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ON THE CUT POLYTOPE

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The cut polytope $P_C(G)$ of a graph $G=(V, E)$ is the convex hull of the incidence vectors of all edge sets of cuts of G. We show some classes of facet-defining inequalities of $P_C(G)$. We describe three methods with which new facet-defining inequalities of $P_C(G)$ can be constructed from known ones. In particular, we show that inequalities associated with chordless cycles define facets of this polytope; moreover, for these inequalities a polynomial algorithm to solve the separation problem is presented. We characterize the facet defining inequalities of $P_C(G)$ if G is not contractible to K_5. We give a simple characterization of adjacency in $P_C(G)$ and prove that for complete graphs this polytope has diameter one and that $P_C(G)$ has the Hirsch property. A relationship between $P_C(G)$ and the convex hull of incidence vectors of balancing edge sets of a signed graph is studied.

Key words: Max cut problem, facets of polyhedra, polyhedral combinatorics.

1. Introduction and notation

The graphs we consider are finite, undirected, and without multiple edges. We denote a graph by $G=(V, E)$, where V is the *node set* and E the *edge set* of G. Given $U \subseteq V$ we denote by $\delta(U)$ the set of edges with exactly one extremity in U, and we call this set a cut.

If $F \subseteq E$ the incidence vector of F, x^F is defined by

$$x^F(e)=\begin{cases}1 & \text{if } e \in F,\\ 0 & \text{if } e \in E \setminus F.\end{cases}$$

We denote by $P_C(G)$ the convex hull of incidence vectors of cuts of G. $P_C(G)$ is called the cut polytope of G.

We shall study the facial structure of $P_C(G)$. Our aim is to solve the following discrete quadratic problem

$$\begin{aligned}&\min \quad H=\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} s_i s_j\\ &\text{subject to} \quad s_i \in \{-1, 1\} \text{ for } i=1, \ldots, n.\end{aligned} \tag{1.1}$$

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This problem arises in statistical physics [1] and can be reduced to a maximum cut problem as follows.

Let us define a graph $G=(V, E)$, where

$$V=\{1,\ldots,n\} \quad \text{and} \quad ij\in E \text{ if and only if } J_{ij}\neq 0;$$

the weight J_{ij} is associated to the edge ij. Given a cut C the weight of C is $\sum_{ij\in C} J_{ij}$.

It is easy to see that problem (1.1) is equivalent to the problem of finding a maximum cut in G, cf. [1].

The problem of finding a maximum bipartite subgraph has been studied in [3], and if the weights are non-negative this is equivalent to the maximum cut problem. For general weights this is not true; thus we shall study the cut polytope here.

We shall characterize a class of facet-defining inequalities of $P_C(G)$ where the separation problem can be solved in polynomial time. Therefore, we can optimize a linear function in polynomial time over the polytope defined by these inequalities. This is a way of getting lower bounds for quadratic discrete programming.

The maximum cut problem is NP-hard [4] for general graphs and is polynomially solvable for graphs with no long odd cycles [6], planar graphs [7], and graphs not contractible to K_5 [2].

We shall characterize $P_C(G)$ for graphs not contractible to K_5, we shall study adjacency in $P_C(G)$, and we shall use these results to study the polytope of balancing edges of a signed graph.

If $G=(V, E)$ is a graph, the cardinality of V is called the *order* of G. If $e\in E$ is an edge with endnodes i and j, we also write ij to denote the edge e. If $H=(W, F)$ is a graph with $W\subseteq V$ and $F\subseteq E$, then H is called a subgraph of G.

If $G=(V, E)$ is a graph and $F\subseteq E$, then $V(F)$ denotes the set of nodes of V that occur at least once as an endnode of an edge in F. Similarly, for $W\subseteq V$, $E(W)$ denotes the set of all edges of G with both endnodes in W.

A graph G is called *complete* if every two different nodes of G are linked by an edge. The complete graph with n nodes is denoted by K_n. A graph is called *bipartite* if its node set can be partitioned into two nonempty, disjoint sets V_1 and V_2 such that no two nodes in V_1 and no two nodes in V_2 are linked by an edge. We call V_1, V_2 a *bipartition* of V. If $|V_1|=p$, $|V_2|=q$ and G is a maximal bipartite graph, it is denoted by $K_{p,q}$. If $W\subseteq V$, then $\delta(W)$ is the set of edges with one endnode in W and the other in $V\setminus W$. The edge set $\delta(W)$ is called a *cut*. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v\in V$ and call $\delta(v)$ the *star* of v.

If U, W are disjoint subsets of V, then $[U:W]$ denotes the set of edges of G that have one endnode in U and the other endnode in W. We write $[u:W]$ instead of $[\{u\}:W]$ for $u\in V$.

A *path* P in $G=(V, E)$ is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1=v_0v_1$, $e_2=v_1v_2, \ldots, e_k=v_{k-1}v_k$ and such that $v_i\neq v_j$ for $i\neq j$. The nodes v_0 and v_k are the endnodes of P and we say that P links v_0 and v_k or goes from v_0 to v_k. If $P=e_1$, $e_2,\ldots,e_k$ is a path linking v_0 and v_k and $e_{k+1}=v_0v_k\in E$, then the sequence e_1, $e_2,\ldots,e_k$, e_{k+1} is called a *cycle*.

If P is a cycle and uv an edge of $E\backslash P$ with $u, v \in V(P)$, then uv is called a *chord* of P. A cycle with three edges is called a *triangle*.

If v is a node of a graph G, then $G\backslash v$ denotes the subgraph of G obtained by removing node v and all edges incident to v from G.

A graph G is *contractible* to G' if G' can be obtained from G by a sequence of elementary contractions, in which a pair of adjacent vertices is identified and all other adjacencies between vertices are preserved (multiple edges arising from the identification being replaced by single edges).

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces in $\mathbb{R}^m$. A *polytope* is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, denoted by dim P, is the maximum number of affinely independent points in P minus one.

If $a \in \mathbb{R}^m \backslash \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^T x \leq a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leq a_0\}$. We say that a valid inequality $a^T x \leq a_0$ *supports* P or defines a *face* of P if $\emptyset \neq P \cap \{x \mid a^T x = a_0\} \neq P$. A valid inequality $a^T x \leq a_0$ defines a *facet* of P if it defines a face of P and if there exist dim P affinely independent points in $P \cap \{a^T x = a_0\}$.

If $P \subseteq \mathbb{R}^m$ is a full dimensional polyhedron, i.e., dim $P = m$, a linear system $Ax \leq b$ that defines P is minimal if and only if there is a bijection between the inequalities of the system and the facets of P. Moreover, these facet-defining inequalities are unique up to positive multiples.

Given $b : E \to \mathbb{R}$, and $F \subseteq E$, $b(F)$ will denote $\sum_{e \in F} b(e)$. The support of b, E_b will be $E_b = \{e \mid b(e) \neq 0\}$.

The bipartite subgraph polytope $P_B(G)$ is the convex hull of incidence vectors of bipartite subgraphs of G. It is clear that $P_C(G) \subseteq P_B(G)$, but in general $P_C(G) \neq P_B(G)$.

Barahona, Grötschel and Mahjoub [3] show that $P_C(G)$ is full dimensional; moreover, some of the facet-defining inequalities of $P_B(G)$ studied by them are also facet-defining inequalities of $P_C(G)$.

If x is a real number, then $\lceil x \rceil$ resp. $\lfloor x \rfloor$ denotes the smallest resp. largest integer not smaller resp. larger than x.

2. Construction of facets

First, we will state three theorems that characterize some facet-defining inequalities of $P_C(G)$. We will omit the proofs because they are analogous to those that appear in [3].

Theorem 2.1. *Let $G = (V, E)$ be a graph and let (W, F) be a complete subgraph of order $p \geq 3$ of G. Then*

$$x(F) \leq \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor \tag{2.2}$$

is a valid inequality with respect to $P_C(G)$. Furthermore, (2.2) defines a facet of $P_C(G)$ if and only if p is odd. □

A graph is called a bicycle p-wheel if G consists of a cycle of length p and two nodes that are adjacent to each other and to every node in the cycle.

Theorem 2.3. *Let $G=(V,E)$ be a graph and let (W,F) be a bicycle $(2k+1)$-wheel, $k \geq 1$, contained in G. Then the inequality*

$$x(F) \leq 2(2k+1)$$

defines a facet of $P_C(G)$. □

Theorem 2.4. *Let $H=(W,F)$ be a complete subgraph of order q where $W=\{1,2,\ldots,q\}$. Let positive integers t_i $(1 \leq i \leq q)$ satisfy $\sum_{i=1}^{q} t_i = 2k+1$, $k \geq 3$ and $\sum_{t_i>1} t_i \leq k-1$. Set*

$$a_{ij} := \begin{cases} t_i t_j, & 1 \leq i < j \leq q, \\ 0, & \{i,j\} \not\subset W. \end{cases}$$

Then

$$a^{\mathrm{T}} x \leq \alpha := k(k+1)$$

defines a facet of $P_C(G)$.

To simplify technical details in subsequent proofs, we first state a lemma.

Lemma 2.5. *Let $b^{\mathrm{T}}x \leq \beta$ be a valid inequality with respect to $P_C(G)$. Given adjacent nodes p and q, let S be a proper subset of $V \setminus \{p,q\}$ and $T = V \setminus (S \cup \{p,q\})$. Suppose that the incidence vectors of the edge sets $\delta(S)$, $\delta(T)$, $\delta(S \cup \{p\})$, $\delta(S \cup \{q\})$ satisfy $b^{\mathrm{T}}x \leq \beta$ with equality. Then*

$$b_{pq} = 0.$$

Proof. $0 = \beta - \beta = b^{\mathrm{T}}x^{\delta(T)} - b^{\mathrm{T}}x^{\delta(S \cup \{p\})} = b([q:T]) - b([q:S]) - b_{pq}$, and $0 = \beta - \beta = b^{\mathrm{T}}x^{\delta(S)} - b^{\mathrm{T}}x^{\delta(S\cup\{q\})} = b([q:S]) - b([q:T]) - b_{pq}$. Thus, summing the two equations we obtain

$$-2b_{pq} = 0. \qquad \square$$

Now we shall describe three methods to construct "facets from facets."

Theorem 2.6. (a) (Node splitting) *Let $G=(V,E)$ be a graph and $a^{\mathrm{T}}x \leq \alpha$ be a facet-defining inequality for $P_C(G)$. Let E_a be the support of a, and let v be a node in $V(E_a)$. Let W be a subset of $V(E_a)$ such that $a^{\mathrm{T}}x^{\delta(W)} = \alpha$ and assume that $v \in W$. Choose any nonempty subset $F \subseteq \delta(v) \cap E(W)$ such that $a_e > 0$ for $e \in F$, and construct a new graph $G' = (V', E')$ from G as follows. Split node v into two nodes v_1, v_2 such* *that v_1 is incident to all edges contained in F and v_2 is incident to all edges in $\delta(v)\backslash F$. The edge v_1v_2 is added; in addition, any further edges v_1u with $u \notin V(E_a)$ may be added. The other parts of G remain unchanged. Set*

$$\begin{aligned}
&\bar{a}_{ij} := a_{ij} && \textit{for all } ij \in E\backslash\delta(v),\\
&\bar{a}_{v_1u} := a_{vu} && \textit{for all } v_1u \in E' \textit{ with } vu \in F,\\
&\bar{a}_{v_2u} := a_{vu} && \textit{for all } v_2u \in E' \textit{ with } vu \in (\delta(v)\cap E_a)\backslash F,\\
&\bar{a}_{v_1v_2} := -a(F), &&\\
&\bar{a}_{ij} := 0 && \textit{otherwise},\\
&\bar{\alpha} := \alpha; &&
\end{aligned}$$

then $\bar{a}^{\mathrm{T}}x \leqslant \bar{\alpha}$ defines a facet of $P_{\mathrm{C}}(G')$.

(b) (Contraction of an edge) *Let $G'=(V',E')$ be a graph and $\bar{a}^{\mathrm{T}}x \leqslant \bar{\alpha}$ be an inequality defining a facet of $P_{\mathrm{C}}(G')$. Suppose that $v_1v_2 \in E_{\bar{a}}$, that v_1 and v_2 have no common neighbor in $(V', E_{\bar{a}})$, that $\bar{a}_{v_1u} \geqslant 0$ for $v_1u \in \delta(v_1)\backslash\{v_1v_2\}$ and that $-\bar{a}_{v_1v_2} = \bar{a}(\delta(v_1)\backslash\{v_1v_2\}) \geqslant \bar{a}(\delta(v_2)\backslash\{v_1v_2\})$. Let $G=(V,E)$ be the graph obtained from G' by removing the nodes v_1, v_2 and adding a new node v and the edges $\{vu \mid \bar{a}_{v_1u}>0\} \cup \{vu \mid v_2u \in E'\}$. Set*

$$\begin{aligned}
&a_{uv} := \bar{a}_{uv} && \text{for all } uv \in E \cap E',\\
&a_{vu} := \bar{a}_{v_1u} && \text{if } \bar{a}_{v_1u} > 0,\\
&a_{vu} := \bar{a}_{v_2u} && \text{if } v_2u \in E' \text{ unless } \bar{a}_{v_1u} > 0,\\
&\alpha := \bar{\alpha}; &&
\end{aligned}$$

then $a^{\mathrm{T}}x \leqslant \alpha$ defines a facet of $P_{\mathrm{C}}(G)$.

Proof. The validity of the new inequalities defined in (a) and (b) follows by elementary construction. Then, let us assume that there is a facet-defining inequality $b^{\mathrm{T}}x \leqslant \beta$ for $P_{\mathrm{C}}(G)$ that has the following property. If a vector $x \in P_{\mathrm{C}}(G)$ satisfies $\bar{a}^{\mathrm{T}}x = \alpha$, then x also satisfies $b^{\mathrm{T}}x = \beta$. If we can prove that $\bar{a} = \rho b$ for some $\rho > 0$, then we can conclude that $\bar{a}^{\mathrm{T}}x \leqslant \alpha$ is equivalent to $b^{\mathrm{T}}x \leqslant \beta$, i.e., $a^{\mathrm{T}}x \leqslant \alpha$ defines a facet of $P_{\mathrm{C}}(G)$.

(a) First we will show that, for $u \notin V(E_{\bar{a}})$, $b_{uv_1} = 0$. To prove this apply lemma (2.5) with $S := (W\backslash\{v\}) \cup \{v_2\}$, $p := v_1$, $q := u$.

Since $a^{\mathrm{T}}x \leqslant \alpha$ defines a facet of $P_{\mathrm{C}}(G)$, there are $m = |E|$ cuts $\delta(U_1), \ldots, \delta(U_m) \subseteq E$ whose incidence vectors are affinely independent and satisfy $a^{\mathrm{T}}x^{\delta(U_i)} = \alpha$, $i = 1, \ldots, m$. Set

$$U_i' := \begin{cases} U_i, & \text{if } v \notin U_i,\\ (U_i\backslash\{v\}) \cup \{v_1, v_2\}, & \text{if } v \in U_i. \end{cases}$$

Since $\bar{a}^{\mathrm{T}}x^{\delta(U_i')} = \alpha$, then $b^{\mathrm{T}}x^{\delta(U_i')} = \beta$, $i = 1, \ldots, m$. The vectors $x^{\delta(U_i')}$, $i = 1, \ldots, m$ are affinely independent, thus we can conclude that $b_{uw} = \rho\bar{a}_{uw}$ for $uw \neq v_1v_2$. Let

W' be $(W\backslash\{v\}) \cup \{v_1, v_2\}$, and $W'' = W'\backslash\{v_1\}$. Since $\bar{a}^{\mathrm{T}} x^{\delta(W')} = \bar{a}^{\mathrm{T}} x^{\delta(W'')} = \alpha$, we have that

$$b^{\mathrm{T}} x^{\delta(W')} = b^{\mathrm{T}} x^{\delta(W'')} = \beta,$$

then

$$0 = b^{\mathrm{T}} x^{\delta(W')} - b^{\mathrm{T}} x^{\delta(W'')} = -b_{v_1 v_2} - b(F),$$

hence

$$b_{v_1 v_2} = -b(F) = -\rho \bar{a}(F) = \rho \bar{a}_{v_1 v_2}.$$

$b^{\mathrm{T}} x \leqslant \beta$ is valid, thus $\rho > 0$.

(b) By assumption $\bar{a}^{\mathrm{T}} x \leqslant \bar{\alpha}$ defines a facet of $P_{\mathrm{C}}(G)$, so there are $m := |E'|$ edge sets $\delta(U_1'), \ldots, \delta(U_m')$ whose incidence vectors are affinely independent and satisfy $\bar{a}^{\mathrm{T}} x^{\delta(U_i')} = \bar{\alpha}$, $i = 1, \ldots, m$. We may assume that $N \subseteq V'$ is the set of neighbors of v_1 in $(V', E_{\bar{a}})$ different from v_2, and that $k := |N|$.

Now let M be the (m, m)-matrix whose rows are the incidence vectors $x^{\delta(U_1')}, \ldots, x^{\delta(U_m')}$. We may assume that the last $k+1$ columns correspond to the edges $v_1 i$, $i \in N$ and $v_1 v_2$. Moreover, we assume that the set $\delta(U_1'), \ldots, \delta(U_m')$ are ordered in such a way that only the sets $\delta(U_1'), \ldots, \delta(U_r')$ contain the edge $v_1 v_2$, and that $v_2 \in U_i'$, $i = 1, \ldots, m$.

Note that the assumptions of the theorem imply that if U_i' does not contain v_1 then it necessarily contains all nodes $i \in N$; otherwise, $U'' = U_i' \cup \{v_1\}$ would define a cut $\delta(U'')$ such that $\bar{a}^{\mathrm{T}} x^{\delta(U'')} > \alpha$. Thus, our matrix M looks as follows:

$$
\begin{array}{|c|c|c|c|l}
\multicolumn{1}{c}{\overbrace{\qquad\quad}^{m-t-k}} & \multicolumn{1}{c}{\overbrace{\qquad}^{t-1}} & \multicolumn{1}{c}{\overbrace{\qquad}^{k}} & \multicolumn{1}{c}{\overbrace{\quad}^{1}} & \\
\hline
M_1 & M_2 & M_3 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} & \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} r \\
\hline
M_4 & M_5 & M_6 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} m-r \\
\hline
\end{array}
$$

where M_3 contains only ones and columns s, $m-k-t < s < m-k$, correspond to edges in $\delta(v_1)\backslash E_{\bar{a}}'$ and $v_2 u \in E'$ for which $\bar{a}_{v,u} > 0$. Now we transform the sets $U_i' \subseteq V'$ into sets $U_i \subseteq V$, $i = 1, \ldots, m$ as follows.

$$U_i = (U_i'\backslash\{v_1, v_2\}) \cup \{v\}.$$

This transformation corresponds to contracting the edge $v_1 v_2$. It follows from our remarks above that $a^{\mathrm{T}} x^{\delta(U_i)} = \alpha$ for $i = 1, \ldots, m$. If A is the $(m, m-1)$-matrix whose rows are the incidence vectors $x^{\delta(U_i)}$ $i=1,\ldots,m$, then A looks as follows:

A_1	A_2
A_3	A_4

where

$$A_1 = M_1, \qquad A_3 = M_4, \qquad A_4 = M_6,$$

and A_2 contains zeros only.

To obtain A we can perform the following operations on M. Subtract the last column from the columns $m-k,\ldots,m-1$. Then delete the last column and columns s such that $m-k-t<s<m-k$. It is clear that the rows of this matrix have affine rank $m-t$, and our proof is complete. □

Theorem 2.7 (Replacing a node by a triangle). *Let $G=(V,E)$ be a graph and $a^{\mathrm{T}}x \leqslant \alpha$ be a facet-defining inequality for $P_{\mathrm{C}}(G)$. Let v be a node in $V(E_a)$ such that $a_e \geqslant 0$ for each $e \in \delta(v)$. Let F_1, F_2, F_3 be a partition of $\delta(v)$ and assume that there exist W_1, W_2, $W_3 \subseteq V$ such that $a^{\mathrm{T}}x^{\delta(W_i)} = \alpha$ and $F_i \subseteq E(W_i)$ for $i=1, 2, 3$. (W_i may coincide with W_j for $i \neq j$.) Construct a new graph $G'=(V',E')$ from G as follows. Replace v by v_1, v_2, v_3 such that v_i is incident to all edges contained in F_i, $i=1, 2, 3$. Add edges v_1v_2, v_1v_3 and v_2v_3, in addition any further edge v_iu, $i=1, 2, 3$, with $u \notin V(E_a)$ may be added. The other parts of G remain unchanged. Set*

$$\bar{a}_{ij} := a_{ij} \qquad \text{for all } ij \in E_a \setminus \delta(v),$$

$$\bar{a}_{v_iu} := a_{vu} \qquad \text{for all } v_iu \in E' \text{ with } v_iu \in F_i,\ i=1,2,3,$$

$$\bar{a}_{v_1v_2} := \frac{-a(F_1)-a(F_2)+a(F_3)}{2},$$

$$\bar{a}_{v_1v_3} := \frac{-a(F_1)-a(F_3)+a(F_2)}{2},$$

$$\bar{a}_{v_2v_3} := \frac{-a(F_2)-a(F_3)+a(F_1)}{2},$$

$$\bar{a}_{ij} := 0, \qquad \text{otherwise},$$

$$\bar{\alpha} := \alpha;$$

then $\bar{a}^{\mathrm{T}}x \leqslant \bar{\alpha}$ defines a facet of $P_{\mathrm{C}}(G')$. □

The proof of this theorem is analogous to the proof of Theorem 2.6(a). We leave it to the reader.

To illustrate these constructions we will give some examples of facet-defining inequalities of $P_{\mathrm{C}}(G)$ that are not valid for $P_{\mathrm{B}}(G)$.

Let $G=(V,E)$ be the complete graph K_7; then $X(E)\leqslant 12$ defines a facet of $P_C(G)$, by Theorem 2.1. Choosing $F=\{71,72,73\}$ we split the node 7 into v_1 and v_2 and we obtain a graph $G'=(V',E')$ (cf. Figure 2.1).

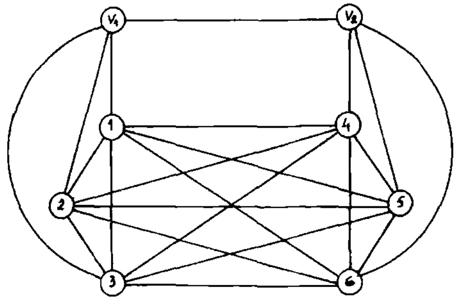

Fig. 2.1.

The inequality

$$x(E'\backslash\{v_1v_2\})-3x(v_1v_2)\leqslant 12$$

defines a facet of $P_C(G)$.

By splitting some nodes of a bicycle 5-wheel as it is shown in Figure 2.2, we obtain the graph of Figure 2.3.

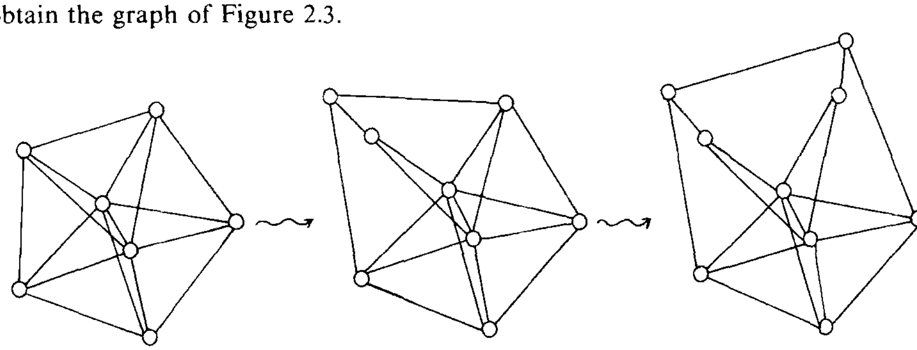

Fig. 2.2.

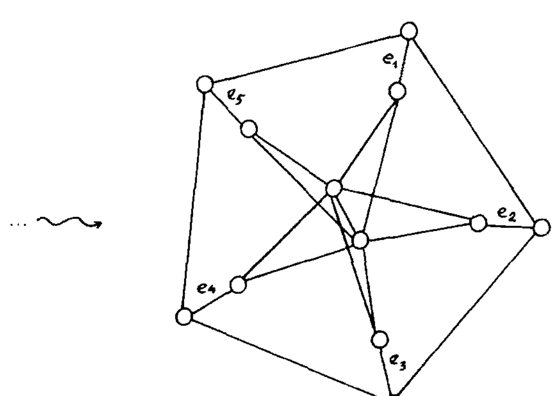

Fig. 2.3.

Let $G=(V, E)$ be this graph and $E'=\{e_1, \ldots, e_5\}$; then the inequality

$$x(E\backslash E')-2x(E')\leqslant 10$$

defines a facet of $P_C(G)$.

Let $G=(V, E)$ be the complete graph K_7. Replacing the node 7 by a triangle we can obtain the graph $G'=(V', E')$ in Figure 2.4.

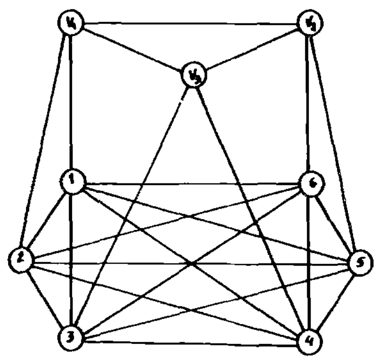

Fig. 2.4.

Since $x(E)\leqslant 12$ defines a facet of $P_C(G)$, by Theorem 2.7, the inequality

$$x(E'\backslash\{v_1v_2, v_2v_3, v_1v_3\})-x(\{v_1v_2, v_2v_3, v_1v_3\})\leqslant 12$$

defines a facet of $P_C(G')$.

The third method for obtaining "facets from facets" is the following.

Theorem 2.8 (Changing the signs of a star). *Let $G=(V, E)$ be a graph and $a^Tx\leqslant\alpha$ be a facet-defining inequality for $P_C(G)$. For any $v\in V$ the inequality*

$$-\sum_{e\in\delta(v)} a_e x(e)+\sum_{e\notin\delta(v)} a_e x(e)\leqslant\alpha-a(\delta(v)) \tag{2.8}$$

defines a facet for $P_C(G)$.

Proof. Let us denote inequality (2.8) by $\bar{a}^Tx\leqslant\bar{\alpha}$. This is valid for $P_C(G)$; otherwise, there exists $U\subseteq V$ with $v\in U$ such that

$$\bar{a}^Tx^{\delta(U)}>\bar{\alpha}.$$

But this implies that, for $U'=U\backslash\{v\}$,

$$a^Tx^{\delta(U')}>\alpha.$$

Since $a^Tx\leqslant\alpha$ defines a facet, there are $m=|E|$ sets U_i, $i=1,\ldots,m$, such that $a^Tx^{\delta(U_i)}=\alpha$, $v\in U_i$, $i=1,\ldots,m$, and the vectors $x^{\delta(U_1)},\ldots,x^{\delta(U_m)}$ are affinely independent. Set $U_i' = U_i \setminus \{v\}$; then

$$x^{\delta(U_i')}(e) = \begin{cases} x^{\delta(U_i)}(e), & \text{if } e \notin \delta(v), \\ 1 - x^{\delta(U_i)}(e), & \text{if } e \in \delta(v). \end{cases}$$

Thus the vectors $x^{\delta(U_1')}, \ldots, x^{\delta(U_m')}$ are affinely independent and satisfy $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$ with equality. $\square$

Theorems 2.6 and 2.8 can be combined to give new procedures of construction of facet-defining inequalities. The following corollaries illustrate this fact.

Corollary 2.9. *Let $G = (V, E)$ be a graph and $a^{\mathrm{T}}x \leq \alpha$ be a facet-defining inequality for $P_{\mathrm{C}}(G)$. Let $W \subseteq V$, and set*

$$\bar{a}_{ij} := a_{ij} \qquad \text{for } ij \in E \setminus \delta(W),$$

$$\bar{a}_{ij} := -a_{ij} \qquad \text{for } ij \in \delta(W),$$

$$\bar{\alpha} = \alpha - a(\delta(W));$$

then $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$ defines a facet of $P_{\mathrm{C}}(G)$.

Proof. Using Theorem 2.8 we change the signs of the coefficients associated with $\delta(v)$ for each $v \in W$. $\square$

Corollary 2.10. (a) (Subdivision of an edge) *Let $G = (V, E)$ be a graph and $a^{\mathrm{T}}x \leq \alpha$ be a facet-defining inequality for $P_{\mathrm{C}}(G)$. Let $ij \in E$ be an edge with $a_{ij} \neq 0$. Let $G' = (V', E')$ be a graph obtained from G in the following way. Nodes $i_1, \ldots, i_k$ are added. The edge set $P = ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj$ is added. Any further edge i_lu with $u \notin V(E_a)$ may be added. The edge ij is removed. Let P^+, P^- define a partition of P with $|P^+|$ odd if $a_{ij} > 0$, and $|P^-|$ even if $a_{ij} < 0$. Let $\bar{a} \in \mathbb{R}^{E'}$ be defined as follows*

$$\bar{a}_{uv} = a_{uv} \qquad \text{for all } uv \in E \cap E',$$

$$\bar{a}_{uv} = a_{ij} \qquad \text{for all } uv \in P^+,$$

$$\bar{a}_{uv} = -a_{ij} \qquad \text{for all } uv \in P^-,$$

$$\bar{a}_{uv} = 0 \qquad \text{otherwise.}$$

Then

$$\bar{a}^{\mathrm{T}}x \leq \alpha + (|P^+| - 1)a_{ij} \quad \text{if } a_{ij} > 0,$$

$$\bar{a}^{\mathrm{T}}x \leq \alpha - |P^-|a_{ij} \qquad \text{if } a_{ij} < 0,$$

defines a facet of $P_{\mathrm{C}}(G')$.

(b) (Replacing a path by an edge) *Let $G' = (V', E')$ be a graph and $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$ be a facet-defining inequality for $P_{\mathrm{C}}(G')$. Let $E_{\bar{a}}$ be the support of $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$. Suppose that $E_{\bar{a}}$ contains a path $P = \{ii_1, i_2i_3, \ldots, i_kj\}$ such that*

(i) *i_l has degree two in $E_{\bar{a}}$, for $l = 1, \ldots, k$.*

(ii) *$ij \notin E_{\bar{a}}$.*

(iii) *$\bar{a}_{uv} = \alpha'$ for $uv \in P^+ \subseteq P$, $\bar{a}_{uv} = -\alpha'$ for $uv \in P^- = P \setminus P^+$, with $|P^+|$ odd if $\alpha' > 0$ and $|P^-|$ even if $\alpha' < 0$.*

Let $G=(V, E)$ be the graph obtained from G' by removing the nodes $i_1, \ldots, i_k$ and adding the edge ij. Let $a \in \mathbb{R}^E$ be defined as follows:

$$a_{uv} = \bar{a}_{uv} \quad \textit{for all } uv \in E \cap E',$$

$$a_{ij} = \alpha'.$$

Then

$$a^{\mathrm{T}}x \leqslant \alpha - (|P^+|-1)\alpha' \quad \textit{if } \alpha' > 0,$$

$$a^{\mathrm{T}}x \leqslant \alpha + |P^-|\alpha' \qquad \textit{if } \alpha' < 0,$$

defines a facet of $P_C(G)$. □

Proof. (a) If $a_{ij} > 0$ we apply Theorem 2.6(a) and Theorem 2.8 repeatedly. If $a_{ij} < 0$ we first change the signs of $\delta(i)$ using Theorem 2.8, then we apply Theorem 2.6(a) and Theorem 2.8, and finally we change the signs of $\delta(i)$ again.

(b) If $\alpha' > 0$ we apply Theorem 2.6(b) and Theorem 2.8 repeatedly. If $\alpha' < 0$ we first change the signs of $\delta(i)$ by Theorem 2.8, then we apply Theorem 2.6(b) and Theorem 2.8, and we change the signs of $\delta(i)$ again. □

Let $I\Delta J$ denote $(I\backslash J) \cup (J\backslash I)$, the symmetric difference of I and J. Note that if I and J are cuts then $I\Delta J$ is a cut.

Corollary 2.11. *For any pair of cuts, C and D, there is a one-to-one correspondence between the facets adjacent to x^C and to x^D.*

Proof. Let $ax \leqslant \alpha$ be a facet-defining inequality such that $ax^C = \alpha$. If we apply Corollary 2.9 with $\delta(W) = C\Delta D$, we obtain a facet-defining inequality $bx \leqslant \beta$, such that $bx^D = \beta$. □

Let us define

$$\mathrm{CONE}(P_C(G)) = \{y \mid y = \lambda x, x \in P_C(G), \lambda \in \mathbb{R}_+\}.$$

If $P_C(G) = \{x \mid Ax \geqslant b, Ex \geqslant 0\}$, $b < 0$, then $\mathrm{CONE}(P_C(G)) = \{x \mid Ex \geqslant 0\}$.

Corollary 2.11 shows that a set of inequalities defining $P_C(G)$ can be obtained from a set of inequalities defining the facets adjacent to one extreme point of $P_C(G)$. Hence, getting a characterization of $\mathrm{CONE}(P_C(G))$ is as hard as getting a characterization of $P_C(G)$.

3. Facets associated with edges and cycles

Given a graph $G=(V, E)$, an incidence vector x must verify the inequalities

$$0 \leqslant x(e) \leqslant 1 \quad \text{for } e \in E. \tag{3.1}$$

Moreover, if x is an incidence vector of a cut, then for each cycle C, $x(C)$ is an even number. These conditions imply the inequalities

$$x(F)-x(C\backslash F)\leq |F|-1 \quad \text{for each cycle } C, \qquad F\subseteq C,\ |F| \text{ odd.} \tag{3.2}$$

In what follows we shall study when the above inequalities define facets of $P_C(G)$.

Theorem 3.3. *Inequality* (3.2) *defines a facet of $P_C(G)$ if and only if C is a chordless cycle.*

Proof. If C is a chordless cycle, inequality (3.2) can be obtained from an inequality associated with a triangle (Theorem 2.1), by applying Theorems 2.6 and 2.8 repeatedly as in Corollary 2.10.

If $C=\{v_1v_2, v_2v_3, \ldots, v_kv_1\}$ has a chord, say v_1v_l, it is easy to see that an inequality (3.2) associated with C can be obtained by summing inequalities (3.2) associated to $C_1=\{v_1v_2, v_2v_3, \ldots, v_{l-1}v_l, v_lv_1\}$ and $C_2=\{v_1v_l, \ldots, v_{k-1}v_k, v_kv_1\}$. This completes our proof. □

Theorem 3.4. *Inequality* (3.1) *defines a facet if and only if e does not belong to a triangle.*

Proof. Let us suppose that e does not belong to a triangle and let us study the inequality

$$x(e)\geq 0. \tag{3.5}$$

Let us assume that there is a facet-defining inequality $b^{\mathrm{T}}x\geq\beta$ such that

$$S=\{x\in P_C(G)\,|\,x(e)=0\}\subseteq\{x\in P_C(G)\,|\,b^{\mathrm{T}}x=\beta\}.$$

S is the cut polytope of the graph G' obtained by contracting e. Since e does not belong to a triangle, then G' does not contain multiple edges and thus $P_C(G)$ is full dimensional. We can conclude that $b_f=0$ for $f\in E\backslash\{e\}$. The inequality $b^{\mathrm{T}}x\geq\beta$ is valid; hence $b_e>0$. Now we can conclude that

$$x(e)\leq 1$$

defines a facet by applying Theorem 2.8 to inequality (3.5). This completes the first part of the proof.

Let us suppose that e belongs to a triangle, say $\{e, f, g\}$. By Theorem 3.3 we have the following facet-defining inequalities.

$$x(e)+x(f)+x(g)\leq 2, \tag{3.6}$$

$$x(e)-x(f)-x(g)\leq 0, \tag{3.7}$$

$$-x(e)+x(f)-x(g)\leq 0, \tag{3.8}$$

$$-x(e)-x(f)+x(g)\leq 0. \tag{3.9}$$

Summing (3.6) and (3.7) we obtain

$$2x(e)\leq 2,$$

and (3.8) plus (3.9) gives

$$2x(e) \geq 0.$$

Hence, in this case (3.1) does not define a facet. □

Grötschel, Lovász and Schrijver [5] have shown that there exists a polynomially bounded algorithm for a linear optimization problem over a polyhedron if and only if there is a polynomially bounded algorithm for the associated separation problem: given a point x, either verify that it belongs to the polyhedron or else find a hyperplane that separates it from the polyhedron.

The knowledge of an efficient method to solve the separation problem gives an answer to some theoretical and practical questions. It proves that a problem is polynomially solvable, and it permits the design of linear programming based cutting plane algorithms.

In what follows we shall give a polynomial algorithm to solve the separation problem for inequalities (3.2). Let us write these inequalities as

$$\sum_{e \in C \setminus F} x(e) + \sum_{e \in F} (1 - x(e)) \geq 1 \quad \text{for a cycle } C,\ F \subseteq C,\ |F| \text{ odd.}$$

Given x, we are looking for a minimum weighted cycle, where some edges have the weight $x(\cdot)$, and an odd number of edges have weight $1 - x(\cdot)$.

From the given graph G we form a new graph G' with two nodes i' and i'', for every node i of G. For every edge ij of G we put edges $i'j'$ and $i''j''$ with weight $x(ij)$, and edges $i'j''$ and $i''j'$ with weight $1 - x(ij)$. Now, for node i of G we find a shortest path from i' to i''. The minimum over the nodes of the lengths of the corresponding shortest path is the weight of the required cycle. As the computation of a shortest path takes $O(n^2)$ calculations, this procedure has a time complexity of $O(n^3)$.

The existence of a good algorithm for the separation problem associated with inequalities (3.1) and (3.2) leads us to ask which are the graphs such that these inequalities suffice to define $P_C(G)$. In what follows we shall see that those are the graphs not contractible to K_5.

Corollary 2.11 implies that $P_C(G)$ is defined by the inequalities associated to cycles and to edges if and only if $\mathrm{CONE}(P_C(G))$ is defined by:

$$x(f) \leq x(C \setminus \{f\}), \quad C \text{ a cycle, } f \in C,$$

$$x(e) \geq 0, \quad e \in E.$$

This is called the "sum of circuits" property by Seymour [9]. Actually, he proved that the "sum of circuits" property holds if and only if G is not contractible to K_5.

We can state the following:

Corollary 3.10. *A graph G is not contractible to K_5 if and only if* $P_C(G)$ is defined by:

$$0 \leq x(e) \leq 1, \quad \textit{for each edge } e \textit{ that does not belong to a triangle,}$$

$$x(F) - x(C \setminus F) \leq |F| - 1, \quad \textit{for each chordless cycle } C,\ F \subseteq C,\ |F| \textit{ odd.}$$

4. Adjacency in $P_C(G)$

In this section we will give a simple characterization of adjacency in $P_C(G)$, and we will derive a bound for the diameter of $P_C(G)$.

Theorem 4.1. *Let $G=(V,E)$, be a graph. Let x^I, x^J be extreme points of $P_C(G)$; let I, J be the corresponding cuts. Let $F=E\backslash(I\Delta J)$. Then x^I and x^J are adjacent in $P_C(G)$ if and only if $H=(V,F)$ has two connected components.*

Proof. We shall use the fact that x^I and x^J are adjacent in $P_C(G)$ if and only if there is a vector $c=(c_e: e\in E)$ such that x^I and x^J are the only two extreme points that maximize cx over $P_C(G)$.

(i) Let us suppose that $G_i=(V_i,E_i)$, $i=1,2$ are the connected components of H. Let T_i be a spanning tree of G_i, $i=1,2$. Set

$$c_e=\begin{cases} 1 & \text{if } e\in T_i\cap(I\cap J),\ i=1,2,\\ -1 & \text{if } e\in T_i\backslash(I\cap J),\ i=1,2,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$cx\leqslant|T_1\cap(I\cap J)|+|T_2\cap(I\cap J)|$$

for all $x\in P_C(G)$. The equality holds for an extreme point x if and only if $x=x^I$ or $x=x^J$.

(ii) Let us suppose that $G_i=(V_i,E_i)$, $i=1,\ldots,k$, $k\geqslant 3$, are the connected components of H. Assume that there is a vector c with the desired properties. Setting

$$a(e)=\begin{cases} 1-x(e)^I, & \text{if } e\in\delta(V_1),\\ x(e)^I, & \text{otherwise,}\end{cases}$$

$$b(e)=\begin{cases} 1-x(e)^J, & \text{if } e\in\delta(V_1),\\ x(e)^J, & \text{otherwise,}\end{cases}$$

a and b belong to $P_C(G)$, and $a+b=x^I+x^J$, so $cx^I+cx^J=ca+cb$. Consequently, $\max\{ca,cb\}\geqslant cx^I=cx^J$, which is a contradiction as both a and b are extreme points of $P_C(G)$. This completes the proof. □

The graph of a polyhedron is the graph whose nodes correspond to the extreme points of this polyhedron and that has an edge joining each pair of nodes for which the corresponding extreme points are adjacent.

Given two cuts I and J, let i and j be the corresponding nodes of the graph of $P_C(G)$. The distance from I to J, $d(I,J)$ is the number of edges in the shortest path from i to j. The diameter of $P_C(G)$ is

$$\max\{d(I, J)\colon I, J \text{ are cuts of } G\}.$$

Theorem 4.1 implies the following:

Corollary 4.2. *If G is a complete graph, then $P_C(G)$ has diameter one.* □

If G is connected and C is a cut, let us define $T(C)$ as the graph obtained by contracting edges not in C and replacing multiple edges by single edges. Let $m(C)$ be the number of edges of $T(C)$.

Theorem 4.3. *If I and J are cuts of G, then*

$$d(I, J) \leqslant m(I \Delta J).$$

Proof. Let P be a minimal cut included in $I\Delta J$.
Set $L = I\Delta P$. L is a cut and x^L is adjacent to x^I. Furthermore, $m(L\Delta J) < m(I\Delta J)$, and the theorem follows by induction on $m(I\Delta J)$. □

The bound in Theorem 4.3 may be realized. For instance, if $G = (V, E)$ is the graph $K_{1,p}$, $I = \emptyset$ and $J = E$, then $d(I, J) = p$.

Corollary 4.4. *The diameter of $P_C(G)$ is at most*

$$\max\{m(C)\colon C \text{ is a cut}\}. \quad \square$$

Again, the graph $K_{1,p}$ shows that the bound of the corollary may be realized.
A d-dimensional polyhedron P with k facets has the Hirsch property if the diameter of P is at most $k - d$.

Theorem 4.5. *$P_C(G)$ has the Hirsch property.*

Proof. $P_C(G)$ has dimension $|E|$. Let us partition E into E_1 and E_2, where E_1 contains the edges that belong to a triangle. Let $T_1, \ldots, T_r$ be the triangles of G. By Theorems 3.3 and 3.4, $P_C(G)$ has at least $4r + 2|E_2|$ facets. Since $r \geqslant |E_1|/3$, $4r + 2|E_2| - |E| \geqslant |E_1|/3 + |E_2|$.
On the other hand, if C is a cut

$$m(C) \leqslant |E_1|/3 + |E_2|,$$

and our proof is complete. □

5. Signed graphs

A signed graph is a graph $G=(V, E)$ where E is partitioned into E_- and E_+. Elements of E_- (E_+) are called negative (positive). For instance, if this structure represents a social group, we can think of positive and negative relationships between the members of the group.

A signed graph is called balanced if there exists $U \subseteq V$ such that $E_- = \delta(U)$. A graph is balanced if and only if each of its cycles includes an even number of negative edges (cf. Harary [8]).

A balancing set is an edge set $S \subseteq E$ such that when the signs of the elements of S are changed the resulting graph is balanced.

It is easy to see that S is a balancing set if and only if there exists $U \subseteq V$, such that $S \cap E_+ = S \cap \delta(U) = E_+ \cap \delta(U)$ and $S \cap E_- = S \cap (E \backslash \delta(U)) = E_- \cap (E \backslash \delta(U))$. Then y is the incidence vector of a balancing set if and only if there exists an incidence vector of a cut x, such that

$$y(e) = \begin{cases} x(e) & \text{if } e \in E_+, \\ 1 - x(e) & \text{if } e \in E_-. \end{cases} \tag{5.1}$$

Thus, a minimum weighted balancing set can be found by solving a minimum (maximum) cut problem.

Let us call $P_{\mathrm{BS}}(G)$ the convex hull of incidence vectors of balancing sets of a signed graph G. Facet-defining inequalities of $P_{\mathrm{BS}}(G)$ can be obtained from facet-defining inequalities of $P_{\mathrm{C}}(G)$ by using relation (5.1). In particular, Theorem 3.5 can be stated as

Remark 5.2. A signed graph G is not contractible to K_5 if and only if $P_{\mathrm{BS}}(G)$ is defined by

$$x(C\backslash F) - x(F) \geq 1 - |F|, \quad \text{for each chordless cycle } C,\ F \subseteq C,\ |C \cap E_-| + |F| \text{ odd},$$

$$0 \leq x(e) \leq 1, \quad \text{for each edge } e \text{ that does not belong to a triangle.}$$

Again relation (5.1) enables us to translate adjacency results in $P_{\mathrm{C}}(G)$ into adjacency results in $P_{\mathrm{BS}}(G)$. For instance, Corollary 4.2 gives us

Remark 5.3. If G is a complete signed graph, then $P_{\mathrm{BS}}(G)$ has diameter one.

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