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THE STEINER TRAVELING SALESMAN POLYTOPE AND RELATED POLYHEDRA*

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Abstract. In this paper we consider an extended formulation of the Steiner traveling salesman problem, that is, when variables are associated with both the edges and the nodes of the graph. We give a complete linear description of the associated polytope when the underlying graph is series-parallel. By projecting this polytope onto the edge variables, we obtain a characterization of the Steiner traveling salesman polytope in the same class of graphs. Both descriptions yield polynomial time (cutting plane) algorithms for the corresponding problems in that class of graphs.

Key words. Steiner traveling salesman problem, polyhedral combinatorics

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1. Introduction.

A cycle of a graph G is called *simple* if no node is incident to more than two of its edges. Given a graph $G = (V, E)$, a weight vector $w \in \mathbb{R}^{|E|}$ associated with the edges of G, and a subset of distinguished nodes $T \subseteq V$, called *terminals*, the *Steiner traveling salesman problem* (StSP) is the problem of finding a minimum weight simple cycle of G spanning T. Such a cycle is called a *Steiner tour*. The nodes not in T are called *Steiner nodes*. Given a weight vector $c \in \mathbb{R}^{|V|}$ associated with the nodes of G (in addition to the edge weights), and a root vertex $r \in V$, the *r-traveling salesman problem* (r-TSP) is to find a simple cycle containing r and whose total weight of both nodes and edges is minimized. Such a cycle is called an *r-tour*. An r-tour will be called *trivial* if it is reduced to the node r. The r-TSP is also called the *extended formulation* of the StSP.

In this paper we give a complete description, in $\mathbb{R}^{|E|+|V|}$, of the polytope associated with the solutions to the r-TSP in the class of series-parallel graphs. By projecting this polytope onto $\mathbb{R}^{|E|}$, we obtain a complete characterization of the polytope associated with the solutions to the StSP in the same class of graphs. This yields polynomial cutting plane algorithms to solve both the r-TSP and the StSP in that class of graphs.

The StSP and r-TSP are both NP-hard. They contain as a special case the well-known traveling salesman problem (TSP). The TSP has been shown to be polynomial in special classes of graphs. In [9], Cornuéjols, Fonlupt, and Naddef consider the graphical Steiner TSP, that is, when the Steiner tour can go through a node more than once. They give a linear time algorithm for this problem on series-parallel graphs. Their algorithm is an extension of an algorithm of Ratliff and Rosenthal [25] for graphs that model rectangular warehouses (a particular class of series-parallel graphs).

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Let $G = (V, E)$ be a graph. If $F \subseteq E$, $U \subseteq V$, then $(x^F, y^U) \in \mathbb{R}^{|E|+|V|}$ denotes the *incidence vector* of the subgraph (U, F) of G, i.e., $x^F(e) = 1$ if $e \in F$ and 0 otherwise, and $y^U(v) = 1$ if $v \in U$ and 0 otherwise. The *r-traveling* (resp., *Steiner traveling*) *salesman polytope* of G, denoted by r-TSP(G) (resp., StSP(G, T)), is the convex hull of the incidence vectors of the r-tours (resp., Steiner tours) of G, i.e.,

$$r\text{-TSP}(G) = \text{conv}\{(x^F, y^U) \in \mathbb{R}^{|E|+|V|} \mid (U, F) \text{ is an } r\text{-tour of } G\},$$
$$\text{StSP}(G, T) = \text{conv}\{x^F \in \mathbb{R}^{|E|} \mid F \subseteq E \text{ is a Steiner tour}\}.$$

Let TSP(G) denote the polytope associated with the TSP.

To the best of our knowledge, neither the r-TSP(G) nor the StSP(G, T) has been considered in the literature. However, the traveling salesman polytope, TSP(G), has been one of the most attractive subjects in polyhedral combinatorics in the past three decades [20], [21]. In particular, several classes of facet defining inequalities of TSP(G) have been identified, and efficient separation algorithms have been devised.

Complete descriptions of the TSP(G) have been obtained for some classes of graphs. Cornuéjols, Naddef, and Pulleyblank [8] describe the TSP(G) for Halin graphs. In [3], Barahona and Grötschel characterize the TSP(G) for graphs not contractible to $K_5 \setminus \{e\}$. A complete description of a minimal system of inequalities defining TSP(G), when G is complete, is known for graphs having no more than 8 nodes. Norman [24] describes the TSP(G) for complete graphs on 6 nodes. Boyd and Cunningham [5] give that description for graphs on 7 nodes, and Christof, Jünger, and Reinelt [10] give a description of the TSP(G) for graphs on 8 nodes.

A graph $G = (V, E)$ is said to be *k-edge connected* (for k fixed) if, for any pair of nodes $i, j \in V$, there are at least k edge-disjoint paths from i to j. Given weights on the edges of G and a set of terminals $T \subseteq V$, the *Steiner 2-edge connected subgraph problem* is the problem of finding a minimum 2-edge connected subgraph of G, spanning T. This problem is closely related to the StSP. In fact, as is pointed out in [13], when $T = V$, the problem of determining if a graph $G = (V, E)$ contains a Steiner tour (Hamiltonian cycle) can be reduced to the Steiner 2-edge connected subgraph problem. The relation between the two problems has been widely investigated in the metric case, that is, when the underlying graph $G = (V, E)$ is complete and the weight function satisfies the triangle inequalities (i.e., $w(e_1) \leq w(e_2) + w(e_3)$ for every three edges e_1, e_2, e_3 defining a triangle in G). In particular, Monma, Munson, and Pulleyblank [23] showed that $\tau \leq \frac{4}{3}Q_2$ when $T = V$, where τ is the weight of an optimal Steiner tour and Q_2 is the weight of an optimal 2-edge connected subgraph. Then it follows that the value τ' of an optimal solution of the classical linear relaxation of the TSP(G) provides a lower bound on τ. Cunningham (see [23]) shows that τ' also provides a lower bound on Q_2. Further structural properties and worst case analysis are given in Frederickson and Ja'Ja' [15], Bienstock, Brickell, and Monma [4], and Goemans and Bertsimas [17].

Given a graph $G = (V, E)$ with weights on its edges and a set of terminals $S \subseteq V$, the *Steiner tree problem* is to find a minimum weight tree in G which spans S. This problem, which is known to be NP-hard, is closely related to the StSP. Although a polynomial time algorithm in series-parallel graphs is known for this problem, still we do not have a complete description of the associated polytope in that class of graphs. In [16], Goemans gives an extended formulation for that problem and characterizes the associated polytope when the graph is series-parallel. By projecting that polytope onto the edge variables, he also obtains a large class of facet-defining inequalities for the Steiner tree polytope. For more details on the polyhedral aspect of that problem, see [6], [7], [22], and [11].

In the next section we present an integer programming formulation of the r-TSP and give some basic properties of the relaxation of our formulation. In section 3, we prove that the linear inequalities in our formulation are sufficient to completely characterize the r-TSP(G) when G is series-parallel. In section 4, we give a complete description of the StSP(G,T) in series-parallel graphs; this is done by projecting r-TSP(G) on the edge variables. The remainder of this section is devoted to more definitions and notations.

The graphs we consider are finite, undirected, and connected and may have multiple edges and loops. We denote a graph by $G = (V, E)$, where V is the *node set* and E is the *edge set* of G. If e is an edge with endnodes u and v, then we write $e = uv$.

A graph G is said to be *contractible* to a graph H if H may be obtained from G by a sequence of elementary removals and contractions of edges. A contraction consists of identifying a pair of adjacent vertices, preserving all other vertices, and preserving all other adjacencies between vertices. A graph is called *series-parallel* [12] if it is not contractible to K_4 (the complete graph on four nodes). Clearly, series-parallel graphs have the following property.

REMARK 1. *If G is a series-parallel graph contractible to a graph H, then H is series-parallel.*

Given a graph $G = (V, E)$ and a node subset $W \subseteq V$ of G, the set of edges having one endnode in W and the other in $V \setminus W$ is called a *cut* of G and denoted by $\delta(W)$. If $v \in V$ is a node of G, then we write $\delta(v)$ for the cut $\delta(\{v\})$. We denote by $G(W)$ the subgraph of G induced by W, and by $E(W)$ its edges. For $W, W' \subseteq V$, (W, W') denotes the set of edges having one endnode in W and the other in W'. If $W \subseteq V$, we let $\overline{W} = V \setminus W$. Given a constraint $ax \geq \alpha$, a^T, $x \in \mathbb{R}^n$, and a solution $x^* \in \mathbb{R}^n$, we will say that $ax \geq \alpha$ is *tight* for x^* if $ax^* = \alpha$.

2. The polytope r-TSP(G). Let $G = (V, E)$ be a graph and $r \in V$ a root vertex. Let $x(e)$, $y(v)$ be variables associated with each edge e and node v. For any subset of edges $F \subseteq E$, we let $x(F) = \sum_{e \in F} x(e)$.

The r-TSP can then be formulated as the following integer program:

$$\text{Minimize} \sum_{e \in E} w(e)x(e) + \sum_{v \in V} c(v)y(v)$$

subject to

$$
\begin{array}{lll}
(1) & x(\delta(W)) \geq 2y(v) & \text{for all } W \subset V,\ |\overline{W}| \geq 2,\ r \in W,\ v \in \overline{W}, \\
(2) & x(\delta(r)) \leq 2y(r), & \\
(3) & x(\delta(v)) = 2y(v) & \text{for all } v \in V \setminus \{r\}, \\
(4) & x(e) \leq y(v) & \text{for all } v \in V,\ e \in \delta(v), \\
(5) & y(v) \leq 1 & \text{for all } v \in V, \\
(6) & x(e) \geq 0 & \text{for all } e \in E, \\
(7) & x(e), y(v) \in \mathbb{N} & \text{for all } e \in E,\ v \in V.
\end{array}
$$

Constraints (1) and (3) will be called *generalized cut constraints*. A generalized cut constraint is associated with a cut $\delta(W)$ and a node $v \in \overline{W}$. The pair $(\delta(W), v)$ will be called a *generalized cut*. A generalized cut will be called *tight* for a solution (x, y) if the corresponding constraint is tight for (x, y). Notice that the generalized cuts $(\delta(W), v)$ with $\overline{W} = \{v\}$ (equations (3)) are tight for all solutions of $H(G)$. The case where $|\overline{W}| \geq 2$ will be specified if necessary. Inequalities (5) and (6) are called *trivial inequalities*. Inequalities (4) combined with the trivial inequalities (5) imply that if $x(e) = 1$ for some $e \in \delta(v)$, then $y(v) = 1$. Let $H(G)$ denote the polytope defined by inequalities (1)–(6). We have the following.

THEOREM 2. *If G is series-parallel, then r-TSP$(G) = H(G)$.*

The proof of this theorem will be given in the following section. In what follows we are going to discuss some properties of the solutions of $H(G)$, which will be useful in the rest of the paper.

LEMMA 3. *Let $(x, y) \in \mathbb{R}^{|E|+|V|}$ be a solution of $H(G)$ such that $x(e) > 0$ for all $e \in E$. If $(\delta(W), v)$ is a generalized cut tight for (x, y), then $G(\overline{W})$ is connected.*

Proof. This is clear if $\overline{W} = \{v\}$. So suppose that $|\overline{W}| \geq 2$, and let us assume, on the contrary, that there is a partition $\overline{W}_1$, $\overline{W}_2$ of $\overline{W}$ such that $(\overline{W}_1, \overline{W}_2) = \emptyset$. Without loss of generality, we may suppose that $v \in \overline{W}_1$. Since G is connected, it follows that $(W, \overline{W}_1) \neq \emptyset \neq (W, \overline{W}_2)$. By our hypothesis, we have $x(W, \overline{W}_2) > 0$. As $(\delta(W), v)$ is tight for (x, y), it follows that $x(\delta(W)) = x(W, \overline{W}_1) + x(W, \overline{W}_2) = 2y(v)$. This implies that $x(\delta(W \cup \overline{W}_2)) = x(W, \overline{W}_1) < 2y(v)$, and thus the generalized cut $(\delta(W \cup \overline{W}_2), v)$ is violated by (x, y). But this contradicts the fact that $(x, y) \in H(G)$. □

LEMMA 4. *Let $(x, y) \in H(G)$, and let $(\delta(W), v)$ and $(\delta(W'), v')$ be two generalized cuts tight for (x, y). Then the following hold:*

(i) *If $v \in \overline{W \cup W'}$, then $(\delta(W \cap W'), v')$ and $(\delta(W \cup W'), v)$ are both generalized cuts tight for (x, y).*

(ii) *If $v \in W' \setminus W$ and $v' \in W \setminus W'$, then $(\delta(\overline{W' \setminus W}), v)$ and $(\delta(\overline{W \setminus W'}), v')$ are both generalized cuts tight for (x, y).*

Proof. The proof follows from the submodularity of the cuts, that is,

$$x(\delta(W)) + x(\delta(W')) \geq x(\delta(W \cap W')) + x(\delta(W \cup W')) \text{ for any } W, W' \subset V. \qquad \square$$

3. Proof of Theorem 2. Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals. A *Steiner* 2-*edge connected subgraph* of G is a 2-edge connected subgraph of G spanning T. Denote by STECP(G, T) the convex hull of the incidence vectors of the Steiner 2-edge connected subgraphs of G, and let $P(G, T)$ be the polytope given by the following linear inequalities:

$$
\begin{array}{lll}
(8) \qquad & 0 \leq x(e) \leq 1 & \text{for all } e \in E, \\
(9) \qquad & x(\delta(W)) \geq 2 & \text{for all } W \subseteq V,\ T \neq W \bigcap T \neq \emptyset, \\
(10) \qquad & x(\delta(W)) \geq 2x(e) & \text{for all } W \subseteq V,\ T \subseteq W,\ e \notin E(W).
\end{array}
$$

Inequalities (9) and (10) are called *Steiner* and *left-Steiner cut inequalities*, respectively. In [2], Baïou and Mahjoub state the following.

THEOREM 5. *If G is series-parallel, then STECP$(G, T) = P(G, T)$.*

For a complete proof of this theorem, see [1]. In what follows we are going to use that description to prove Theorem 2.

The proof of Theorem 2 is by induction on the number of edges. The theorem is trivially true for a graph with no more than two edges. Suppose it is true for any series-parallel graph with no more than m edges and suppose that G contains exactly $m+1$ edges. Let us assume, on the contrary, that r-TSP$(G, S) \neq H(G)$, and let (x, y) be a fractional extreme point of $H(G)$. We have the following lemmas.

LEMMA 6. *$x(e)$ and $y(v)$ are positive for all $e \in E$ and $v \in V$.*

Proof. By inequalities (4) it suffices to prove that $x(e) > 0$ for all $e \in E$. If e_0 is an edge such that $x(e_0) = 0$, then let $x' \in \mathbb{R}^{|E|-1}$ be given by $x'(e) = x(e)$ for all $e \in E \setminus \{e_0\}$. Clearly, (x', y) belongs to $H(G')$, where G' is the graph obtained from G by deleting e. Moreover (x', y) is an extreme point of $H(G')$. Since (x', y) is fractional and G' is series-parallel, we have a contradiction. □

LEMMA 7. *If $(\delta(W), v)$ is a generalized cut tight for (x, y) with $|\overline{W}| \geq 2$, then $y(v) = 1$.*

Proof. Suppose, on the contrary, that $y(v) < 1$. Suppose that $|W|$ is minimum. That is, for every generalized tight cut $(\delta(W'), w')$ with $|\overline{W'}| \geq 2$ and $|W'| < |W|$, we have $y(w') = 1$. Now remark that by constraints (1)

$$y(v) \geq y(v') \quad \text{for all } v' \in \overline{W}. \tag{11}$$

Let $G' = (V', E')$ be the graph obtained from G by contracting $\overline{W}$, and denote by $\bar{w}$ the node resulting from this contraction. By Lemma 3 together with Remark 1, it follows that G' is series-parallel. Let x' be the restriction of x on E', and $y' \in \mathbb{R}^{|W|+1}$ such that $y'(u) = y(u)$ if $u \in W$ and $y'(\bar{w}) = y(v)$.

It is easy to see that (x', y') is a solution of $H(G')$. As G' is series-parallel and $|E'| < |E|$, by the induction hypothesis, $H(G')$ is integral. In consequence, (x', y') can be written as a convex combination of (integral) extreme points of $H(G')$. Thus there are t extreme points of $H(G')$, $(x'_1, y'_1), \ldots, (x'_t, y'_t)$ and $\lambda_1, \ldots, \lambda_t \geq 0$, such that

$$(x', y') = \sum_{i=1}^{t} \lambda_i (x'_i, y'_i), \qquad \sum_{i=1}^{t} \lambda_i = 1.$$

Since $y'(\bar{w}) = y(v) < 1$, there must exist a solution among $(x'_1, y'_1), \ldots, (x'_t, y'_t)$, say (x'_1, y'_1) such that $y'_1(\bar{w}) = 0$. By equality (3) associated with $\bar{w}$, it follows that $x'_1(\delta(W)) = x'_1(\delta(\bar{w})) = 2y'_1(\bar{w})) = 0$. Let $(x^*, y^*) \in \mathbb{R}^{|E|+|V|}$ be the solution such that

$$x^*(e) = \begin{cases} x'_1(e) & \text{if } e \in E(W), \\ 0 & \text{otherwise}, \end{cases} \qquad y^*(v) = \begin{cases} y'_1(v) & \text{if } v \in W, \\ 0 & \text{otherwise}. \end{cases}$$

In what follows we are going to show that every constraint of $H(G)$ that is tight for (x, y) is also tight for (x^*, y^*). Since $(x, y) \neq (x^*, y^*)$, this contradicts the extremality of (x, y).

First, it can be easily seen that every inequality among (2)–(6) that is tight for (x, y) is also tight for (x^*, y^*). So let us consider a generalized cut $(\delta(W'), v')$ tight for (x, y) with $|\overline{W'}| \geq 2$. Suppose first that $W' \subseteq W$.

If $v' \in W$, then $(\delta(W'), v')$ is also a generalized cut in G', and thus it is tight for both (x', y') and (x'_1, y'_1). Hence $x^*(\delta(W')) = x'_1(\delta(W')) = 2y'_1(v') = 2y^*(v')$.

If $v' \in \overline{W}$, then $2y(v) \leq x(\delta(W')) = 2y(v')$. By (11) this implies that $y(v') = y(v) = y'(\bar{w})$. Thus $(\delta(W'), \bar{w})$ is a generalized cut in G' tight for (x', y'), and hence $x^*(\delta(W')) = x'_1(\delta(W')) = 2y'_1(\bar{w}) = 0 = 2y^*(v')$. Now if $W \subseteq W'$, by the definition of (x^*, y^*), we have $x^*(\delta(W')) = 2y^*(v') = 0$. Thus we can suppose that $W \setminus W' \neq \emptyset \neq W' \setminus W$. We consider two cases.

Case 1. $v \in \overline{W \cup W'}$. From Lemma 4(i) we have that $(\delta(W \cap W'), v')$ is a generalized cut tight for (x, y). Since $(W \cap W') \subset W$, it follows from above that $(\delta(W \cap W'), v')$ is also tight for (x^*, y^*) and thus $x^*(\delta(W')) = x^*(\delta(W \cap W')) = 2y^*(v')$.

Case 2. $v \in W' \setminus W$. Then $v' \notin \overline{W \cup W'}$; otherwise by Lemma 4(i), by exchanging v' and v, $(\delta(W \cap W'), v)$ would be a generalized cut tight for (x, y), which contradicts the minimality of $|W|$. Thus suppose that $v' \in W \setminus W'$. By Lemma 4(ii), $(\delta(\overline{W \setminus W'}), v')$ is a generalized cut tight for (x, y). Since $(\delta(\overline{W \setminus W'}), v')$ is also a generalized cut in G', it is also tight for (x', y') and hence for (x'_1, y'_1). Thus

$$x^*(\delta(W')) = x'_1(\delta(\overline{W \setminus W'})) = 2y'_1(v') = 2y^*(v'). \qquad \square$$

Now, let

$$T = \{v \in V : \ y(v) = 1\}.$$

LEMMA 8. $|T| \geq 2$.

Proof. Assume the contrary, that is, $|T| \leq 1$. Then by inequalities (4), it follows that $x(e) < 1$ for all $e \in E$. And by the inequality of type (1) corresponding to $W = \{r\}$ together with inequality (2), it follows that $y(v) < 1$ for all $v \in V \setminus \{r\}$. If $T = \emptyset$, then we also have $y(r) < 1$. In consequence, if we consider the solution $(0, 0)$, we will have that all the constraints of $H(G)$ that are tight for (x, y) are also tight for that solution. But this contradicts the extremality of (x, y).

Now let us assume that $|T| = 1$. Hence $y(r) = 1$ (and $y(v) < 1$ for all $v \in V \setminus \{r\}$).

If $x(\delta(r)) < 2$, then, by considering the incidence vector of the trivial r-tour, we will also have a solution that satisfies with equality all the constraints of $H(G)$ tight for (x, y), which again yields a contradiction.

So suppose that $x(\delta(r)) = 2$. Since $0 < y(v) < 1$ for all $v \in V \setminus \{r\}$, by Lemma 7 no inequality (1) is tight for (x, y).

Claim. No inequality (4) is tight for (x, y).

Proof of the claim. As (x, y) is an extreme point of $H(G)$ and $0 < x(e) < 1$ for all $e \in E$, it follows that there is a set of pairs $(e_1, v_1), \dots, (e_l, v_l)$, $e_i \in \delta(v_i)$ for $i = 1, \dots, l$, such that (x, y) is the unique solution of the system

$$(\mathrm{L}) \begin{cases} y(r) = 1, & \\ x(\delta(v)) = 2y(v) & \text{for all } v \in V, \\ x(e_i) = y(v_i) & \text{for } i = 1, \dots, l. \end{cases}$$

Let $f = uv \in E$. Suppose that $x(f) = y(u)$. Let $G' = (V', E')$ be the graph obtained from G by contracting f. Let x' be the restriction of x on E' and $y' \in \mathbb{R}^{|V'|}$ such that $y'(w) = y(w)$ if $w \in V' \setminus \{w_0\}$ and $y'(w_0) = y(v)$, where w_0 is the node of V' that arises from the contraction of f. Now as $x(f) < 1$ and $y(r) = 1$, we have $u \neq r$. If $v = r$, we let w_0 be the root vertex in G'. Note that $x'(\delta(w_0)) = 2y'(w_0)$. It can be, in fact, easily seen that $(x', y') \in H(G')$. In what follows we will show that (x', y') is also an extreme point of $H(G')$. Indeed, if this is not the case, as by the induction hypothesis $H(G')$ is integral, there are integral extreme points $(x^1, y^1), \dots, (x^k, y^k)$ and scalars $\lambda_1, \dots, \lambda_k \geq 0$ such that

$$(x', y') = \sum_{i=1}^{k} \lambda_i (x^i, y^i), \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

Note that any constraint of $H(G')$ tight for (x', y') is also tight for (x^i, y^i) for $i = 1, \dots, k$. We distinguish two cases.

Case 1. $v \neq r$. Thus $y'(w_0) < 1$. In consequence, there must exist one of the extreme points (x^i, y^i), say (x^1, y^1), such that $y^1(w_0) = 0$. Since $x^1(\delta(w_0)) = 2y^1(w_0)$ and $x^1(e) \geq 0$ for all $e \in E$, it follows that $x^1(e) = 0$ for all $e \in \delta(w_0)$. Consider the solution $(\bar{x}, \bar{y}) \in \mathbb{R}^{|E|+|V|}$ given by

$$\bar{x}(e) = \begin{cases} x^1(e) & \text{if } e \in E \setminus (\delta(u) \cup \delta(v)), \\ 0 & \text{otherwise,} \end{cases} \qquad \bar{y}(w) = \begin{cases} y^1(w) & \text{if } w \in V \setminus \{u, v\}, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $(\bar{x}, \bar{y})$ is a solution of (L). In fact, clearly equalities (3) as well as the equality $x(\delta(r)) = 2y(r)$ are satisfied by $(\bar{x}, \bar{y})$. Moreover, as $y'(r) = 1$, we have $\bar{y}(r) = y^1(r) = 1$. Now for a pair (e_i, v_i), as $\bar{x}(e) = 0$ for $e \in \delta(u) \cup \delta(v)$ and $\bar{y}(u) = \bar{y}(v) = 0$, it follows that the corresponding inequality is satisfied with equality if $v_i \in \{u, v\}$. If $v_i \in V \setminus \{u, v\}$, as $x'(e_i) = y'(v_i)$, we should have $x^1(e_i) = y^1(v_i)$. Hence $(\bar{x}, \bar{y})$ satisfies system (L). As $(\bar{x}, \bar{y}) \neq (x, y)$, this is a contradiction.

Case 2. $v = r$. Thus $y'(w_0) = 1$. Since $x'(\delta(u) \setminus \{f\}) < 1$, there must exist one of the extreme points (x^i, y^i), say (x^1, y^1), such that $x^1(\delta(u) \setminus \{f\}) = 0$. Consider the solution $(\bar{x}, \bar{y}) \in \mathbb{R}^{|E|+|V|}$ given by

$$\bar{x}(e) = \begin{cases} 0 & \text{if } e = f, \\ x^1(e) & \text{otherwise,} \end{cases} \qquad \bar{y}(w) = \begin{cases} y^1(w) & \text{if } w \in V \setminus \{u, v\}, \\ 0 & \text{if } w = u, \\ 1 & \text{if } w = v. \end{cases}$$

As above, one can easily verify that $(\bar{x}, \bar{y})$ is a solution of (L), which is again a contradiction and this ends the proof of the claim.

From the claim above and Lemma 6, it follows that the only inequalities tight for (x, y) are

$$(\mathrm{L}') \begin{cases} y(r) = 1, \\ x(\delta(v)) = 2y(v) \quad \text{for all } v \in V. \end{cases}$$

Now we claim that G contains at least one nontrivial r-tour. In fact, as $x(\delta(r)) = 2$ and $x(e) \leq 1$ for all $e \in E$, r is adjacent to at least two nodes. If there is no nontrivial tour, then G must contain a cut $\delta(S)$ separating r and one of its neighbors, say u, such that $\delta(S) = \{ru\}$. On the other hand, we have $x(ru) \leq x(\delta(u)) = 2y(u)$. If $|\bar{S}| \geq 2$, as $y(u) < 1$, by Lemma 7 it follows that $x(ru) = x(\delta(S)) \neq 2y(u)$. Hence $x(\delta(S)) < 2y(u)$, a contradiction. Now suppose that $|\bar{S}| = 1$, that is, $\bar{S} = \{u\}$. Thus $x(ru) = x(\delta(u)) = 2y(u)$. However, by inequalities (4) one should have $x(ru) \leq y(u)$. Since $y(u) > 0$, this is also impossible.

Now the incidence vector of any nontrivial r-tour verifies equalities (L′). This yields a contradiction with the fact that (x, y) is an extreme point, which finishes the proof of our lemma. ∎

In what follows, we will show that the projection of (x, y) onto $\mathbb{R}^{|E|}$, i.e., x, is an extreme point of $P(G, T)$. It is clear that every constraint of $P(G, T)$ can be obtained from some linear combination of constraints of $H(G)$. Thus $x \in P(G, T)$. Now to prove that x is an extreme point of $P(G, T)$, it suffices to display a system of equalities from $P(G, T)$, where x is the unique solution.

If there exists an inequality of type (4) that is tight for (x, y) with $y(v) = 1$, then this equality corresponds to an inequality $x(e) \leq 1$ of $P(G, T)$ that is tight for x. Denote such equalities by (8′). Let $(\delta(W), v)$ be a generalized cut tight for (x, y). Then by Lemma 7 we have $y(v) = 1$ and hence $v \in T$. Thus the equation yielded by $(\delta(W), v)$ corresponds to the Steiner cut inequality $x(\delta(W)) \geq 2$ of $P(G, T)$ that is tight for x. Let us denote by (9′) such equalities.

Now consider an equality of type (3). If $y(v) = 1$ for $v \neq r$, then, as before, this equality corresponds to a Steiner cut of $P(G, T)$ that is tight for x. If inequality (2) is tight for (x, y)—that is, $x(\delta(r)) = 2$—then, by Lemma 8, $|T| \geq 2$, and $r \in T$, this equality also corresponds to a Steiner cut tight for x in $P(G, T)$. We will also denote these equalities by (9′). If $y(v) < 1$ and there exists $e \in \delta(v)$ such that $x(e) = y(v)$, then this yields a left-Steiner cut $x(\delta(v)) \geq 2x(e)$ tight for x. We let (10′) be the set of these equalities. Let (S) be the system of equalities defined by (8′), (9′), and (10′).

We claim that x is the unique solution of (S). Indeed, if there is a further solution x' of (S), then by considering $y' \in \mathbb{R}^{|V|}$ such that $y'(v) = \frac{1}{2}x'(\delta(v))$ for all $v \in V$, the solution (x', y') would verify with equality all the constraints tight for (x, y). As $x' \neq x$ and (x, y) is an extreme point of $H(G)$, this is impossible.

Now since the equalities of (S) all come from inequalities of $P(G, T)$, x is an extreme point of $P(G, T)$. Since x is fractional and G is series-parallel, this contradicts Theorem 5.

4. The polytope $\mathbf{StSP}(G, T)$. Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals. Let $N = V \setminus T$ be the set of Steiner vertices.

Let $P_{E,N}(G) \subseteq \mathbb{R}^{|E|+|N|}$ be the polytope obtained from $H(G)$ by selecting a root vertex $r \in T$ and setting $y(v) = 1$ for all $v \in T$. Thus $P_{E,N}(G)$ is given by the following system:

$$\begin{array}{lll}
(12) & x(\delta(W)) \geq 2y(v) & \text{for all } W \subset V,\ T \subseteq W,\ v \notin W, \\
(13) & x(\delta(v)) \leq 2y(v) & \text{for all } v \in N, \\
(14) & x(e) \leq y(v) & \text{for all } v \in N,\ e \in \delta(v), \\
(15) & y(v) \leq 1 & \text{for all } v \in N, \\
(16) & x(\delta(W)) \geq 2 & \text{for all } W \subseteq V,\ T \neq W \bigcap T \neq \emptyset, \\
(17) & x(\delta(v)) = 2 & \text{for all } v \in T \setminus \{r\}, \\
(18) & x(\delta(r)) \leq 2, & \\
(19) & x(e) \leq 1 & \text{for all } e \in \delta(v),\ v \in T, \\
(20) & x(e) \geq 0 & \text{for all } e \in E.
\end{array}$$

As $P_{E,N}(G)$ is a face of $H(G)$, by Theorem 2, we have the following.

Corollary 9. *$P_{E,N}(G)$ is integral if G is series-parallel.*

Now, to describe the polytope $\mathrm{StSP}(G, T)$, we are going to project onto the subspace of the edge variables. To do this we use Fourier–Motzkin elimination [26] to eliminate the node variables $y(v)$ from $P_{E,N}(G)$. For every node $v \in N$, we will combine inequalities containing $+y(v)$ with the ones containing $-y(v)$ as follows:

- By combining inequalities (12) and (13), we obtain the inequalities

$$(21) \qquad x(\delta(W)) \geq x(\delta(v)) \quad \text{for all } W \subset V,\ T \subseteq W,\ v \notin W;$$

- combining inequalities (12) and (14), we obtain the left-Steiner cut inequalities (10);
- combining inequalities (13) and (15), we obtain

$$(22) \qquad x(\delta(v)) \leq 2 \quad \text{for all } v \in N;$$

- and finally, the combination of inequalities (14) and (15) gives the inequalities $x(e) \leq 1$ for all $e \in \delta(v)$, $v \in N$. This, together with inequalities (19), yields

$$(23) \qquad x(e) \leq 1 \quad \text{for all } e \in E.$$

Lemma 10. *The left-Steiner cut inequalities* (10), $x(\delta(W)) \geq 2x(e)$, *with* $|\overline{W}| \geq 2$, *are redundant for StSP(G, T).*

Proof. As $e \notin E(W)$, there is a node, say v, of e that belongs to $\overline{W}$. By inequality (21) associated with W and v together with the left-Steiner cut associated with $\delta(V \setminus \{v\})$ and the edge e, we have

$$x(\delta(W)) \geq x(\delta(v)) = x(\delta(V \setminus \{v\})) \geq 2x(e). \qquad \square$$

By Lemma 10, the left-Steiner cut inequalities that may be essential in the description of $\mathrm{StSP}(G, T)$ can be written as follows:

$$(24) \qquad x(\delta(v)) \geq 2x(e) \quad \text{for all } v \in N,\ e \in \delta(v).$$

Now from the development above and Corollary 9 we obtain the following result.

THEOREM 11. *If G is series-parallel, then inequalities* (16)–(18), (20), *and* (21)–(24) *completely describe $StSP(G, T)$.*

5. Concluding remarks. We have studied an extended formulation of the StSP and have given a complete linear description of the associated polytope when the underlying graph is series-parallel. By projecting this polytope onto the edge variables space, we have obtained a description of the Steiner traveling salesman polytope in that class of graphs.

It would be interesting to have such a description for the graphical Steiner traveling salesman polyhedron in that class of graphs. A complete characterization of that polyhedron in series-parallel graphs is, unfortunately, still unknown even when $T = V$. In fact, as shown by Cornuéjols, Fonlupt, and Naddef [9], the traveling salesman polyhedron in this case may contain constraints which do not come from cuts. In [14], Fonlupt and Naddef characterize the graphs for which the graphical traveling salesman polyhedron is given by the nonnegativity and the cut constraints.

Given a graph $G = (V, E)$ and two nodes u, v of V, let $G_{u,v}$ be the graph obtained from G by identifying u and v. Let w be the node resulting from the identification of u and v. Let $P_{u,v}(G)$ be the polytope, the extreme points of which are the incidence vectors of the paths of G between u and v, different from uv (if $uv \in E$). Clearly, $P_{u,v}(G) = \mathrm{StSP}(G_{u,v}, \{w\})$. Thus Theorem 11 provides at the same time a description of $P_{u,v}(G)$ when G is series-parallel and $uv \in E$.

We conclude by mentioning that, as inequalities (1), (16), and (21) can be separated in polynomial time, by the ellipsoid method [18], Theorems 2 and 11 provide polynomial cutting plane algorithms for both the r-TSP and StSP problems on series-parallel graphs. These are, to the best of our knowledge, the first polynomial time algorithms for these problems in that class of graphs.

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