.R. Mahjoub, "On two-connected subgraph polytopes," *Discrete Mathematics*, vol. 147, pp. 19–34, 1995.

D. Bienstock, E.F. Brickell, and C.L. Monma, "On the structure of minimum weigth k-connected spanning networks," *SIAM J. Discrete Mathematics*, vol. 3, pp. 320–329, 1990.

S. Chopra, "The k-edge connected spanning subgraph polyhedron," *SIAM J. Discrete Mathematics*, vol. 7, pp. 245–259, 1994.

S. Chopra and M.R. Rao, "The Steiner tree problem II: Properties and classes of facets," *Mathematical Programming*, vol. 64, pp. 231–246, 1994.

N. Christofides and C.A. Whitlock, "Network synthesis with connectivity constraints—A survey," in *Operational Research*, vol. 81, J.P. Brans (Ed.), North Holland, 1981, pp. 705–723.

G. Cornuéjols, J. Fonlupt, and D. Naddef, "The traveling salesman problem on a graph and some related integer polyhedra," *Mathematical Programming*, vol. 33, pp. 1–27, 1985.

R. Coullard, A. Rais, R.L. Rardin, and D.K. Wagner, "The 2-Connected Steiner Subgraph Polytope for Series-Parallel Graphs," Report No. CC-91-23, School of Industrial Engineering, Purdue University, 1991a.

R. Coullard, A. Rais, R.L. Rardin, and D.K. Wagner, "The Dominant of the 2-Connected-Steiner Subgraph Polytope for W_4-Free Graphs," Report No. 91-28, School of Industrial Engineering, Purdue University, 1991b.

M. Didi Biha, "Graphes k-arête connexes et polyèdres," Ph.D. Dissertation, Université de Bretagne Occidentale, Brest, France, 1998.

M. Didi Biha, H. Kerivin, and A.R. Mahjoub, "Steiner trees and polyhedra," *Discrete Applied Mathematics*, 1999.

M. Didi Biha and A.R. Mahjoub, "k-edge connected polyhedra on series-parallel graphs," *Operations Research Letters*, vol. 19, pp. 71–78, 1996.

R.J. Duffin, "Topology of series-parallel networks," *Journal of Mathematical Analysis and Applications*, vol. 10, pp. 303–318, 1965.

J. Edmonds, "Submodular functions, matroids and certain polyhedra," in *Combinatorial Structures and Their Applications*, R. Guy et al. (Ed.), Gordon and Breach: New York, 1970, pp. 69–87.

J. Edmonds, "Matroids and the greedy algorithm," *Mathematical Programming*, vol. 1, pp. 127–136, 1971.

R.E. Erickson, C.L. Monma, and A.F. Veinott, "Send-and-split method for minimum-concave-cost network flows," *Mathematics of Operations Research*, vol. 12, pp. 634–664, 1987.

J. Fonlupt and D. Naddef, "The traveling salesman problem in graphs with some excluded minors," *Mathematical Programming*, vol. 53, pp. 147–172, 1992.

J. Fonlupt and A.R. Mahjoub, "Critical Extreme Points of the 2-Edge Connected Subgraph Polytope," in *Proceedings IPCO'99*, Springer-Verlag, 1999, pp. 166–183. LNCS 1610.

M.X. Goemans, "The Steiner tree polytope and related polyhedra," *Mathematical Programming*, vol. 63, pp. 157–182, 1994.

M. Grötschel, L. Lovász, and A. Schrijver, "The ellipsoid method and its consequences in combinatorial optimization," *Combinatorica*, vol. 1, pp. 70–89, 1981.

M. Grötschel and C. Monma, "Integer polyhedra arising from certain network design problems with connectivity constraints," *SIAM J. Discrete Mathematics*, vol. 3, pp. 502–523, 1990.

M. Grötschel, C. Monma, and M. Stoer, "Polyhedral Approaches to Network Survivability," in *Reliability of Computer and Communication Networks*, F. Roberts, F. Hwang, and C.L. Monma (Eds.), AMS/ACM, 1991, pp. 121–141. Series Discrete Mathematics and Computer Science, vol. 5.

M. Grötschel, C. Monma, and M. Stoer, "Facets for polyhedra arising in the design of communication networks with low-connectivity constraints," *SIAM J. Optimization*, vol. 2, pp. 474–504, 1992a.

M. Grötschel, C. Monma, and M. Stoer, "Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints," *Operations Research*, vol. 40, no. 2, pp. 309–330, 1992b.

R. Halin, "Studies on minimally n-connected graphs," in *Combinatorial Mathematics and its Applications*, J.A. Welsh (Ed.), Academic Press: New York, 1971, pp. 129–136.

A.R. Mahjoub, "Two-edge connected spanning subgraphs and polyhedra," *Mathematical Programming*, vol. 64, pp. 199–208, 1994.

A.R. Mahjoub, "On perfectly two-edge connected graphs," *Discrete Mathematics*, vol. 170, pp. 153–172, 1997.

F. Margot, A. Prodon, and Th.M. Liebling, "Tree polytope on 2-trees," *Mathematical Programming*, vol. 63, pp. 183–191, 1994.

C.L. Monma, B.S. Munson, and W.R. Pulleyblank, "Minimum-weight two connected spanning networks," *Mathematical Programming*, vol. 46, pp. 153–171, 1990.

H. Nagamochi and T. Ibaraki, "Computing edge connectivity in multigraphs and capacitated graphs," *SIAM. J. Discrete Mathematics*, vol. 5, pp. 54–66, 1992.

K. Steiglitz, P. Weiner, and D.J. Kleitman, "The design of minimum cost survivable networks," *IEEE Transactions on Circuit Theory*, vol. 16, pp. 455–460, 1969.

M. Stoer, "Design of survivable networks," in *Lecture Notes in Mathematics, vol. 1531,* Springer-Verlag, 1992.

M. Stoer and F. Wagner, "A Simple Min Cut Algorithm," in *Proceedings of the 1994 European Symposioum on Algorithms ESA'94*, Springer Verlag, 1994, pp. 141–147. LNCS 855.

S. Voss, "Steiner-probleme in graphen," in *Mathematical Systems in Economics*, Hain, 1990.

P. Winter, "Generalized Steiner Problem in Halin Networks," in *Proceedings of the 12th International Symposium on Mathematical Programming*, MIT, 1985.

P. Winter, "Generalized Steiner problem in series-parallel networks," *Journal of Algorithms*, vol. 7, pp. 549–566, 1986.

P. Winter, "Steiner problems in networks: A survey," *Networks*, vol. 17, pp. 129–167, 1987.