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COMPOSITIONS OF GRAPHS AND POLYHEDRA II: STABLE SETS*

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Abstract. A graph G with a two-node cutset decomposes into two pieces. A technique to describe the stable set polytope for G based on stable set polytopes associated with the pieces is studied. This gives a way to characterize this polytope for classes of graphs that can be recursively decomposed. This also gives a procedure to describe new facets of this polytope. A compact system for the stable set problem in series-parallel graphs is derived. This technique is also applied to characterize facet-defining inequalities for graphs with no $K_5\backslash e$ minor. The stable set problem is polynomially solvable for this class of graphs. Compositions of h-perfect graphs are also studied.

Key words. polyhedral combinatorics, composition of polyhedra, stable set polytope, compact systems

AMS subject classifications. 05C85, 90C27

1. Introduction. Given a graph G, let $P(G)$ be the stable set polytope of G. If G has a one- or two-node cutset, then G decomposes into G_1 and G_2. We study a technique to derive a system of inequalities that defines $P(G)$ from systems related to G_1 and G_2. In a companion paper [2], we studied the same technique for the polytopes of balanced and acyclic subgraphs. We can use this to characterize the stable set polytope for classes of graphs that can be decomposed by two-vertex cuts, provided that the pieces are "easy" to handle. It also gives a procedure for characterizing facets of the stable set polytope by composition of facets for the pieces. We use this method in [3] to characterize the stable set polytope for graphs with no W_4 minor.

In §2 we study the structure of the facets of $P(G)$ and show some facet-defining inequalities for subdivisions of a wheel. In §3 we study the composition of polyhedra. In §4 we study the algorithmic aspects of this kind of composition. In §5 we study series-parallel graphs. We derive a compact system for the stable set problem in this class of graphs; i.e., we show that $P(G)$ is a projection of a polyhedron that is defined by a system whose number of variables and number of inequalities is linear in the number of nodes of the graph. In §6 we study some facets of $P(G)$ for graphs with no $K_5\backslash e$ minor. Based on a decomposition theorem of Wagner, we can derive a polynomial algorithm for finding a maximum weighted stable set in this class of graphs. Using composition of facets, we show that, for any positive integer p, we can find a graph G with no $K_5\backslash e$ minor such that $P(G)$ has a facet-defining inequality with coefficients $1, 2, \ldots, p$. In §7 we study compositions of h-perfect graphs.

We finish this introduction with a few definitions. Given a graph $G = (V, E)$, a stable set $S \subseteq V$ is a node set such that there is no edge with both endnodes in S. If $S \subseteq V$, let $x^S \in \Re^V$, where $x^S(u) = 1$ if $u \in S$, and $x^S(u) = 0$ if $u \notin S$; x^S is called the *incidence vector* of S.

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The *stable set polytope* $P(G)$ is the convex hull of incidence vectors of all stable sets of G, i.e.,

$$P(G) = \text{conv}\, \{x^S \in \Re^V \mid S \text{ is a stable set of } G\}.$$

The polytope $P(G)$ is full-dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system $Ax \leq b$ such that $P(G) = \{x \mid Ax \leq b\}$; moreover, there is a natural bijection among the facets of $P(G)$ and the inequalities of that system.

2. On the facets of $P(G)$. The facets of $P(G)$ have been studied in [11], [5], [10], [13]–[15]. In this section, we present some properties of those inequalities that will be used later. We also present some facets for subdivisions of a wheel.

Let $ax \leq \alpha$ be an inequality that defines a facet of $P(G)$. If a contains at least two nonzero components, we say that $ax \leq \alpha$ defines a nontrivial facet. In this section, we study only nontrivial facets, so we have that $a \geq 0$ and $\alpha > 0$. We denote by V_a the set

$$V_a = \{v \mid a_v > 0\}.$$

The subgraph induced by V_a is denoted by G_a. Let us remark that G_a is a two-connected graph.

We now present two lemmas about the structure of G_a; their proofs appear in [9].

LEMMA 2.1. *If G_a contains a path with vertices p, u, v, q, where u and v are of degree 2, then $a_u = a_v$.*

LEMMA 2.2. *If G_a is different from an odd hole (and from K_3), then it does not contain between two given nodes p and q two edge-disjoint paths such that each node of them different from p, q is of degree 2.*

In what follows, we give two procedures of construction of facets of the stable set polytope from known facets. The first procedure consists of subdividing a star.

THEOREM 2.3 (subdivision of a star). *Let $G = (V, E)$ be a graph and $ax \leq \alpha$ be a nontrivial facet-defining inequality. Let v be a vertex of G and $N = \{v_0, \ldots, v_{k-1}\}$ be the neighbor set of v. Suppose that, for each $i = 0, \ldots, k-1$, there exists a stable set $\tilde{S}_i$ such that $ax^{\tilde{S}_i} = \alpha$ and $\tilde{S}_i \cap N = \{v_i, v_{i+1}, \ldots, v_{i+p-1}\}$, where $p \geq 1$ is a fixed integer and the indices are numbered modulo k. Suppose also that p and k are relatively prime and $a_{v_0} = a_{v_1} = \cdots = a_{v_{k-1}} = a_v/p$. Let $G' = (V', E')$ be the graph obtained from G by adding on each edge vv_i a new node v'_i for $i = 0, \ldots, k-1$. Set*

$$\bar{a}_u = a_u \quad \text{for } u \in V \setminus \{v\},$$

$$\bar{a}_v = a_v(k-p)/p,$$

$$\bar{a}_{v'_i} = a_v/p \quad \text{for } i = 0, 1, \ldots, k-1,$$

$$\bar{\alpha} = \alpha + a_v(k-p)/p.$$

Then $\bar{a}x \leq \bar{\alpha}$ defines a facet of $P(G')$.

Proof. First, let us show that $\bar{a}x \leq \bar{\alpha}$ is valid for $P(G')$. Let S' be a maximal stable set of G'.

Case 1. The node v belongs to S'.

Then $S = S' \setminus \{v\}$ is a stable set in G, which implies that $ax^S \leq \alpha$ and then $\bar{a}x^{S'} \leq \bar{\alpha}$.

Case 2. The node v does not belong to S'.

Let $T = S' \cap N$ and $T' = \{v'_i \mid v_i \notin T,\ 0 \leq i \leq k-1\}$. Note that $T' \subseteq S'$. Let $S = (S' \setminus (T \cup T')) \cup \{v\}$. It is clear that S is a stable set of G; then $ax^S \leq \alpha$. Since $|T \cup T'| = k$, we have that $\bar{a}x^{S'} \leq \bar{\alpha}$.

Now we have to show that $\bar{a}x \le \bar{\alpha}$ is facet-inducing. Let $n = |V|$ and $m = n + k$. There are n stable sets $S_1, \ldots, S_n$ of G whose incidence vectors are linearly independent and satisfy $ax = \alpha$. Consider the following sets:

$$S_i' = (S_i \setminus \{v\}) \cup \{v_0', \ldots, v_{k-1}'\} \quad \text{if } v \in S_i,$$
$$S_i' = S_i \cup \{v\} \quad \text{if } v \notin S_i, \text{ for } i = 1, \ldots, n,$$

and

$$S_{n+j+1}' = \tilde{S}_j \cup \{v_i' \mid v_i \notin \tilde{S}_j\} \quad \text{for } j = 0, 1, \ldots, k-1.$$

The incidence vectors of $S_1', \ldots, S_m'$ satisfy $\bar{a}x = \bar{\alpha}$. Let us assume that they also satisfy $bx = \bar{\alpha}$, where $bx \le \bar{\alpha}$ is a facet-defining inequality of $P(G')$. We prove that $b = \bar{a}$.

Since the incidence vectors of $\tilde{S}_0, \ldots, \tilde{S}_{k-1}$ are linearly independent, we can assume that $S_1 = \tilde{S}_0, \ldots, S_k = \tilde{S}_{k-1}$.

Consider the equations $bx^{S_i'} - bx^{S_{n+i}'} = 0$, $i = 1, \ldots, k$. This is a system like

$$[\lambda\gamma]\begin{pmatrix} u \\ -C \end{pmatrix} = 0,$$

where u is a row of 1's, and C is the $k \times k$ cyclic matrix having $(k-p)$ 1's in each row and column. Hence, we have that

$$b_{v_i'} = b_v/(k-p) \quad \text{for } i = 0, \ldots, k-1.$$

There is some number $\delta > 0$ such that $\bar{\alpha} - b_v = \delta\alpha$.

Consider the equations $bx^{S_i'} = \bar{\alpha}$ (or $bx^{S_i'} - b_v = \bar{\alpha} - b_v$), $i = 1, \ldots, n$. Since a is the unique solution of $ax^{S_i} = \alpha$, $i = 1, \ldots, n$, and $\bar{\alpha} - b_v = \delta\alpha$, we have that

$$b_u = \delta a_u \quad \text{for } u \in V \setminus \{v\} \quad \text{and} \quad b_v p/(k-p) = \delta a_v.$$

Therefore,

$$b_v = \delta a_v(k-p)/p = \delta \bar{a}_v \quad \text{and} \quad b_{v_i'} = \delta a_v/p = \delta \bar{a}_{v_i'}, \quad i = 0, \ldots, k-1.$$

Since $\bar{\alpha} = \delta\alpha + b_v = \delta\alpha + \delta a_v(k-p)/p$, we have that $\delta = 1$. The proof is complete. □

Wolsey [15] gave some methods to construct facets of $P(G)$ from known ones. One of those methods is the following, which consists of replacing one edge by a chordless path of length 3.

THEOREM 2.4 (subdivision of an edge). *Given a graph $G = (V, E)$ and $uv \in E$, let $ax \le \alpha$ be a nontrivial facet-defining inequality of $P(G)$, different from $x(u) + x(v) \le 1$. Let G' be the graph G without the edge uv, if $\beta = \max\{ax \mid x \in P(G')\}$ has a solution with $x(u) = x(v) = 1$, then*

$$ax + \lambda x(s) + \lambda x(t) \le \beta$$

defines a facet of $P(G'')$, where $\lambda = \beta - \alpha$, and G'' has been obtained by adding the nodes s and t to G', and the edges us, st, and tv.

In the following, we show a converse transformation.

THEOREM 2.5. *Let $G = (V, E)$ be a graph. Let $ax \le \alpha$ be a facet-defining inequality of $P(G)$. Suppose that G contains a path (pu, uv, vq) such that u and v are of degree 2. Assume also that $a_p = a_u = a_v = \beta$. Let $G' = (V', E')$ be the graph obtained from G by* *replacing that path by the edge pq. Let*

$$\bar{a}_u = a_u \quad \textit{for } u \in V',$$

$$\bar{\alpha} = \alpha - \beta,$$

then $\bar{a}x \leq \bar{\alpha}$ defines a face of $P(G')$.

Proof. First, we show that $\bar{a}x \leq \bar{\alpha}$ is valid for $P(G')$. Let S' be a stable set of G'. If $\{p, q\} \cap S' \neq \emptyset$, say $p \in S'$, then $S = S' \cup \{v\}$ is a stable set in G; hence $\bar{a}x^{S'} \leq \alpha - \beta = \bar{\alpha}$. If $\{p, q\} \cap S' = \emptyset$, then $S = S' \cup \{v\}$ is a stable set of G and thus $\bar{a}x^{S'} \leq \alpha - \beta = \bar{\alpha}$.

Let $n = |V|$ and $m = n - 2$. We must exhibit m stable sets of G' whose incidence vectors are linearly independent and satisfy $\bar{a}x = \bar{\alpha}$.

Since $ax \leq \alpha$ defines a facet of $P(G)$, there are n stable sets $S_1, \ldots, S_n$ of G such that $ax^{S_i} = \alpha$, $1 \leq i \leq n$, and this set of vectors is linearly independent. Consider the following sets:

1) $S'_i = S_i \backslash \{v\}$ if $\{p, v\} \subseteq S_i$,
2) $S'_i = S_i \backslash \{p\}$ if $\{p, q\} \subseteq S_i$,
3) $S'_i = S_i \backslash \{u\}$ if $\{q, u\} \subseteq S_i$,
4) $S'_i = S_i \backslash \{u\}$ if $u \in S_i$, $q \notin S_i$,
5) $S'_i = S_i \backslash \{v\}$ if $v \in S_i$, $p \notin S_i$

for $i = 1, \ldots, n$.

Note that the sets S'_i for $i = 1, \ldots, n$ are all stable sets of G'. Let us denote by M (respectively, M') the matrix whose columns are the incidence vectors of $S_1, \ldots, S_n$ $(S'_1, \ldots, S'_m)$. The matrices M and M' look like

$$M = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 1\cdots1 & 1\cdots1 & 0\cdots0 & 0\cdots0 & 0\cdots0 \\ 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1 & 0\cdots0 \\ 1\cdots1 & 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 \\ 0\cdots0 & 1\cdots1 & 1\cdots1 & 0\cdots0 & 0\cdots0 \end{pmatrix},$$

$$M' = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 1\cdots1 & 0\cdots0 & 0\cdots0 & 0\cdots0 & 0\cdots0 \\ 0\cdots0 & 1\cdots1 & 1\cdots1 & 0\cdots0 & 0\cdots0 \end{pmatrix}.$$

We must show that the rank of M' is $n - 2$. Let $\bar{M}$ be the following matrix:

$$\bar{M} = \begin{pmatrix} & 0 \\ & \vdots \\ M & 0 \\ & 1 \\ & 0 \\ & 0 \\ & 0 \\ 1\cdots1 & \end{pmatrix}.$$

This is nonsingular. In fact, if $\bar{M}$ is singular, then its last row should be linearly dependent of the others. Since a is the only solution of $tM = (\alpha, \ldots, \alpha)$, we should have $\beta/\alpha = 1$. However, $\alpha \geq 2\beta$, a contradiction.

Now let us add the rows corresponding to u and v to the row corresponding to p and subtract from the resulting row the last row of $\bar{M}$. We obtain the following:

$$\begin{pmatrix} A_1 & & A_2 & A_3 & A_4 & A_5 & & 0 \\ 1\cdots 1 & & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & & \vdots \\ 0\cdots 0 & & 0\cdots 0 & 1\cdots 1 & 1\cdots 1 & 0\cdots 0 & & 0 \\ 1\cdots 1 & & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & & 0 \\ 0\cdots 0 & & 1\cdots 1 & 1\cdots 1 & 0\cdots 0 & 0\cdots 0 & & 0 \\ 1111 & \cdots & & & & & \cdots & 1 \end{pmatrix}.$$

Since this matrix is nonsingular, we can conclude that M' is of rank $n-2$. □

We finish this section by showing some facet-defining inequalities of $P(G)$, when G is a subdivision of a wheel.

Let G be the graph of Fig. 1(a); it is well known that the inequality

$$\sum_{j=1}^{5} x(j) + 2x(6) \leq 2$$

defines a facet of $P(G)$. By applying Theorem 2.3 to the star of node 6 and then to the star of node 5, we obtain the graph of Fig. 1(b) and a facet-defining inequality; the coefficients different from 1 appear in the figure. The right-hand side is 7. Again, if we apply Theorem 2.3 to the stars of nodes 1, 2, 3, 4, and 5 in Fig. 1(a), we obtain a facet-defining inequality whose right-hand side is 12 and whose coefficients different from 1 appear in Fig. 1(c). Finally, if we apply Theorem 2.3 to the star of 6 in Fig. 1(a) and then Theorem 2.4 to subdivide some edges, we also obtain the graph in Fig. 1(c) but a different inequality whose right-hand side is 10 and whose coefficient different from 1 appears in Fig. 1(d).

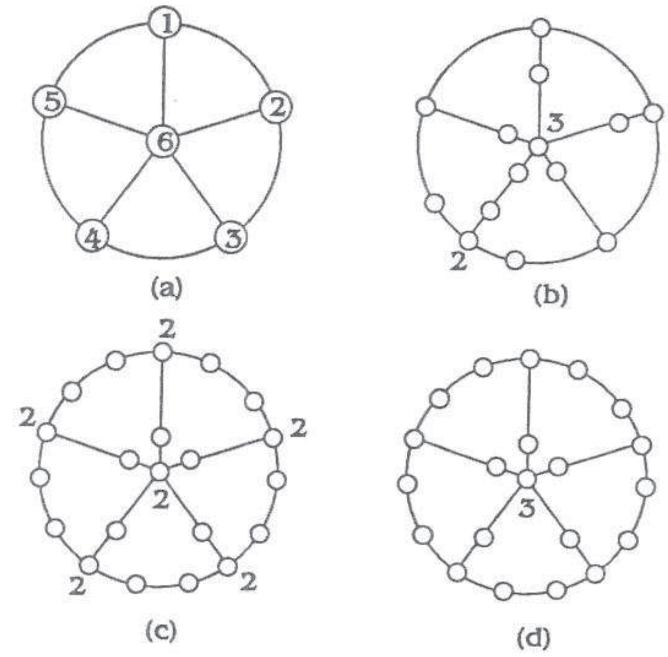

FIG. 1

3. Compositions of graphs. Let $G = (V, E)$ be a graph such that $V = V_1 \cup V_2$, $W = V_1 \cap V_2 \neq \emptyset$ and $(W, E(W))$ is a clique and $(V \backslash W, E(V \backslash W))$ is disconnected. Chvátal [5] proved the following.

THEOREM 3.1. *If* $G_1 = (V_1, E(V_1))$, $G_2 = (V_2, E(V_2))$, *then a system that defines* $P(G)$ *is obtained by taking the union of the systems that define* $P(G_1)$ *and* $P(G_2)$.

This theorem applies to the case where G has a one-node cutset or a two-node cutset $\{u, v\}$ with $uv \in E$; we refer to this as Case 1.

In the remainder of this section, we assume that

(i) $V = V_1 \cup V_2$,
(ii) $V_1 \cap V_2 = \{u, v\}$,
(iii) $G \backslash \{u, v\}$ is disconnected,
(iv) The nodes u and v are not adjacent.

This will be called Case 2. We add a five-cycle to each piece. We shall see that we can easily derive a description of the polytope for the original graph from the polytopes of the modified pieces. Let $\bar{G}_k = (\bar{V}_k, \bar{E}_k)$ be defined as follows:

(i) $\bar{V}_k = V_k \cup \{w_1, w_2, w_3\}$,
(ii) $\bar{E}_k = E(V_k) \cup \{uw_1, vw_1, uw_2, w_2w_3, w_3v\}$ for $k = 1, 2$. Let $\bar{G} = (\bar{V}, \bar{E})$ be the union of $\bar{G}_1$ and $\bar{G}_2$, i.e., $\bar{V} = \bar{V}_1 \cup \bar{V}_2$, $\bar{E} = \bar{E}_1 \cup \bar{E}_2$.

The inequality

$$\sum_{i=1}^{3} x(w_i) + x(u) + x(v) \leq 2 \tag{3.1}$$

defines a facet $F(\bar{G}_k)$ of $P(\bar{G}_k)$, $k = 1, 2$ and a facet $F(\bar{G})$ of $P(\bar{G})$. Furthermore, the polytope $P(G)$ is the projection of $F(\bar{G})$ along the variables $\{x(w_i)\}$, i.e.,

$$P(G) = \{y \mid (y, x(w_1), x(w_2), x(w_3))^t \in F(\bar{G})\}.$$

The next lemma gives a system that defines $F(\bar{G})$.

LEMMA 3.2. *Given two systems of inequalities defining* $F(\bar{G}_1)$ *and* $F(\bar{G}_2)$, *the union of these two systems defines* $F(\bar{G})$.

Proof. See Theorem 2.4 in [2]. □

Lemmas 2.1 and 2.2 show that the facet-defining inequalities of $P(\bar{G}_k)$ can be classified as follows, for $k = 1, 2$:

$$\sum_{j \in V_k} a_{ij}^k x(j) \leq \alpha_i^k, \qquad i \in I_1^k, \tag{3.2a}$$

$$\sum_{j \in V_k} a_{ij}^k x(j) + x(w_1) \leq \alpha_i^k, \qquad i \in I_2^k, \tag{3.2b}$$

$$\sum_{j \in V_k} a_{ij}^k x(j) + x(w_2) + x(w_3) \leq \alpha_i^k, \qquad i \in I_3^k, \tag{3.2c}$$

$$x(u) + x(w_1) \leq 1, \tag{3.2d}$$

$$x(u) + x(w_2) \leq 1, \tag{3.2e}$$

$$x(v) + x(w_1) \leq 1, \tag{3.2f}$$

$$x(v) + x(w_3) \leq 1, \tag{3.2g}$$

$$x(w_2) + x(w_3) \leq 1, \tag{3.2h}$$

$$x(u) + x(v) + x(w_1) + x(w_2) + x(w_3) \leq 2, \tag{3.2i}$$

$$x(j) \geq 0, \qquad j \in \bar{V}_k. \tag{3.2j}$$

The set I_1^k consists of the inequalities whose support does not intersect $\{w_1, w_2, w_3\}$. The set I_2^k contains the inequalities whose support includes $\{u, v, w_1\}$ and has empty intersection with $\{w_2, w_3\}$. The inequalities in I_3^k have a support that contains $\{u, v, w_2, w_3\}$ and does not include w_1.

Then $F(\bar{G})$ is defined by both systems together with the inequality

$$-x(u) - x(v) - x(w_1) - x(w_2) - x(w_3) \leq -2. \tag{3.3}$$

Now we project the variables $\{x(w_i)\}$ using the following result of Balas and Pulleyblank [1].

THEOREM 3.3. *Let* $Z = \{(w, x) | Aw + Bx \leq b,\ w \geq 0,\ x \geq 0\}$ *the projection of* Z *along the subspace of the* w *variables is*

$$X = \{x | (vB)x \leq vb,\ \forall v \in \text{extr } \Psi,\ x \geq 0\},$$

where extr Ψ *denotes the set of extreme rays of*

$$\Psi = \{y | yA \geq 0,\ y \geq 0\}.$$

In our case, the rows of A are of the following types:

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{matrix}.$$

The first nine rows correspond to inequalities (3.1a)–(3.1i), and the last row corresponds to (3.3). The extreme rays of Ψ correspond to the extreme points of

$$\{y | yA \geq 0,\ \textstyle\sum y_i = 1,\ y \geq 0\},$$

so we enumerate the extreme points of

$$\{z | Bz \geq 0,\ \textstyle\sum z_i = 1,\ z \geq 0\},$$

where B is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

The extreme points are the columns of the matrix below:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & & \vdots & \frac{1}{3} & \frac{1}{4} & 0 \\ & 0 & \ddots & & \frac{1}{3} & 0 & 0 \\ & & & & 0 & \frac{1}{4} & 0 \\ \vdots & \vdots & & & 0 & \frac{1}{4} & 0 \\ & & & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

Therefore the inequalities given by Theorem 3.3 are obtained by performing the following steps:

(i) Keep inequalities (3.2a)–(3.2i) but delete the variables $\{x(w_i)\}$,

(ii) Add three inequalities, one of type (3.2b), (3.2d), or (3.2f), one of type (3.2c) or (3.2h), and (3.3),

(iii) Add four inequalities, one of type (3.2b), (3.2d), or (3.2f), one of type (3.2e), one of type (3.2g), and (3.3),

(iv) Add (3.2i) and (3.3) (this gives the redundant inequality $0 \le 0$).

The next lemma shows that some of those inequalities are redundant.

LEMMA 3.4. *Let $ax \le \alpha$ be an inequality that defines a nontrivial facet of $P(G)$. If $V_a \subseteq V_k$, then this inequality also defines a facet of $P(\bar{G}_k)$, $k = 1, 2$.*

Proof. Let H be the graph obtained by replacing the edge uw_1 in $\bar{G}_k$ by the path us, st, tw_1, where s and t are new nodes. First, we prove that $ax \le \alpha$ defines a facet of $P(H)$. It is clear that this inequality is valid for $P(H)$.

By hypothesis, there are $|V_k|$ linearly independent incidence vectors of stable sets of G_k that satisfy $ax = \alpha$. Let us form a matrix M with them. Among these vectors, there is one vector $\bar{x}$ such that $\bar{x}(u) = 0$ and one vector $\tilde{x}$ such that $\tilde{x}(v) = 0$. Consider the matrix

$$M' = \begin{pmatrix} M & \bar{x} & \bar{x} & \tilde{x} & \bar{x} & \tilde{x} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last five rows correspond to s, t, w_1, w_2, and w_3, respectively.

The columns of M' are linearly independent incidence vectors of stable sets of H and satisfy $ax = \alpha$. Now, by Theorem 2.5, we can replace the path us, st, tw_1 by the edge uw_1 and we have that $ax \le \alpha$ defines a facet of the stable set polytope of $\bar{G}_k$. □

This lemma states that, if an inequality is essential in the definition of $P(G)$ and its support is included in V_k, then a multiple of this inequality already appears in (3.2a). So when we apply (i) we should keep only the inequality (3.2a); in (ii) we should keep only those constraints whose support intersects both $V_1 \backslash \{u, v\}$ and $V_2 \backslash \{u, v\}$; for the same reasons the inequalities produced in (iii) are redundant. We can state the main result of this paper.

THEOREM 3.5. *The polytope $P(G)$ is defined by* (3.2a), *together with $x(j) \ge 0$, for $j \in V$, and the mixed inequalities*

$$\sum_{j \in V_k} a^k_{ij} x(j) + \sum_{j \in V_l} a^l_{sj} x(j) - x(u) - x(v) \le \alpha^k_i + \alpha^l_s - 2 \tag{3.4}$$

$$\text{for } k = 1, 2;\ l = 1, 2;\ k \ne l;\ i \in I^k_2,\ s \in I^l_3.$$

To prove that this system is minimal, we introduce the following lemma.

LEMMA 3.6. *Inequalities* (3.2b) *and* (3.2c) *define facets of $F(\bar{G}_k)$.*

Proof. First, we study inequality (3.2b). Let $ax \le \alpha$ be one of them and let $S = \{x_1, \ldots, x_p\}$ be the set of extreme points of $P(\bar{G}_k)$ that satisfy $ax = \alpha$.

Let x_j be a vector in S; if $x_j \in F(\bar{G}_k)$, we define $x'_j = x_j$; otherwise, we define $x'_j(i) = x_j(i)$ if $i \ne w_2, w_3$. We set to 1 $x'_j(w_2)$ or $x'_j(w_3)$ to obtain an extreme point $x'_j \in F(\bar{G}_k)$. Now let us assume that $ax \le \alpha$ is inessential in the definition of $F(\bar{G}_k)$. This implies that $a = \sum \lambda_i b_i + \gamma d$, $\alpha \ge \sum \lambda_i \beta_i + \gamma \varepsilon$, with $\lambda_i \ge 0$, where $b_i x \le \beta_i$ denotes an inequality of (3.2) different from $ax \leq \alpha$ and from (3.2i); $dx = \varepsilon$ denotes the equation obtained from (3.2i).

For each inequality $b_i x \leq \beta_i$, there is a vector $x_j \in S$ such that $b_i x_j < \beta_i$ and $b_i x'_j = \beta_i$. Hence $b_{iw_j} > 0$ for $j = 2$ or 3; then $b_{iw_1} = 0$. This implies $\gamma = 1$ and then some $\lambda_i < 0$, a contradiction.

Now we consider an inequality $ax \leq \alpha$ of the type (3.2c).

Since $\dim (F(\bar{G}_k)) = |\bar{V}_k| - 1$, we must show $|\bar{V}_k| - 1$ linearly independent vectors in $F(\bar{G}_k)$ that satisfy $ax = \alpha$. Let $S = \{x_1, \ldots, x_p\}$ be the set of extreme points of $P(\bar{G}_k)$ that satisfy $ax = \alpha$. Let x_j be a vector in S; if $x_j \in F(\bar{G}_k)$, we set $x'_j = x_j$; otherwise, we set $x'_j(i) = x_j(i)$, if $i \neq w_1$, $x'_j(w_1) = 1$. Since S contains $|\bar{V}_k|$ linearly independent vectors and we modified only one component of the vectors in S, the set $S' = \{x'_1, \ldots, x'_p\}$ contains $|\bar{V}_k| - 1$ linearly independent vectors in $F(\bar{G}_k)$ that satisfy $ax = \alpha$. □

So if $ax \leq \alpha$ is of type (3.2b) or (3.2c) and another inequality $bx \leq \beta$ defines the same face of $F(\bar{G}_k)$, then $b = \lambda a + \gamma d$, where $dx = \varepsilon$ denotes (3.2i) and $\lambda \geq 0$. The constraint $bx \leq \beta$ cannot be in (3.2) because of the structure of the inequalities in this system.

COROLLARY 3.7. *Inequalities* (3.4) *define facets of* $P(G)$.

Proof. Assume that $k = 1$, $l = 2$. We show that there exists a vector $\bar{x} \in P(G)$ that satisfies this inequality and all others as strict inequalities.

For the inequality

$$\sum_{j \in V_1} a^1_{ij} x(j) + x(w_1) \leq \alpha^1_i, \qquad i \in I^1_2,$$

let $S = \{x_1, \ldots, x_n\}$ be the set of extreme points of $F(\bar{G}_1)$ that satisfy it. For

$$\sum_{j \in V_2} a^2_{ij} x(j) + x(w_2) + x(w_3) \leq \alpha^2_i, \qquad i \in I^2_3,$$

let $T = \{y_1, \ldots, y_m\}$ be the set of extreme points of $F(\bar{G}_2)$ that satisfy it. First, for each vector x_i, we find a vector y_{j_i} such that together they give a vector $z_i \in F(\bar{G})$, $i = 1, \ldots, n$. Similarly, for each vector y_i, we find a vector x_{j_i} that gives a vector $z_{n+i} \in F(\bar{G})$, $i = 1, \ldots, m$. Let $\{z_1, \ldots, z_r\}$ be the set of vectors thus obtained, ($r = n + m$). Let z'_k be the vector obtained by dropping the components $z_k(w_j)$, $1 \leq j \leq 3$ from z_k; then

$$\bar{x} = \frac{1}{r}(z'_1 + \cdots + z'_r)$$

is the required vector. □

4. Algorithmic aspects. The optimization problem can be also decomposed. The following algorithm appeared in Boulala and Uhry [4] and Sbihi and Uhry [12].

Let $G = (V, E)$ be a graph and $c : V \to \Re_+$ a weight function. Let us assume that G is the graph of Theorem 3.1, let $W = \{w_1, \ldots, w_l\}$, and let β_i be the maximum weight of a stable set of G_2 that contains w_i for $1 \leq i \leq l$. Let β_0 be the maximum weight of a stable set of G_2 that does not contain any node of W. Let us redefine the weights in G_1 as follows:

$$c'(u) = c(u) \qquad \text{if } u \notin W,$$

$$c'(w_i) = \max\{0, \beta_i - \beta_0\} \quad \text{for } 1 \leq i \leq l.$$

Let α be the maximum weight of a stable set of G_1; then the maximum weight of a stable set of G is $\alpha + \beta_0$.

Now let us study Case 2 of §3. Let $y_0 = u$, $y_1 = w_2$, $y_2 = w_3$, $y_3 = v$, $y_4 = w_1$. For $0 \le i \le 4$, let β_i be the maximum weight of a stable set of $\bar{G}_2$ whose node set contains y_i and y_{i+2} (indices taken mod 5); the weights of the nodes $\{w_i\}$ are zero.

Let $[\gamma_0, \ldots, \gamma_4]$ be the solution of the system

$$(\gamma_0, \ldots, \gamma_4)\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = (\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4).$$

We have that

$$\gamma_0 = \frac{\beta_0 - \beta_1 - \beta_2 + \beta_3 + \beta_4}{2},$$

$$\gamma_1 = \frac{\beta_0 + \beta_1 - \beta_2 - \beta_3 + \beta_4}{2},$$

$$\gamma_2 = \frac{\beta_0 + \beta_1 + \beta_2 - \beta_3 - \beta_4}{2},$$

$$\gamma_3 = \frac{-\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4}{2},$$

$$\gamma_4 = \frac{-\beta_0 - \beta_1 + \beta_2 + \beta_3 + \beta_4}{2}.$$

Let $M = \min\{\gamma_i\}$; if the numbers $\{\beta_i\}$ are integers, then $\gamma_i - M$ is a nonnegative integer for $0 \le i \le 4$. Now we define

$$c'(j) = c(j) \quad \text{if } j \in V_1 \backslash \{u, v\},$$

$$c'(y_i) = \gamma_i - M \quad \text{for } 0 \le i \le 4.$$

Let α be the maximum weight of a stable set of $\bar{G}_1$; then the maximum weight of a stable set of G is $\alpha + 2M$. Let us remark that one of the weights $c'(y_k)$ is zero; we can then delete that node from $\bar{G}_1$.

5. Application to series-parallel graphs. Boulala and Uhry [4] proved the following.

THEOREM 5.1. *If $G = (V, E)$ is a series-parallel graph, then $P(G)$ is defined by*

$$0 \le x(u) \quad \textit{for all } u \in V,$$

$$(5.1) \qquad x(u) + x(v) \le 1 \quad \textit{for all } uv \in E,$$

$$\sum_{u \in C} x(u) \le \frac{|C| - 1}{2} \quad \textit{for all odd holes } C.$$

They also gave a linear time algorithm to find a maximum weighted stable set in a series-parallel graph. A short proof of Theorem 5.1 appears in [9].

A connected series-parallel graph can be decomposed into paths and triangles using one-node and two-node cutsets. If G consists of a cycle and a path joining two nodes of the cycle, then it is easily seen that $P(G)$ is defined by (5.2). Thus Theorem 5.1 can be derived from Theorems 3.1 and 3.5.

Since a series-parallel graph may have exponentially many odd cycles, the stable set polytope may have exponentially many facets.

Hassin and Tamir [7] proved that, if $G = (V, E)$ is a series-parallel graph, then it contains a two-node cutset such that, when the decomposition of §3 is carried out, $|V_i| \leq 2/3|V| + 2$ for $i = 1, 2$. Hence we can recursively decompose a series-parallel graph until each piece has at most fifteen nodes. By Lemma 3.2, we can describe a polytope Q such that $P(G)$ is a projection of Q. If G has n nodes, then the number of inequalities and the number of variables in the system that defines Q is $O(n)$. Such a system is compact.

6. Graphs with no $K_5 \backslash e$ minor. A graph G is said to contain a graph H as a minor if a graph isomorphic to H can be obtained from G by repeated deletion and contraction of edges of G. Let us denote by $\mathscr{C}$ the class of all connected graphs that do not contain $K_5 \backslash e$ as a minor; this is the graph K_5 minus one edge. Wagner [16] gave a characterization of the graphs in $\mathscr{C}$.

If G_1 and G_2 are node-disjoint graphs with at least two nodes, v_1 a node of G_1 and v_2 a node of G_2 then the 1-sum of G_1 and G_2 (with respect to v_1 and v_2) is obtained by identifying the nodes v_1 and v_2. If e_1 is an edge of G_1 and e_2 is an edge of G_2, then the 2-sum of G_1 and G_2 (with respect to e_1 and e_2) is obtained by identifying e_1 and e_2 (and, of course, the endnodes of e_1 and e_2).

Wagner proved that each maximal graph G in $\mathscr{C}$ (i.e., by adding a further edge to G, the new graph will contain $K_5 \backslash e$ as a minor) can be obtained by starting with the graphs of Fig. 2 and taking repeated 1-sums or 2-sums. Equivalently, if we have a maximal graph $G \in \mathscr{C}$, we can decompose it into the graphs of Fig. 2. If the graph $G \in \mathscr{C}$ is not maximal, then we may use also spanning subgraphs of the graphs in Fig. 2. In this case, the nodes of the two-node cutset may be nonadjacent in G.

Let n be the number of nodes of $G \in \mathscr{C}$; a two-vertex cutset can be found in $O(n)$ time. The pieces that we obtain are the graphs of Fig. 2, where some edges are replaced by a five-cycle. The max stable set problem in a graph of this type obtained from a wheel can be solved in linear time, so, using the procedure of §4, we can find a maximum weighted stable set in G in $O(n^2)$ time.

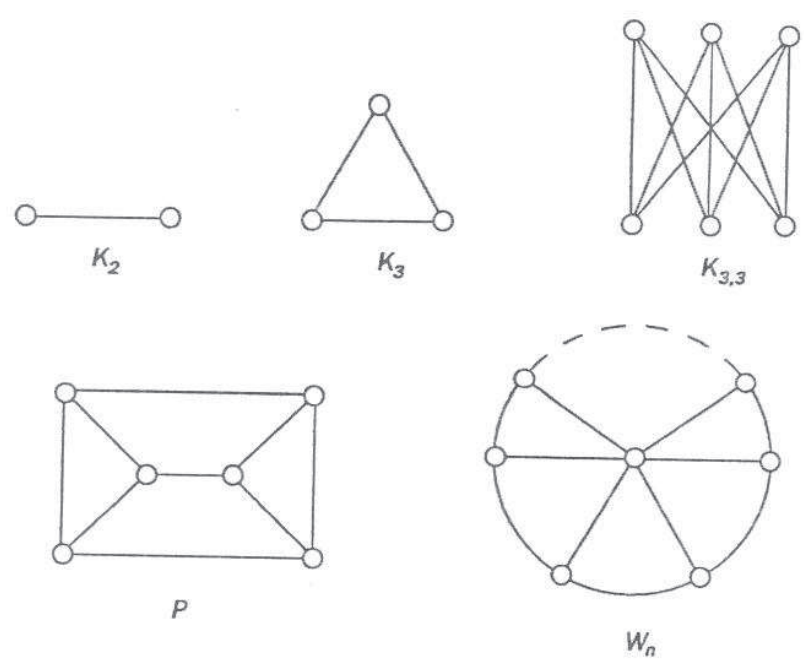

FIG. 2

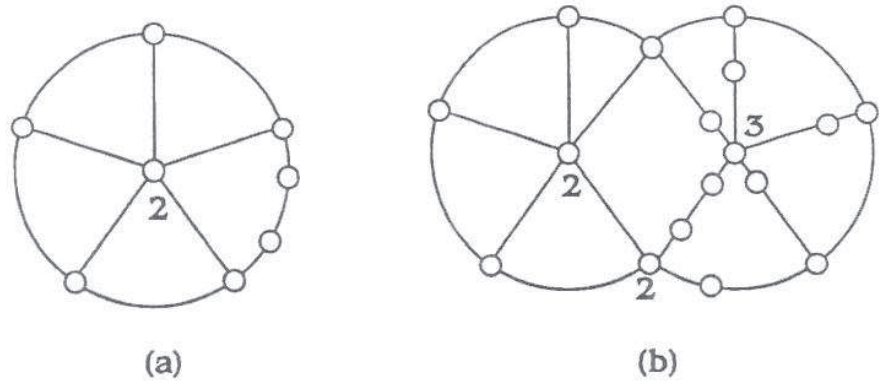

FIG. 3

The development of a polynomial algorithm for combinatorial optimization problems has often been closely related to the characterization of a system of linear inequalities that defines the corresponding polytope. This is the case for the stable set polytope of series-parallel graphs [4]. The missing piece here is a characterization of the polytope for subdivisions of wheels; in §2 we gave procedures to produce some of these inequalities. In [3] we used a tour de force to characterize the polytope for subdivisions of K_4. In what follows, we present some examples of facet-defining inequalities of $P(G)$ for $G \in \mathscr{C}$.

The graph of Fig. 3(a) has been obtained by subdividing one edge of the graph in Fig. 1(a). Theorem 2.4 gives us a facet-defining inequality whose right-hand side is 3 and whose coefficient different from 1 is shown in the figure. We can compose this graph with the graph of Fig. 1(b) to obtain the graph in Fig. 3(b). By Corollary 3.7, we can see that there is facet-defining inequality whose right-hand side is 8 and whose coefficients different from 1 appear in the figure. Given any positive integer p, we can construct a graph $G \in \mathscr{C}$ and an inequality that defines a facet of $P(G)$ with coefficients $1, 2, \ldots, p$. For this, it is enough to compose subdivisions of wheels of different sizes.

7. Composition of *h*-perfect graphs. A graph G is said to be h-perfect if $P(G)$ is defined by the constraints corresponding to cliques, odd holes, and the nonnegativity constraints.

Let G be the graph of Theorem 3.5; we can derive h-perfectness of G as follows:

(a) If $\bar{G}_1$ and $\bar{G}_2$ are h-perfect then G is also h-perfect;

(b) If $\bar{G}_2 \setminus \{w_1\}$ is h-perfect, $\bar{G}_1$ is h-perfect and the set of inequalities (3.2c) for $P(\bar{G}_1)$ is empty then G is h-perfect;

(c) If $\bar{G}_2 \setminus \{w_2, w_3\}$ is h-perfect, $\bar{G}_1$ is h-perfect and the set of inequalities (3.2b) for $P(\bar{G}_1)$ is empty then G is h-perfect.

Sbihi and Uhry [12] studied graphs G, which are the union of a bipartite graph G_1 and graph G_2 having exactly two common nodes u and v and no edge in common. They proved that the graph G is h-perfect if the graph obtained from G by replacing G_1 by an $u - v$ chain is h-perfect. They also proved that the graph obtained by substituting bipartite graphs for edges of a series parallel graph is h-perfect. Their results follow from remarks (a)–(c).

Gerards [6] studied graphs with not odd K_4. Those graphs can be decomposed by two-node cutsets as shown by Lovász et al. [8]. Gerards used compositions similar to those of Boulala and Uhry [4] and Sbihi and Uhry [12].

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