Compositions of Graphs and Polyhedra I: Balanced Induced Subgraphs and Acyclic Subgraphs

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COMPOSITIONS OF GRAPHS AND POLYHEDRA I: BALANCED INDUCED SUBGRAPHS AND ACYCLIC SUBGRAPHS*

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Abstract. Let $P(G)$ be the balanced induced subgraph polytope of G. If G has a two-node cutset, then G decomposes into G_1 and G_2. It is shown that $P(G)$ can be obtained as a projection of a polytope defined by a system of inequalities that decomposes into two pieces associated with G_1 and G_2. The problem max cx, $x \in P(G)$ is decomposed in the same way. This is applied to series-parallel graphs to show that, in this case, $P(G)$ is a projection of a polytope defined by a system with $O(n)$ inequalities and $O(n)$ variables, where n is the number of nodes in G. Also for this class of graphs, an algorithm is given that finds a maximum weighted balanced induced subgraph in $O(n \log n)$ time. This approach is also used to obtain composition of facets of $P(G)$. Analogous results are presented for acyclic induced subgraphs.

Key words. polyhedral combinatorics, composition of polyhedra, balanced subgraphs, acyclic subgraphs, compact systems

AMS subject classifications. 05C85, 90C27

1. Introduction. Given a graph G, let $P(G)$ be a polytope associated with G. If G has a one- or two-node cutset, then G decomposes into G_1 and G_2. We study a technique to derive $P(G)$, provided that we know two polytopes related to G_1 and G_2. We use the same ideas to decompose the problem

$$\text{Maximize } cx,$$
$$x \in P(G)$$

into two optimization problems related to G_1 and G_2. Similar compositions of polyhedra have been studied in [8], [3], [10], [5], and [9]. First, we study the polytopes of balanced induced subgraphs and acyclic induced subgraphs. In a subsequent paper, we apply a simplification of this technique to the stable set polytope.

The graphs we consider are finite, undirected, and may have multiple edges. We denote a graph by $G = (V, E)$, where V is the *node set* and E is the *edge set* of G. If $W \subseteq V$, then $E(W)$ denotes the set of all edges of G with both endnodes in W, and the graph $H = (W, E(W))$ is the subgraph of G induced by W.

A signed graph $G = (V, E)$ is a graph whose edges are labeled positive or negative [11]. A positive (negative) edge $\{u, v\}$ is denoted by $\{u, v, +\}$, $(\{u, v, -\})$. A signed graph is said to be *balanced* if the set of negative edges form a cut, that is, if the node set V can be partitioned into U and $\bar{U}$ in such a way that $E(U) \cup E(\bar{U})$ is the set of positive edges. Also, a signed graph is balanced if it does not contain a cycle with an odd number of negative edges. Suppose that from a node i we send a signal $s_i \in \{-1, 1\}$ to all the adjacent nodes; if the edge $\{i, j\}$ is positive, then j receives the signal s_i; if $\{i, j\}$ is negative, then j receives $-s_i$. A signed graph is balanced if and only if, when we send a signal from a node, this node receives the same signal in return.

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If $W \subseteq V$ and $H = (W, E(W))$ is balanced, then H is called a *balanced induced subgraph* (BIS) of G. If $W \subseteq V$, let $x^W \in \Re^V$, where $x^W(u) = 1$ if $u \in W$ and where $x^W(u) = 0$ if $u \notin W$; x^W is called the *incidence vector* of W.

The convex hull of incidence vectors of all balanced induced subgraphs of G is denoted by $P(G)$ and called the BIS *polytope* of G, i.e.,

$$P(G) = \text{conv}\ \{x^W \in \Re^V \mid (W, E(W)) \text{ is a BIS of } G\}.$$

Given a signed graph $G = (V, E)$ with node weights $c(v)$ for all $v \in V$, the *maximum* BIS *problem* is to find a BIS $(W, E(W))$ such that $c(W) = \sum \{c(v) : v \in W\}$ is as large as possible.

Every optimum basic solution of the linear program

$$\max cx,$$
$$x \in P(G)$$

is the incidence vector of a maximum weighted BIS of G.

The edge problem of finding a maximum balanced spanning subgraph can be reduced to a max-cut problem [4]. This problem is polynomially solvable for graphs not contractible to K_5 [3] and for toroidal graphs [2].

If $H = (V, F)$ is a graph, then the maximum stable set problem in H can be reduced to a maximum BIS problem in a signed graph $G = (V, E)$ that is obtained by replacing each edge in F by a positive edge and a negative edge. Thus the maximum BIS problem can be viewed as a generalization of the maximum stable set problem. This shows that the maximum BIS problem is NP-hard even for signed planar graphs. When all the edges are negative, a BIS coincides with a bipartite induced subgraph.

The polytope $P(G)$ is full-dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system $Ax \leq b$ such that $P(G) = \{x \mid Ax \leq b\}$; moreover, there is a natural bijection among the facets of $P(G)$ and the inequalities of that system.

In §2 we show that, if G admits a two-vertex decomposition into G_1 and G_2, then $P(G)$ is a projection of a polytope defined by a system of inequalities that decomposes into pieces associated with G_1 and G_2. In §3 the optimization problem is decomposed in a similar way. In §4 we apply this technique to series-parallel graphs and we show that, in this case, $P(G)$ is a projection of a polytope defined by a system with $O(n)$ inequalities and $O(n)$ variables, where n is the number of nodes in G. Also for this class of graphs, we give an algorithm that finds a maximum BIS in $O(n \log n)$ time. In §5 we use the same approach for finding compositions of facets of $P(G)$. Analogous results about acyclic induced subgraphs are mentioned in §6.

2. Compositions of graphs. In this section, we derive a system of inequalities that defines a polytope having $P(G)$ as a projection, provided that G is a composition of two graphs and such a system is known for each piece.

The next theorem is a generalization of a result of Chvátal [7] about the stable set polytope.

THEOREM 2.1. *Let $G = (V, E)$ be a signed graph such that there exists node sets V_1 and V_2 with the following properties*:

(i) $V = V_1 \cup V_2$,

(ii) $W = V_1 \cap V_2 \neq \emptyset$,

(iii) *Between each pair $\{i, j\} \subseteq W$, there exists a positive edge and a negative edge in E*,

(iv) *The induced subgraph $(V \setminus W, E(V \setminus W))$ is disconnected.*

If $G_1 = (V_1, E(V_1))$, $G_2 = (V_2, E(V_2))$, then a system of inequalities that defines $P(G)$ is obtained by the juxtaposition of such systems defining $P(G_1)$ and $P(G_2)$.

Proof. Let $Ax \le b$ be the system obtained by juxtaposing both systems. Let $\bar{x}$ be a point in the polyhedron defined by $Ax \le b$. Let $\bar{x}_1$ ($\bar{x}_2$) be the restriction of $\bar{x}$ to the set of indices associated with nodes in V_1 (V_2). Since $\bar{x}_1 \in P(G_1)$, we can write

$$\bar{x}_1 = \sum_{i=1}^{p} \lambda_i y_i,$$

$$\lambda_i \ge 0 \quad \text{for } 1 \le i \le p, \ \sum_{i=1}^{p} \lambda_i = 1,$$

where y_i is an extreme point of $P(G_1)$ for $i = 1, \ldots, p$. We can also write

$$\bar{x}_2 = \sum_{i=1}^{k} \alpha_i z_i,$$

$$\alpha_i \ge 0 \quad \text{for } 1 \le i \le k, \ \sum_{i=1}^{k} \alpha_i = 1,$$

where z_i is an extreme point of $P(G_2)$ for $i = 1, \ldots, k$.

Let $W = \{w_1, \ldots, w_l\}$. Then

$$\bar{x}(w_j) = \sum_i \{\lambda_i \mid y_i(w_j) = 1\} = \sum_i \{\alpha_i \mid z_i(w_j) = 1\}, \qquad 1 \le j \le l.$$

We can match a vector y_i with $y_i(w_j) = 1$ with a vector z_r with $z_r(w_j) = 1$ for $1 \le j \le l$ and we can match a vector y_i with $y_i(w_j) = 0$ for $1 \le j \le l$ with a vector z_r with $z_r(w_j) = 0$ for $1 \le j \le l$. We obtain an incidence vector of a balanced induced subgraph of G. Thus $\bar{x}$ can be written as a convex combination of incidence vectors of balanced induced subgraphs of G. □

Now we study graphs with a two-vertex cutset. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \{u, v\}$ and let $G = (V, E)$ be the union of G_1 and G_2, i.e., $V = V_1 \cup V_2$, $E = E_1 \cup E_2$. We cover four cases.

Case 1. There is a positive edge and a negative edge between u and v in G.

Case 2. There is only a positive edge between u and v in G.

Case 3. There is only a negative edge between u and v in G.

Case 4. Nodes u and v are not adjacent in G.

Case 1 is covered by Theorem 2.1; thus we restrict ourselves to the three other cases. In Case 2, we define $\bar{G}_i = (\bar{V}_i, \bar{E}_i)$, $i = 1, 2$ as follows; see Fig. 1:

$$\bar{V}_i = V_i \cup \{w_1, w_2, w_3, w_4\},$$

$$\bar{E}_i = E_i \cup \{\{w_1, u, -\}, \{w_1, v, +\}, \{w_2, u, -\}, \{w_3, u, -\}, \{w_3, v, -\}, \{w_4, v, -\}, \{w_3, w_4, -\}, \{w_3, w_2, -\}\}.$$

In Case 3, $\bar{G}_i = (\bar{V}_i, \bar{E}_i)$, $i = 1, 2$ are defined as in Case 2, but $\{w_1, v\}$ is labeled negative. In Case 4, $\bar{G}_i = (\bar{V}_i, \bar{E}_i)$, $i = 1, 2$ are defined as follows:

$$\bar{V}_i = V_i \cup \{w_1, w_2, w_3, w_4, w_5\},$$

$$\bar{E}_i = E_i \cup \{\{w_1, u, -\}, \{w_2, u, -\}, \{w_3, u, -\}, \{w_4, u, -\}, \{w_1, v, -\}, \{w_2, v, +\}, \{v, w_4, -\}, \{v, w_5, -\}, \{w_3, w_4, -\}, \{w_4, w_5, -\}\}.$$

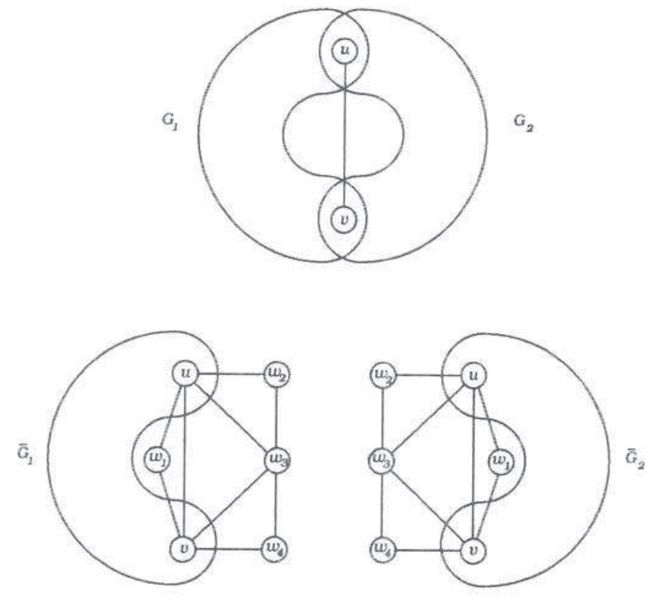

FIG. 1

Our aim is to derive a system of inequalities for $P(G)$ from systems defining $P(\bar{G}_1)$ and $P(\bar{G}_2)$. Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph obtained as union of $\bar{G}_1$ and $\bar{G}_2$, i.e., $\bar{V} = \bar{V}_1 \cup \bar{V}_2$, $\bar{E} = \bar{E}_1 \cup \bar{E}_2$.

In Cases 2 and 3, the inequality

$$\sum_{i=1}^{4} x(w_i) + x(u) + x(v) \leq 4 \tag{2.1}$$

defines a facet $F(\bar{G}_i)$ of $P(\bar{G}_i)$, $i = 1, 2$ and a facet $F(\bar{G})$ of $P(\bar{G})$. In Case 4, the inequality

$$\sum_{i=1}^{5} x(w_i) + x(u) + x(v) \leq 5 \tag{2.2}$$

plays the same role.

The polytope $P(G)$ is the projection of $F(\bar{G})$ along the variables $\{x(w_i)\}$. The next theorem gives us a system defining $F(\bar{G})$.

THEOREM 2.2. *The juxtaposition of a system that defines* $F(\bar{G}_1)$ *and a system that defines* $F(\bar{G}_2)$ *gives a system that defines* $F(\bar{G})$.

Proof. Let $Ax \leq b$ be such a system and let $\bar{x}$ be a vector that satisfies it. Let $\bar{x}_1$ ($\bar{x}_2$) be the restriction of $\bar{x}$ to the set of indices associated with nodes in $\bar{V}_1$ ($\bar{V}_2$). Since $\bar{x}_1 \in F(\bar{G}_1)$, we have that

$$\bar{x}_1 = \sum_{i=1}^{p} \lambda_i y_i,$$

$$\lambda_i \geq 0 \quad \text{for } 0 \leq i \leq p, \ \sum_{i=1}^{p} \lambda_i = 1,$$

where y_i is an extreme point of $F(\bar{G}_1)$ for $i = 1, \ldots, p$.

We can also write

$$\bar{x}_2 = \sum_{i=1}^{k} \alpha_i z_i,$$

$$\alpha_i \geq 0 \quad \text{for } 0 \leq i \leq k, \ \sum_{i=1}^{k} \alpha_i = 1,$$

where z_i is an extreme point of $F(\bar{G}_2)$ for $i = 1, \ldots, k$.

Let us study Case 2. For a vector $x \in F(\bar{G})$, let us assume that its last six components are $x(w_i)$, $i = 1, \ldots, 4$, $x(u)$, and $x(v)$. Then, for each vector y_i, its last six components form one column of the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Set $\beta_j = \sum \{\lambda_i \mid y_i$ is associated with the jth column of $M\}$. The vector β satisfies

$$M\beta = \begin{bmatrix} \bar{x}(w_1) \\ \vdots \\ \bar{x}(w_4) \\ \bar{x}(u) \\ \bar{x}(v) \end{bmatrix}.$$

In the same way, we can associate the vectors $\{z_i\}$ with the columns of M.

We can define $\gamma_j = \sum \{\alpha_i \mid z_i$ is associated with the jth column of $M\}$, since γ satisfies

$$M\gamma = \begin{bmatrix} \bar{x}(w_1) \\ \vdots \\ \bar{x}(w_4) \\ \bar{x}(u) \\ \bar{x}(v) \end{bmatrix}$$

and M is nonsingular; we have that $\beta = \gamma$. Hence vectors $\{y_i\}$ can be matched with vectors $\{z_i\}$ in such a way that $\bar{x}$ can be written as a convex combination of extreme points of $F(\bar{G})$.

Cases 3 and 4 are analogous. □

3. Algorithmic aspects of the compositions. In this section, we use the compositions of §2 to obtain a maximum weighted BIS, provided that we have an algorithm to solve the problem in each piece.

Let $G = (V, E)$ be a signed graph and $c : V \to \Re_+$ a weight function.

First, let us assume that G is the graph of Theorem 2.1, let $W = \{w_1, \ldots, w_l\}$, and let β_i be the maximum weight of a BIS of G_2 that contains w_i for $1 \leq i \leq l$. Let β_0 be the maximum weight of a BIS of G_2 that does not contain any node of W. Let us redefine the weights in G_1 as follows:

$$c'(u) = c(u) \quad \text{if } u \notin W,$$

$$c'(w_i) = \max\{0, \beta_i - \beta_0\} \quad \text{for } 1 \leq i \leq l.$$

Let α be the maximum weight of a BIS of G_1; then the maximum weight of a BIS of G is $\alpha + \beta_0$.

Now let us study Cases 2–4 of §2. Set $W = \bar{V}_1 \cap \bar{V}_2$ and let

$$\sum_{u \in W} x(u) \leq r(W) \tag{3.1}$$

be inequality (2.1) or (2.2). If $\bar{x}$ is the incidence vector of a BIS of G, we can always complete it to an incidence vector of a BIS of $\bar{G}$ that satisfies (3.1).

Let $W = \{v_1, \ldots, v_p\}$. There are p BIS of $(W, \bar{E}(W))$ whose incidence vectors satisfy (3.1); moreover, they are linearly independent. Let $U_1, \ldots, U_p$ be the node sets of them. Let β_i be the maximum weight of a BIS of $\bar{G}_2$ whose node set contains U_i for $1 \leq i \leq p$. The weights for nodes of $W \backslash V_2$ are zero. Let $[\gamma_1 \cdots \gamma_p]$ be the solution of the system of equations

$$[\gamma_1 \cdots \gamma_p][x^{U_1} \cdots x^{U_p}] = [\beta_1 \cdots \beta_p].$$

Let us redefine the weights in $\bar{G}_1$ as follows:

$$c'(u) = c(u) \quad \text{if } u \in V_1 \backslash W,$$

$$c'(v_i) = \gamma_i + M \quad \text{for } 1 \leq i \leq p,$$

where M is a big number ($M = \sum_{u \in V} c(u)$). Let α be the maximum weight of a BIS of $\bar{G}_1$. Then the maximum weight of a BIS of G is $\alpha - r(W)M$.

4. Application to series-parallel graphs. These decomposition techniques are useful for classes of graphs that can be decomposed by two-node cutsets. For series-parallel graphs, Hassin and Tamir [12] proved the following result.

THEOREM 4.1. *Let $G = (V, E)$ be a series-parallel graph. There exist two nodes u, $v \in V$ and two subsets $V_1, V_2 \subseteq V$, such that*

(i) $|V_i| \leq \frac{2}{3}|V| + 2$, $i = 1, 2$,

(ii) $V_1 \cap V_2 = \{u, v\}$,

(iii) $V = V_1 \cup V_2$, $E = E(V_1) \cup E(V_2)$.

Furthermore, the vertices u, v and the sets V_1 and V_2 can be found in linear time.

To find a maximum weighted BIS, we should recursively decompose the graph. We might have to add five nodes to each piece so we do not decompose if the pieces are not smaller than the graph; i.e., we want $\frac{2}{3}|V| + 7 \leq |V|$. Thus we stop when each piece has at most 21 nodes. Figure 2 shows an example of a graph that cannot be further decomposed. If the graph has 21 nodes or less, we solve the problem by enumeration.

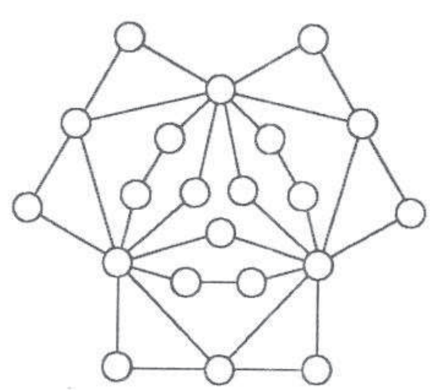

FIG. 2

Let $T(n)$ be the number of operations to solve the problem in a graph with n nodes. Then

$$T(n) \leq cn + T(n_1) + T(n_2),$$

where $n_i \leq \frac{2}{3}n + 2$, $i = 1, 2$ and $n_1 + n_2 = n + 2$. Therefore $T(n) = O(n \log n)$. We can state the following result.

THEOREM 4.2. *A maximum weighted* BIS *in a series-parallel graph with n nodes can be found in* $O(n \log n)$ *time.*

In [6] we showed that, even for series-parallel graphs, $P(G)$ may have facet-defining inequalities that are not simple to describe. However, the theorem below shows that, by allowing some extra variables, we obtain $P(G)$ as a projection of a polytope that is much easier to represent.

THEOREM 4.3. *If G is series-parallel and has n nodes, then* $P(G)$ *is a projection of a polytope defined by a system with* $O(n)$ *inequalities and* $O(n)$ *variables.*

Proof. Applying Theorems 2.1 and 2.2 to the decomposition of the graph, we obtain a polytope Q such that $P(G)$ is a projection of Q. If G has n nodes, then the number of variables in the system that defines Q is $O(n)$, and the number of inequalities is $O(n)$. For this, it is sufficient to know a characterization of $P(G)$ for series-parallel graphs with at most 21 nodes. □

5. Compositions of facets. Now we see that, in Cases 2 and 3 of §2, we obtain a complete description of the facets of $P(G)$ from the facets coming from the pieces. We must first study the structure of these inequalities.

Let $ax \leq \alpha$ be an inequality that defines a nontrivial facet of $P(G)$; i.e., a contains at least two nonzero components. It is easy to see that $a \geq 0$ and $\alpha > 0$. Also, if a has exactly two nonzero components, then it corresponds to $x(u) + x(v) \leq 1$. This can only be the case when u and v are linked by a positive edge and a negative edge. We denote by V_a the set

$$V_a = \{v \mid a_v > 0\}.$$

The graph $G_a = (V_a, E(V_a))$ is called a facet-inducing graph. We denote by $\beta(G)$ the set

$$\beta(G) = \{W \subseteq V \mid (W, E(W)) \text{ is balanced}\},$$

and β_a is the set

$$\beta_a = \{W \subseteq V \mid W \in \beta(G) \text{ and } ax^W = \alpha\}.$$

Given a path $u, u_1, u_2, \ldots, u_k, v$ between u and v, the nodes $u_1, \ldots, u_k$ are called *internal nodes*. Now we present several lemmas about the inequality $ax \leq \alpha$.

LEMMA 5.1. *The graph* G_a *is connected.*

Proof. Suppose that G_a is the union of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$. Let a_1 (respectively, a_2) be the row obtained from a by setting to zero all the components associated with nodes in G_1 (respectively, G_2). Letting

$$\alpha_i = \max a_i x, \qquad x \in P(G), \quad i = 1, 2,$$

we have that $\alpha = \alpha_1 + \alpha_2$ and $a = a_1 + a_2$; thus $ax \leq \alpha$ can be obtained as the sum of two valid inequalities. This is not possible because it defines a facet. □

LEMMA 5.2. *If* G_a *contains a node u of degree* 2 *and its neighbours are v and w, then* $a_u \leq a_v$, $a_u \leq a_w$.

Proof. Since $ax \leq \alpha$ defines a nontrivial facet, there is a set $W \in \beta_a$ such that $u \notin W$. It implies that $v, w \in W$.

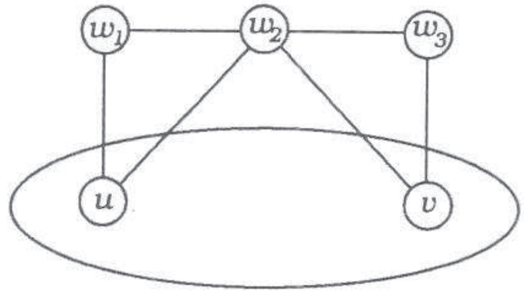

FIG. 3

Let

$$W' = W \backslash \{v\} \cup \{u\}, \qquad W'' = W \backslash \{w\} \cup \{u\}.$$

It is clear that $W' \in \beta(G)$ and $W'' \in \beta(G)$; then $a_u \le a_v$, $a_u \le a_w$. □

COROLLARY 5.3. *If G_a contains a path $u, u_1, \ldots, u_k, v$ whose internal nodes are of degree 2, then $a_{u_i} = a_{u_j}$ for $1 \le i \le k$, $1 \le j \le k$.*

LEMMA 5.4. *If G_a contains the induced subgraph $\Gamma = (U, E(U))$ where*

$$U = \{u, v, w_1, w_2, w_3\},$$

$$E(U) = \{uw_1, uw_2, vw_2, vw_3, w_1w_2, w_2w_3\},$$

and all the edges in $E(U)$ are labeled negative, then $a_{w_1} = a_{w_2} = a_{w_3}$. See Fig. 3.

Proof. Let $T_1 = \{u, w_1, w_2\}$ and $T_2 = \{v, w_2, w_3\}$. If $W \in \beta_a$, $W \cap \{u, w_2\} \ne \varnothing$, and $W \cap \{v, w_2\} \ne \varnothing$, then $|T_1 \cap W| = 2$ and $|T_2 \cap W| = 2$.

Since $ax \le \alpha$ is different from the inequalities

$$x(u) + x(w_1) + x(w_2) \le 2, \qquad x(v) + x(w_2) + x(w_3) \le 2,$$

there are two node sets W_1 and W_2 in β_a such that $\{u, w_2\} \cap W_1 = \varnothing$, $\{v, w_2\} \cap W_2 = \varnothing$. Hence $w_1 \in W_1$ and $w_3 \in W_2$. Let

$$W'_1 = W_1 \backslash \{w_1\} \cup \{w_2\}, \qquad W'_2 = W_2 \backslash \{w_3\} \cup \{w_2\}.$$

Since $\{W'_1, W'_2\} \subseteq \beta(G)$, we have

$$a_{w_2} \le a_{w_1}, \qquad a_{w_2} \le a_{w_3}$$

and, from Lemma 5.2, we have

$$a_{w_2} \ge a_{w_1}, \qquad a_{w_2} \ge a_{w_3},$$

which yields

$$a_{w_1} = a_{w_2} = a_{w_3}. \qquad \square$$

LEMMA 5.5. *Given two nodes u and v, there is at most one path in G_a containing an even (respectively, odd) number of negative edges whose internal nodes are of degree 2 (a path could consist of a single edge).*

Proof. Suppose that G_a contains the paths $u, u_1, \ldots, u_k, v$ and $u, v_1, \ldots, v_l, v$ that satisfy the conditions above. If $W \in \beta_a$, then either (a) $|W \cap \{u_1, \ldots, u_k\}| = k$ and $|W \cap \{v_1, \ldots, v_l\}| = l$ or (b) $|W \cap \{u_1, \ldots, u_k\}| = k - 1$ and $|W \cap \{v_1, \ldots, v_l\}| = l - 1$. Thus

$$\sum_i x^W(u_i) - \sum_j x^W(v_j) = k - l,$$

but this is not possible because $ax \le \alpha$ defines a facet. □

LEMMA 5.6. *Let p be a node of degree 3 in G_a; given any other node q in G_a, there is at most one path between p and q in G_a whose internal nodes are of degree 2.*

Proof. Suppose that there are two paths $p, u_1, \ldots, u_k, q$ and $p, v_1, \ldots, v_l, q$ that satisfy the above conditions. Because of Lemma 5.5, we assume that these paths have different parities.

Let $W \in \beta_a$. We have the two following cases:

(i) If $q \notin W$, then $\{p, u_1, \ldots, u_k, v_1, \ldots, v_l\} \subseteq W$, because p has degree 3 in G_a and $a \geq 0$;

(ii) If $q \in W$, then $|\{p, u_1, \ldots, u_k, v_1, \ldots, v_l\} \cap W| = k + l$, because $a \geq 0$ and W cannot contain a cycle with an odd number of negative edges.

Thus x^W satisfies

$$x(p) + x(q) + \sum_i x(u_i) + \sum_j x(v_j) = k + l + 1,$$

but this is not possible because $ax \leq \alpha$ defines a facet. □

LEMMA 5.7. *For a facet-defining inequality $ax \leq \alpha$, the graph G_a cannot be decomposed as in Theorem 2.1.*

Proof. Suppose that G_a admits such decomposition, since $ax \leq \alpha$ should also define a facet of $P(G_a)$. This contradicts Theorem 2.1. □

Now we study the facet-defining inequalities of $P(\bar{G}_k)$.

LEMMA 5.8. *If $ax \leq \alpha$ defines a facet of $P(\bar{G}_k)$, $k = 1, 2$ and $\{u, v, w_i; 1 \leq i \leq 4\} \subseteq V_a$ in Cases 2 and 3, $\{u, v, w_i; 1 \leq i \leq 5\} \subseteq V_a$, in Case 4, then $ax \leq \alpha$ is of the type (2.1) or (2.2).*

Proof. In Case 2, we can apply Lemma 5.4 and we have that $a_{w_2} = a_{w_3} = a_{w_4}$.

Let $W \in \beta_a$. There are the three following cases:

(i) $W \cap \{u, v\} = \emptyset$; in this case, $\{w_i \mid 1 \leq i \leq 4\} \subseteq W$;

(ii) $\{u, v\} \subseteq W$; in this case, $w_1 \notin W$, because it would create a cycle with one negative edge, and $W \cap \{w_2, w_3, w_4\} = \{w_2, w_4\}$, because $a_{w_2} = a_{w_3} = a_{w_4}$;

(iii) $|\{u, v\} \cap W| = 1$; this implies that $|W \cap \{w_2, w_3, w_4\}| = 2$; also, $w_1 \in W$, because $a_{w_1} > 0$.

Therefore $|W \cap \{u, v, w_i; 1 \leq i \leq 4\}| = 4$. Hence x^W satisfies (2.1); this implies that $ax \leq \alpha$ is of the type (2.1). In Case 3 or Case 4, the proof is analogous. □

Consider now a nontrivial facet-defining inequality $ax \leq \alpha$. In Cases 2 and 3, the structure of G_a falls into one of the types below:

(i) G_a does not contain any of $\{w_i \mid 1 \leq i \leq 4\}$;

(ii) If $w_2 \in G_a$, Lemma 5.7 shows that $u, w_3 \in G_a$; if there is any other node in G_a, then Lemma 5.7 shows that $v \in G_a$, and Lemma 5.6 shows that w_3 should not have degree 3. Hence $w_4 \in G_a$. From Lemma 5.4, we have that $a_{w_2} = a_{w_3} = a_{w_4}$. We can assume without loss of generality that $a_{w_2} = 1$. From Lemma 5.8, we have that $w_1 \in G_a$ only if the inequality is of type (2.1).

(iii) If $\{w_2, w_4\} \cap V_a = \emptyset$, then Lemma 5.5 shows that, in Case 2, w_3 cannot be in V_a; however, w_1 could be in V_a. In Case 3, Lemma 5.5 shows that $|\{w_1, w_3\} \cap V_a| \leq 1$.

Thus we have that, in Cases 2 and 3, the facet-defining inequalities of $P(\bar{G}_k)$ are classified as follows, for $k = 1, 2$:

$$\sum_{j \in V_k} a_{ij}^k x(j) \leq \alpha_i^k, \qquad i \in I_1^k, \tag{5.1a}$$

$$\sum_{j \in V_k} a_{ij}^k x(j) + x(w_2) + x(w_3) + x(w_4) \leq \alpha_i^k, \qquad i \in I_2^k, \tag{5.1b}$$

(5.1c) $$\sum_{j \in V_k} a_{ij}^k x(j) + x(w_3) \leq \alpha_i^k, \qquad i \in I_3^k,$$

(5.1d) $$\sum_{j \in V_k} a_{ij}^k x(j) + x(w_1) \leq \alpha_i^k, \qquad i \in I_4^k,$$

(5.1e) $$x(u) + x(w_2) + x(w_3) \leq 2,$$

(5.1f) $$x(v) + x(w_3) + x(w_4) \leq 2,$$

(5.1g) $$x(u) + x(v) + x(w_1) + x(w_2) + x(w_3) + x(w_4) \leq 4,$$

(5.1h) $$x(w_j) \leq 1, \qquad 1 \leq j \leq 4,$$

(5.1i) $$x(j) \geq 0, \qquad j \in \bar{V}_k.$$

Then $F(\bar{G})$ is defined by both systems, together with the inequality

(5.2) $$-x(u) - x(v) - x(w_1) - x(w_2) - x(w_3) - x(w_4) \leq -4.$$

To project the variables $\{x(w_i)\}$, we use the following theorem of Balas and Pulleyblank [1].

THEOREM 5.9. *Let* $Z = \{(w, x) \mid Aw + Bx \leq b\}$; *the projection of* Z *along the subspace of the* w *variables is*

$$X = \{x \mid (vB)x \leq vb,\ \forall v \in \text{extr } Y,\ x \geq 0\},$$

where extr Y *denotes the set of extreme rays of*

$$Y = \{y \mid yA \geq 0,\ y \geq 0\}.$$

In our case, the matrix A has the following twelve types of rows:

$$\begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1\,. \end{array}$$

Column j corresponds to $x(w_j)$ for $1 \leq j \leq 4$; the first seven rows correspond to inequalities (5.1a)–(5.1g), respectively; the next four rows correspond to inequalities (5.1h); the last row corresponds to inequality (5.2).

The extreme rays of Y correspond to the extreme points of

$$\{y \mid yA \geq 0,\ \sum y_i = 1,\ y \geq 0\},$$

so we enumerate the extreme points of

$$\{z \mid Bz \geq 0,\ \sum z_i = 1,\ z \geq 0\},$$

where B is the matrix below:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

The extreme points are the columns of

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & \vdots & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ & 0 & \ddots & & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ & & & & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{5} \\ \vdots & \vdots & & & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ & & & & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ & & & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{5} \\ & & & 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}.$$

So the inequalities given by Theorem 5.9 are obtained by performing the following steps:

(i) Keep inequalities (5.1a)–(5.1g), but delete the variables $\{x(w_i)\}$;

(ii) Add three inequalities, one of type (5.1b), one of type (5.1d) or $x(w_1) \le 1$, and (5.2),

(iii) Add four inequalities, one of type (5.1d) or $x(w_1) \le 1$, one of type (5.1e), one of type (5.1f), and (5.2), from the result delete $x(w_3)$;

(iv) Add four inequalities, one of type (5.1d) or $x(w_1) \le 1$, one of type (5.1e), $x(w_4) \le 1$, and (5.2);

(v) Add four inequalities, one of type (5.1d) or $x(w_1) \le 1$, one of type (5.1f), $x(w_2) \le 1$, and (5.2);

(vi) Add (5.1g) and (5.2) (which gives the redundant inequality $0 \le 0$);

(vii) Add five inequalities, one of type (5.1c) or $x(w_3) \le 1$, one of type (5.1d) or $x(w_1) \le 1$, $x(w_2) \le 1$, $x(w_4) \le 1$, and (5.2).

The next lemma shows that some of these inequalities are redundant.

LEMMA 5.10. *Let $ax \le \alpha$ be a facet-defining inequality of $P(G)$, $\alpha \ge 0$. If $V_a \subseteq V_k$, then this inequality also defines a facet of $P(\bar{G}_k)$, $k = 1, 2$.*

Proof. It is clear that this inequality is valid for $P(\bar{G}_k)$.

By hypothesis, there are $|\bar{V}_k|$ linearly independent incidence vectors of balanced induced subgraphs that satisfy $ax = \alpha$. Let us form a matrix M with them. Among these vectors, there is one vector $\bar{x}$ with $\bar{x}(u) = 0$ or $\bar{x}(v) = 0$. Hence the columns of

$$M' = \begin{bmatrix} M & \bar{x} & \bar{x} & \bar{x} & \bar{x} \\ 0 & 1 & & & \\ 0 & & 1 & & \\ 0 & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$$

form a set of $|\bar{V}_k|$ linearly independent incidence vectors of balanced induced subgraphs of $\bar{G}_k$. Moreover, they satisfy $ax = \alpha$. □

This lemma states that, if an inequality is essential in the definition of $P(G)$ and its support is included in V_k, then a multiple of this inequality already appears in (5.1a). So in (i) we should only keep (5.1a) and in (ii) we should keep the constraints whose support intersects both $V_1 \backslash \{u, v\}$ and $V_2 \backslash \{u, v\}$. For the same reasons, the inequalities produced in (iii)–(v) are redundant. Thus we obtain the following result.

THEOREM 5.11. *In Cases* 2 *and* 3, $P(G)$ *is defined by* (5.1a), *together with* $x(j) \geq 0$ *for* $j \in V$ *and the mixed inequalities*

$$\sum_{j \in V_k} a_{ij}^k x(j) + \sum_{j \in V_l} a_{sj}^l x(j) - x(u) - x(v) \leq \alpha_i^k + \alpha_s^l - 4 \tag{5.3}$$

for $k = 1, 2$; $l = 1, 2$; $i \in I_2^k$, $s \in I_4^l$.

In what follows, we prove that, if $k \neq l$, inequalities (5.3) define facets of $P(G)$. First, we must introduce a technical lemma.

LEMMA 5.12. *Inequalities* (5.1b) *and* (5.1d) *define facets of* $F(\bar{G}_k)$.

Proof. First, let us study inequalities (5.1b). Since $\dim(F(\bar{G}_k)) = |\bar{V}_k| - 1$, we must show that there are $|\bar{V}_k| - 1$ linearly independent vectors in $F(\bar{G})$ that satisfy (5.1b). Let $n = |\bar{V}_k|$. There is a linearly independent set $\{x_1, \ldots, x_n\}$ of extreme points of $P(\bar{G}_k)$ that satisfy (5.1b).

For a vector x_j, we do the following. If x_j satisfies (5.2), we keep it; otherwise, we set the component $x_j(w_1)$ equal to 1. We obtain a set $S = \{x'_1, \ldots, x'_n\}$ of vectors that satisfy both (5.1b) and (5.2). Since we only modified one component, there are $n - 1$ linearly independent vectors in S.

Now let us study an inequality of type (5.1d), say $ax \leq \alpha$. Let $S = \{x_1, \ldots, x_n\}$ be a set of linearly independent extreme points of $P(\bar{G}_k)$ that satisfy $ax = \alpha$. Let x_j be one of these vectors; if x_j satisfies (5.2), we define $x'_j = x_j$; otherwise, we define $x'_j(u) = x_j(u)$ if $u \neq w_2, w_3, w_4$. We set to 1 some of the components $x'_j(w_2)$, $x'_j(w_3)$, $x'_j(w_4)$ to obtain an extreme point x'_j that satisfies (5.2).

Let us assume that $ax \leq \alpha$ is inessential in the definition of $F(\bar{G})$. This implies that

$$a = \sum \lambda_i b_i + \gamma d, \quad \alpha \geq \sum \lambda_i \beta_i + \gamma \varepsilon, \quad \lambda_i \geq 0,$$

where $b_i x \leq \beta_i$ denotes an inequality in (5.1), other than $ax \leq \alpha$, but not (5.1g), and $dx = \varepsilon$ denotes the equation derived from (5.1g).

For each inequality $b_i x \leq \beta_i$, there is a vector $x_j \in S$ such that $b_i x_j < \beta_i$ and $b_i x'_j = \beta_i$. Hence $b_{iw_j} > 0$, $j = 2, 3, 4$; then $b_{iw_1} = 0$. This implies $\gamma = 1$ and some $\lambda_i < 0$, a contradiction. □

So if $ax \leq \alpha$ is of type (5.1b) or (5.1d) and another inequality $bx \leq \beta$ defines the same face of $F(\bar{G}_k)$, then $b = \lambda a + \gamma d$, where $dx = \varepsilon$ denotes (5.1g) as equation and $\lambda \geq 0$. The constraint $bx \leq \beta$ cannot be in (5.1), because of the structure of the inequalities in this system.

LEMMA 5.13. *If* $k \neq l$, *inequalities* (5.3) *define facets of* $P(G)$.

Proof. Suppose that $k = 1$, $l = 2$. We show that there exists a vector $\bar{x}$ in $P(G)$ that satisfies this inequality as equation, and all the others as strict inequality.

For the inequality

$$\sum_{j \in V_1} a_{ij}^1 x(j) + x(w_2) + x(w_3) + x(w_4) \leq \alpha_i^1, \qquad i \in I_2^1,$$

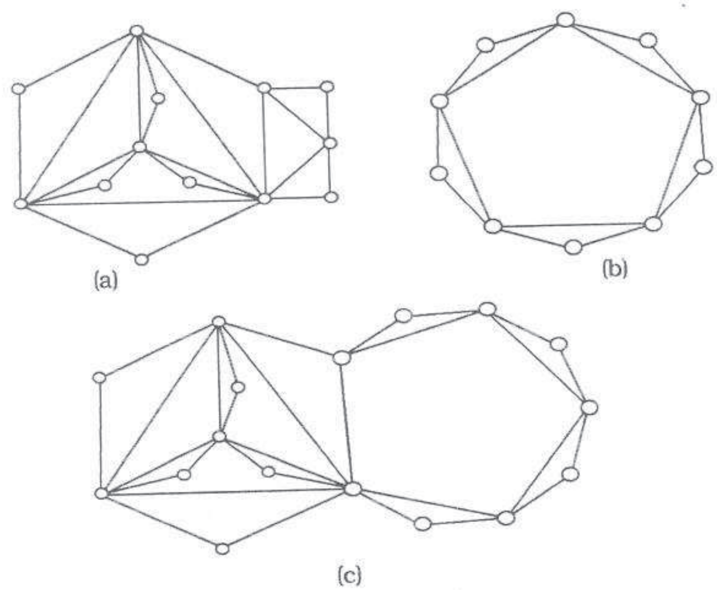

FIG. 4

let $S = \{x_1, \ldots, x_l\}$ be the set of extreme points of $F(\bar{G}_1)$ that satisfy it as equation. For

$$\sum_{j \in V_2} a_{ij}^2 x(j) + x(w_1) \le \alpha_i^2, \qquad i \in I_4^2,$$

let $T = \{y_1, \ldots, y_m\}$ be the set of extreme points of $F(\bar{G}_2)$ that satisfy it as equation.

First, for each vector x_i, we can find a vector y_{j_i} such that both together give a vector in $F(\bar{G})$, $0 \le i \le l$. Next, for each vector y_j, we can find a vector x_{i_j} such that together they give a vector in $F(\bar{G})$, $0 \le j \le m$.

Let $\{z_1, \ldots, z_r\}$ be the set of vectors thus obtained ($r = l + m$). Let z'_k be the vector obtained by dropping the components $z_k(w_j)$, $1 \le j \le 4$.

Then

$$\bar{x} = \frac{1}{r}(z'_1 + \cdots + z'_r)$$

is the required vector. □

This theorem gives a way to describe facets of $P(G)$ by composition of facets for the pieces. For instance, consider the graphs in Figs. 4(a) and 4(b), with all the edges labeled negative. The inequalities

$$\sum x(i) \le 9 \quad \text{and} \quad \sum x(i) \le 7$$

define facets for the first and second graph respectively; see [6]. Theorem 5.16 shows that

$$\sum x(i) \le 12$$

defines a facet for the graph in Fig. 4(c).

The techniques of this section also apply to Case 4 of §2. The only missing piece in this case is a characterization of the extreme rays of the set Y, defined in Theorem 5.9.

6. Acyclic induced subgraphs. In this section, our aim is to show how the same ideas apply to the acyclic induced subgraph polytope. Similar compositions for the polytope of acyclic spanning subgraphs have been studied in [5].

Let $D = (V, A)$ be a directed graph; the induced subgraph $(W, A(W))$ is called acyclic if it does not have a directed cycle. The *acyclic induced subgraph* (AIS) polytope is

$$P'(D) = \operatorname{conv}\{x^W \in \Re^V \mid (W, A(W)) \text{ is acyclic}\},$$

and the *maximum* AIS *problem* is

$$\max cx, \quad x \in P'(D).$$

If $G = (V, E)$ is an undirected graph, the maximum stable set problem in G can be reduced to a maximum AIS problem in a directed graph $D = (V, A)$, where each edge $e \in E$ is replaced by the arcs (i, j) and (j, i). This shows that the maximum AIS problem is NP-hard for planar digraphs.

The analogue of Theorem 2.1 is the following result.

THEOREM 6.1. *Let $D = (V, A)$ be a directed graph such that there exist two node sets V_1 and V_2 with the following properties:*

(i) $V = V_1 \cup V_2$,

(ii) $W = V_1 \cap V_2 \neq \varnothing$,

(iii) *For $\{i, j\} \subseteq W$, the arc $(i, j) \in A$ and $(j, i) \in A$,*

(iv) *The induced subgraph $(V \setminus W, A(V \setminus W))$ is disconnected.*

If $D_1 = (V_1, A(V_1))$ and $D_2 = (V_2, A(V_2))$, then a system of inequalities that defines $P'(D)$ is obtained by the juxtaposition of such systems defining $P'(D_1)$ and $P'(D_2)$.

Now let us study digraphs with a two-vertex cutset.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs such that $V_1 \cap V_2 = \{u, v\}$ and let $D = (V, A)$ be the union of D_1 and D_2, i.e., $V = V_1 \cup V_2$, $A = A_1 \cup A_2$. There are three cases.

Case 1. The arcs (u, v) and (v, u) belong to A.

Case 2. The arc $(u, v) \in A$.

Case 3. There is no arc between u and v.

Case 1 is covered by Theorem 6.1. In Case 2, we define $\bar{D}_i = (\bar{V}_i, \bar{A}_i)$, $i = 1, 2$ as follows:

$$\bar{V}_i = V_i \cup \{w_1, w_2\},$$

$$\bar{A}_i = A_i \cup \{(w_1, u), (v, w_1), (u, w_2), (w_2, u), (v, w_2), (w_2, v)\}.$$

In Case 3, we define

$$\bar{V}_i = V_i \cup \{w_1, w_2, w_3\},$$

$$\bar{A}_i = A_i \cup \{(v, w_1), (w_1, u), (u, w_2), (w_2, v), (u, w_3), (w_3, u), (w_3, v), (v, w_3)\}.$$

For Case 2, the inequality

$$x(w_1) + x(w_2) + x(u) + x(v) \leq 2$$

plays the role of inequality (2.1). For Case 3, the inequality

$$x(w_1) + x(w_2) + x(w_3) + x(u) + x(v) \leq 3$$

plays the role of (2.2).

So they define a facet $F(\bar{D}_i)$ of $P'(\bar{D}_i)$, $i = 1, 2$ and a facet $F(\bar{D})$ of $P'(\bar{D})$; again, the polytope $P'(D)$ is the projection of $F(\bar{D})$ along the variables $\{x(w_i)\}$, and the following is the analogue of Theorem 2.2.

THEOREM 6.2. *The juxtaposition of a system that defines $F(\bar{D}_1)$ and a system that defines $F(\bar{D}_2)$ gives a system that defines $F(\bar{D})$.*

Algorithmic aspects analogous to those of §3 hold for the AIS problem. For series-parallel digraphs, we have the following result.

THEOREM 6.3. *If D is a series-parallel directed graph with n nodes, then a maximum weighted AIS can be found in $O(n \log n)$ time.*

THEOREM 6.4. *If D is a series-parallel directed graph with n nodes, then $P'(D)$ is a projection of a polytope defined by a system with $O(n)$ inequalities and $O(n)$ variables.*

As for the BIS polytope, $P'(D)$ may have facet-defining inequalities that are not easy to describe even for series-parallel digraphs [6]. The above theorem shows that, if we allow extra variables, then we have a polytope that has a much simpler representation.

The techniques of §5 can also be adapted to Case 2 of the present section to produce compositions of facets of $P(D')$.

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