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COMPOSITIONS OF GRAPHS AND POLYHEDRA IV: ACYCLIC SPANNING SUBGRAPHS*

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Abstract. Given a directed graph D that has a two-vertex cut, this paper describes a technique to derive a linear system that defines the acyclic subgraph polytope of D from systems related to the pieces. It also gives a technique to describe facets of this polytope by composition of facets for the pieces. The authors prove that, if the systems for the pieces are totally dual integral (TDI), then the system for D is also. The authors prove that the "cycle inequalities" form a TDI system for any orientation of K_5. These results are combined with Lucchesi–Younger theorem and a theorem of Wagner to prove that, for graphs with no $K_{3,3}$ minor, the cycle inequalities characterize the acyclic subgraph polytope and form a TDI system. This shows that, for this class of graphs, the cardinality of a minimum feedback set is equal to the maximum number of arc disjoint cycles. For planar graphs, this is a consequence of the Lucchesi–Younger theorem.

Key words. polyhedral combinatorics, compositions of polyhedra, acyclic subgraph polytope

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1. Introduction. Given a directed graph $D = (V, A)$, we say that $D' = (V, A')$ is a subgraph of D if $A' \subseteq A$. Given $S \subseteq A$, the incidence vector of S, $x^S \in \Re^A$ is defined by

$$x^S(i,j) = \begin{cases} 1 & \text{if } (i,j) \in S, \\ 0 & \text{if } (i,j) \in A \backslash S. \end{cases}$$

The *acyclic subgraph polytope* of D, denoted by $P(D)$, is the convex hull of incidence vectors of arc sets S such that $D_S = (V, S)$ has no directed cycle. Given a weight function $w : A \to \Re$, the problem of finding a maximum weighted cyclic subgraph can be formulated as the linear program

$$\text{maximize } wx \quad \text{s.t. } x \in P(D).$$

The polytope $P(D)$ is full-dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system $Ax \leq b$ such that $P(D) = \{x : Ax \leq b\}$. These inequalities define the *facets* of $P(D)$. The acyclic subgraph problem is NP-hard, so finding a complete characterization of $P(D)$ seems to be very difficult. On the other hand, Lucchesi and Younger [9] characterized $P(D)$ for planar graphs. Grötschel, Jünger, and Reinelt [6], [7] characterized several facet-defining inequalities of $P(D)$ and used them to design a cutting plane algorithm.

In this paper, we study directed graphs that have a two-vertex cutset. We show how to derive a system that defines $P(D)$ from systems associated with the pieces. This also gives a technique to derive new facets defining inequalities by composition of known

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facets. We also prove that, if the two systems are totally dual integral (TDI), then the new system is, also.

The most natural system of inequalities that we can think of is

$$(1.1) \qquad \sum_{(i,j)\in C} x(i,j) \le |C| - 1 \quad \text{for every directed cycle } C,$$

$$(1.2) \qquad 0 \le x(i,j) \le 1 \quad \text{for every arc } (i,j).$$

Inequalities (1.1) will be called *cycle inequalities*. Lucchesi and Younger [9] proved that, for planar graphs, (1.1), (1.2) is a TDI system and defines $P(D)$.

Wagner [10] proved that graphs with no $K_{3,3}$ minor can be decomposed into planar graphs and copies of K_5. We prove that, for any orientation of K_5, the system (1.1), (1.2) is TDI. We combine Wagner's theorem with the theorem of Lucchesi and Younger and our composition techniques to prove that, for graphs with no $K_{3,3}$ minor, the system (1.1), (1.2) is TDI and defines $P(D)$. This implies that, for this class of graphs, the cardinality of a minimum feedback set is equal to the maximum number of arc disjoint cycles.

The present paper should be considered as a revision of [3]; this type of composition was studied there, but the results on dual integrality are new.

If $G = (V, E)$ is an undirected graph, we say that G contains H as a minor if H can be obtained from G by a sequence of deletions and contractions of edges. An orientation of G is a directed graph that contains exactly one of the arcs (i, j) or (j, i) whenever $ij \in E$. The symmetric digraph $D(G) = (V, A)$ associated with G has the arcs (i, j) and (j, i) whenever $ij \in E$. Given a directed graph $D = (V, A)$ and $S \subseteq V$, we denote by $\delta^+(S)$ (respectively, $\delta^-(S)$) the set of arcs that enters (respectively, leaves) S. We write *cycle* instead of directed cycle.

This paper is organized as follows. Section 2 is devoted to the composition of polyhedra; §3 deals with the composition of facets; in §4 we study the algorithmic aspects of this composition; in §5 we study compositions of TDI systems; §6 is dedicated to the study of the orientations of K_5; in §7 we study graphs with no $K_{3,3}$ minor.

2. Compositions of polyhedra. In this section, we assume that $D = (V, A)$ is a connected digraph having a two-node cutset $\{u, v\}$, i.e.,

(i) $V = V_1 \cup V_2$,

(ii) $V_1 \cap V_2 = \{u, v\}$,

(iii) $D \backslash \{u, v\}$ is disconnected.

For $k = 1, 2$, we define $\bar{D}_k = (V_k, \bar{A}_k)$, where

$$\bar{A}_k = A(V_k) \cup \{(u, v), (v, u)\},$$

and $\bar{D} = (V, \bar{A})$ with $\bar{A} = A \cup \{(u, v), (v, u)\}$. Note that we could create parallel arcs in this way; this is not a problem in any of the treatments that follow.

We explain how to describe $P(D)$ from systems defining $P(\bar{D}_1)$ and $P(\bar{D}_2)$.

The inequality

$$x(u, v) + x(v, u) \le 1$$

defines a facet of $P(\bar{D})$; it is easy to see that this is the only facet-defining inequality that has nonzero coefficients for both $x(u, v)$ and $x(v, u)$. Therefore the facets of $P(\bar{D}_k)$, for

$k = 1, 2$, can be classified into the five types below:

(2.1a) $$\sum_{(i,j)\in A_k} a_l^k(i,j)x(i,j) \le \alpha_l^k, \qquad l \in I_1^k,$$

(2.1b) $$\sum_{(i,j)\in A_k} a_l^k(i,j)x(i,j) + x(u,v) \le \alpha_l^k, \qquad l \in I_2^k,$$

(2.1c) $$\sum_{(i,j)\in A_k} a_l^k(i,j)x(i,j) + x(v,u) \le \alpha_l^k, \qquad l \in I_3^k,$$

(2.1d) $$x(u,v) + x(v,u) \le 1,$$

(2.1e) $$x(i,j) \ge 0 \quad \text{for } (i,j) \in \bar{A}_k,$$

where I_1^k is the set of inequalities with zero coefficients for $x(u, v)$ and $x(v, u)$; I_2^k is the set of inequalities with a nonzero coefficient for $x(u, v)$ and a zero coefficient for $x(v, u)$; I_3^k is the set of inequalities having a zero coefficient for $x(u, v)$ and a nonzero coefficient for $x(v, u)$.

The equation

$$x(u,v) + x(v,u) = 1$$

defines a facet $F(\bar{D}_k)$ of $P(\bar{D}_k)$ and a facet $F(\bar{D})$ of $P(\bar{D})$. The polytope $P(D)$ is a projection of $F(\bar{D})$ along the variables $x(u, v)$ and $x(v, u)$. The following theorem lets us find a system that describes $F(\bar{D})$.

THEOREM 2.1. *The polytope $F(\bar{D})$ is defined by the union of the systems that define $F(\bar{D}_1)$ and $F(\bar{D}_2)$.*

Proof. Let Q denote the polytope defined by the union of these two systems. Let x be a vector in Q. Let x_1 (respectively, x_2) be the restriction of x to $\bar{A}_1$ (respectively, $\bar{A}_2$); we have that

$$x_1 = \sum \alpha_i y_i, \quad \alpha_i \ge 0, \quad \sum \alpha_i = 1, \quad \text{and}$$
$$x_2 = \sum \beta_i z_i, \quad \beta_i \ge 0, \quad \sum \beta_i = 1,$$

where $\{y_i\}$ and $\{z_i\}$ are integer vectors in $F(\bar{D}_1)$ and $F(\bar{D}_2)$, respectively.

Since

$$\sum \{\alpha_i \mid y_i(u,v) = 1\} = x(u,v) = \sum \{\beta_i \mid z_i(u,v) = 1\},$$
$$\sum \{\alpha_i \mid y_i(v,u) = 1\} = x(v,u) = \sum \{\beta_i \mid z_i(v,u) = 1\},$$

we can match vectors in $\{y_i\}$ with vectors in $\{z_i\}$ to write x as a convex combination of integer vectors in $F(\bar{D})$. □

We now need a way to project $x(u, v)$ and $x(v, u)$; this is given by the following result of Balas and Pulleyblank [2].

THEOREM 2.2. *Let $Z = \{(w, x) \mid Aw + Bx \le b,\ w \ge 0,\ x \ge 0\}$; the projection of Z along the subspace of the w variables is*

$$X = \{x \mid (vB)x \le vb,\ \forall v \in \text{extr } Y,\ x \ge 0\},$$

where extr Y *denotes the set of extreme rays of*

$$Y = \{y \mid yA \ge 0\}.$$

In our case, the rows of the matrix A are of the types below:

$$\begin{array}{rr} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{array}.$$

By observing the extreme rays of the set Y, we can deduce the next theorem.

THEOREM 2.3. *The polytope $P(D)$ is defined by the inequalities* (2.1a), *together with $x(i, j) \geq 0$ and the mixed inequalities*

$$\sum_{(i,j)\in A_1} a_l^1(i, j)x(i, j) + \sum_{(i,j)\in A_2} a_p^2(i, j)x(i, j) \leq \alpha_l^1 + \alpha_p^2 - 1 \tag{2.2}$$

for $(l, p) \in (I_2^1 \times I_3^2) \cup (I_3^1 \times I_2^2)$.

COROLLARY 2.4. *If $P(\bar{D}_1)$ and $P(\bar{D}_2)$ are defined by* (1.1), (1.2), *then $P(D)$ is, also.*

A constraint with coefficients 0 or 1 is called a *rank* inequality. It also follows from Theorem 2.3 that, if $P(\bar{D}_1)$ and $P(\bar{D}_2)$ are defined by rank inequalities, then $P(D)$ is, also.

When we add the arcs (u, v) and (v, u), we could create parallel arcs in $\bar{D}_1$ and $\bar{D}_2$. If this is the case, for every inequality of type (2.1b) or (2.1c), we would have a similar inequality in (2.1a). This observation and Theorem 2.3 imply the next corollary.

COROLLARY 2.5. *The polytope $P(\bar{D})$ is defined by* (2.1) *and* (2.2).

3. Compositions of facets. The purpose of this section is to prove that Theorem 2.3 gives a minimal description of $P(D)$.

LEMMA 3.1. *Inequalities* (2.1b) *and* (2.1c) *define facets of $F(\bar{D}_k)$.*

Proof. Consider (2.1b). Let $ax \leq \alpha$ be one of them and let $S = \{x_1, \ldots, x_p\}$ be the set of extreme points of $P(\bar{D}_k)$ that satisfy $ax = \alpha$. There are $|\bar{A}_k|$ linearly independent vectors in S.

Since $\dim (F(\bar{D}_k)) = |\bar{A}_k| - 1$, we need the same number of linearly independent vectors that satisfy $ax = \alpha$. Let x_r be a vector in S; if $x_r \in F(\bar{D}_k)$, we set $x'_r = x_r$; otherwise, we set $x'_r(i, j) = x(i, j)$ for $(i, j) \neq (v, u)$, and $x'_r(v, u) = 1$. Since we have modified only one component of the vectors in S, the set $S' = \{x'_1, \ldots, x'_p\}$ contains $|\bar{A}_k| - 1$ linearly independent vectors in $F(\bar{D}_k)$ that satisfy $ax = \alpha$. □

Because of the structure of system (2.1) and the lemma above, we can see that, for any inequality (2.1b) or (2.1c), there is no other constraint in (2.1) that defines the same face of $F(\bar{D}_k)$.

THEOREM 3.2. *Inequalities* (2.2) *define facets of $P(D)$.*

Proof. Assume $(l, p) \in I_2^1 \times I_3^2$. We show that there exists a vector $\bar{x} \in P(D)$ that satisfies this inequality as equation and all others as strict inequality.

For the inequality

$$\sum_{(i,j)\in \bar{A}_1} a_l^1(i, j)x(i, j) + x(u, v) \leq \alpha_l^1, \qquad l \in I_2^1,$$

let $S = \{x_1, \ldots, x_r\}$ be the set of extreme points of $F(\bar{D}_1)$ that satisfy it as equation.

For

$$\sum_{(i,j)\in \bar{A}_2} a_p^2(i, j)x(i, j) + x(v, u) \leq \alpha_p^2, \qquad p \in I_3^2,$$

let $T = \{y_1, \ldots, y_s\}$ be the extreme points of $F(\bar{D}_2)$ that satisfy it as equation.

Each vector x_t can be matched with a vector $y_{t'}$ to give a vector in $F(\bar{D})$. Let $\{z_1, \ldots, z_d\}$ be the set thus obtained. Let z'_t be the vector obtained by deleting the

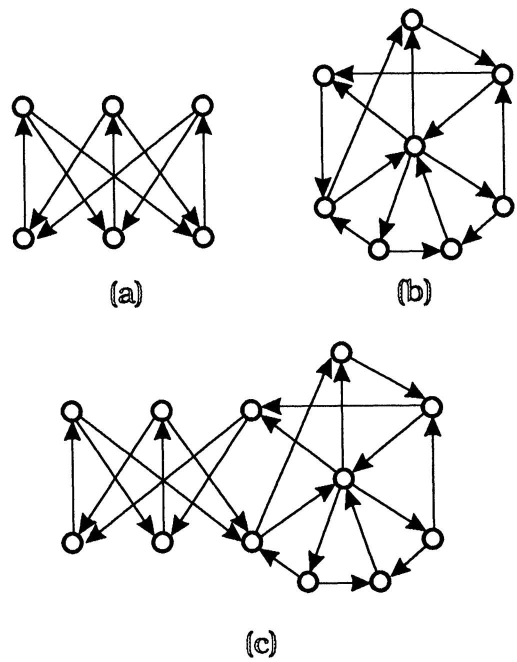

FIG. 3.1

components $z_t'(u, v)$ and $z_t'(v, u)$ from z_t. Then

$$\bar{x} = \frac{1}{d}(z_1' + \cdots + z_d')$$

is the required vector. □

We conclude this section with one example of this composition of facets. Denote by $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ the graphs in Fig. 3.1(a) and 3.1(b), respectively. Let $D = (V, A)$ be the graph in Fig. 3.1(c). Let $\bar{D}_1$ and $\bar{D}_2$ be defined as in §2. Grötschel, Jünger, and Reinelt [7] proved that

$$\sum_{(i,j)\in A_1} x(i,j) \le 7 \tag{3.1}$$

and

$$\sum_{(i,j)\in A_2} x(i,j) \le 11$$

define facets of $P(D_1)$ and $P(D_2)$, respectively; these inequalities also define facets of $P(\bar{D}_1)$ and $P(\bar{D}_2)$, respectively. Theorem 3.2 implies that

$$\sum_{(i,j)\in A} x(i,j) \le 17$$

defines a facet of $P(D)$.

4. Algorithmic aspects. The problem of optimizing a linear function over $P(D)$ can be decomposed in a similar way as shown in this section.

Let $w : \bar{A} \rightarrow \mathfrak{R}_+$ be a weight function. To simplify the notation, we denote by a_1 (respectively, a_2) the arc (u, v) (respectively, (v, u)). Let ω_i^j be the maximum weight of an acyclic subgraph of $\bar{D}_i$ that contains a_j.

Define

$$\alpha = \omega_1^1 - \omega_1^2, \quad \chi = \max\{0, \alpha\}, \quad \kappa = \chi - \alpha.$$

Let $\sigma \in \mathfrak{R}$ such that

$$\chi = \omega_1^1 - \sigma, \qquad \kappa = \omega_1^2 - \sigma. \tag{4.1}$$

Define $w' : \bar{A}_2 \rightarrow \mathfrak{R}_+$ as

$$w'(i,j) = w(i,j) \quad \text{for } (i,j) \in A_2,$$
$$w'(a_1) = \chi,$$
$$w'(a_2) = \kappa.$$

THEOREM 4.1. *The maximum weight of an acyclic subgraph of $\bar{D}$ with respect to w is $\lambda + \sigma$, where λ is the maximum weight of an acyclic subgraph of $\bar{D}_2$ with respect to w'.*

Proof. The maximum weight of an acyclic subgraph of D is

$$\max\{\omega_1^1 + \omega_2^1 - w(a_1), \omega_1^2 + \omega_2^2 - w(a_2)\};$$

it follows from (4.1) that this is equal to

$$\sigma + \max\{\chi + \omega_2^1 - w(a_1), \kappa + \omega_2^2 - w(a_2)\} = \sigma + \lambda. \qquad \square$$

5. Total dual integrality. A system $Ax \leq b$ is called *total dual integral* (TDI) if the dual problem of

$$\max wx \quad \text{s.t. } Ax \leq b$$

has an integer optimal solution for every integer vector w such that the maximum exists. If the system is TDI and b is an integer vector, then $\{x \mid Ax \leq b\}$ has integer extreme points. In this section, we prove that, if systems (2.1) are TDI, then the system given by Corollary 2.5 is also TDI. For $k = 1, 2$, we denote by $\mathscr{P}_k$ the linear program

$$\max wx \quad \text{s.t. (2.1).}$$

The dual problem of $\mathscr{P}_k$ is $\mathscr{D}_k$,

$$\text{minimize} \sum_{p=1,2,3} \sum_{l \in I_p^k} y_l^k \alpha_l^k + y_0^k$$

$$\text{s.t.} \quad \sum_{p=1,2,3} \sum_{l \in I_p^k} a_l^k(i,j) y_l^k \geq w(i,j) \quad \text{for } (i,j) \in A_k,$$

$$\sum_{l \in I_2^k} y_l^k + y_0^k \geq w(u, v),$$

$$\sum_{l \in I_3^k} y_l^k + y_0^k \geq w(v, u),$$

$$y_l^k \geq 0, \qquad y_0^k \geq 0.$$

We denote by $\mathscr{P}_3$ the linear program

$$\max wx \quad \text{s.t. (2.1) and (2.2).}$$

Below is the dual $\mathscr{D}_3$.

$$\text{minimize} \sum_{q=1,2} \sum_{p=1,2,3} \sum_{l \in I_p^q} y_l^q \alpha_l^q + \sum_{(k,l) \in (I_2^1 \times I_3^2) \cup (I_3^1 \times I_2^2)} (\alpha_k^1 + \alpha_l^2 - 1) z_{k,l} + y_0$$

$$\text{s.t.} \quad \sum_{p=1,2,3} \sum_{l \in I_p^1} a_l^1(i,j) y_l^1 + \sum_{(k,l) \in (I_2^1 \times I_3^2) \cup (I_3^1 \times I_2^2)} a_k^1(i,j) z_{k,l} \geq w(i,j) \quad \text{for } (i,j) \in A_1,$$

$$\sum_{p=1,2,3} \sum_{l \in I_p^2} a_l^2(i,j) y_l^2 + \sum_{(k,l) \in (I_2^1 \times I_3^2) \cup (I_3^1 \times I_2^2)} a_l^2(i,j) z_{k,l} \geq w(i,j) \quad \text{for } (i,j) \in A_2,$$

$$\sum_{q=1,2} \sum_{l \in I_2^q} y_l^q + y_0 \geq w(u,v),$$

$$\sum_{q=1,2} \sum_{l \in I_3^q} y_l^q + y_0 \geq w(v,u),$$

$$y_l^q \geq 0, \quad y_0 \geq 0, \quad z_{k,l} \geq 0.$$

Suppose that systems (2.1) are TDI and that the weights w are integer. To prove that $\mathscr{D}_3$ has an integer optimal solution, we must study four cases; we present one of them; the others are similar.

Suppose that we apply the algorithm of §4. Let $\mathscr{A}_2$ be the maximum acyclic arc set in $\bar{D}_2$ with respect to w'. We consider the case where $\alpha \geq 0$ and $(v, u) \notin \mathscr{A}_2$. Thus $\chi = \alpha$, $\kappa = 0$. We can assume that $(u, v) \in \mathscr{A}_2$. There is an integer optimal solution $\bar{y}^2$ of $\mathscr{D}_2$. Complementary slackness implies that

$$\sum_{l \in I_2^2} \bar{y}_l^2 + \bar{y}_0^2 = \alpha.$$

Now let us associate the weights w_1 to the arcs in $\bar{D}_1$, where

$$w_1(i,j) = \begin{cases} w(i,j) & \text{if } (i,j) \neq (v,u), \\ w(v,u) + \alpha & \text{if } (i,j) = (v,u). \end{cases}$$

Since w_1 is integer, there is an integer vector $\bar{y}^1$ that is an optimal solution of $\mathscr{D}_1$. Thus we have

$$\sum_{l \in I_3^1} \bar{y}_l^1 + \bar{y}_0^1 \geq w(v,u) + \alpha,$$

which implies that

$$\sum_{l \in I_3^1} \bar{y}_l^1 + \bar{y}_0^1 \geq \alpha.$$

Suppose that $\bar{y}_0^1 < \alpha$ (the case where $\bar{y}_0^1 \geq \alpha$ is similar). There is a set $\mathscr{I} \subseteq I_3^1$ such that

$$\sum_{l \in \mathscr{I}} \bar{y}_l^1 + \bar{y}_0^1 < \alpha$$

and

$$\sum_{l \in \mathscr{I}} \bar{y}_l^1 + \bar{y}_s^1 + \bar{y}_0^1 \geq \alpha,$$

where $s \in I_3^1 \setminus \mathscr{I}$. Let $\bar{\mathscr{I}} = \mathscr{I} \cup \{s\}$. Define $t = \alpha - (\sum_{l \in \mathscr{I}} \bar{y}_l^1 + \bar{y}_0^1)$. Consider now the system of equations

$$\sum_{k \in \bar{\mathscr{I}}} z_{k,l} + \gamma_l = \bar{y}_l^2 \quad \text{for } l \in I_2^2,$$

$$\sum_{k \in \bar{\mathscr{I}}} \rho_k + \delta = \bar{y}_0^2,$$

$$\sum_{l \in I_2^2} z_{k,l} + \rho_k = \begin{cases} \bar{y}_k^1 & \text{for } k \in \mathscr{I}, \\ t & \text{for } k = s, \end{cases}$$

$$\sum_{l \in I_2^2} \gamma_l + \delta = \bar{y}_0^1.$$

This matrix is totally unimodular; actually, it is a network flow matrix. Therefore the above system has a nonnegative integer solution.

Now consider the vector defined below:

$$\tilde{y}_i^1 = \begin{cases} \bar{y}_i^1 & \text{for } i \in I_1^1 \cup I_2^1 \cup (I_3^1 \setminus \bar{\mathscr{I}}), \\ \rho_i & \text{for } i \in \mathscr{I}, \\ \bar{y}_s^1 - t + \rho_s & \text{if } i = s; \end{cases}$$

$$\tilde{y}_i^2 = \begin{cases} \bar{y}_i^2 & \text{for } i \in I_1^2 \cup I_3^2, \\ \gamma_i & \text{for } i \in I_2^2; \end{cases}$$

$$\tilde{y}_0 = \delta,$$

$$\tilde{z}_{k,l} = \begin{cases} z_{k,l} & \text{for } k \in \bar{\mathscr{I}},\ l \in I_2^2, \\ 0 & \text{otherwise.} \end{cases}$$

The vector $(\tilde{y}, \tilde{z})$ is a feasible solution of $\mathscr{D}_3$ and its value is $\lambda + \sigma$ as defined in Theorem 4.1.

The remaining cases can be treated in a similar way, they are

(1) $\alpha \geq 0$ and $a_2 \in \mathscr{A}_2$,

(2) $\alpha < 0$ and $a_2 \notin \mathscr{A}_2$,

(3) $\alpha < 0$ and $a_2 \in \mathscr{A}_2$.

Therefore the system defined in Corollary 2.5 is TDI.

6. Orientations of K_5. In this section, we prove that system (1.1), (1.2) when associated with $D(K_5)$ is TDI. This has been conjectured by Jünger [8].

Define the linear program

$$\text{(6.1)} \qquad \begin{aligned} &\text{maximize } wx \\ &\text{s.t.} \quad \sum_{(i,j) \in C} x(i,j) \leq |C| - 1 \quad \text{for every directed cycle } C, \\ &\qquad 0 \leq x(i,j) \leq 1 \quad \text{for every arc } (i,j) \end{aligned}$$

and its dual

$$\text{minimize } \sum y_C(|C|-1) + \sum \gamma_{(i,j)}$$

$$\text{(6.2)} \qquad \text{s.t.} \quad \sum_{C\in \mathscr{C}_{(i,j)}} y_C + \gamma_{(i,j)} \geq w(i,j) \quad \text{for each arc } (i,j),$$

$$y \geq 0, \ \gamma \geq 0.$$

Here $\mathscr{C}_{(i,j)}$ denotes the set of cycles that contain (i,j).

Let us denote by $\mathscr{P}$ and $\mathscr{D}$ problems (6.1) and (6.2) when they are associated with $D(K_5)$. We construct $D' = (V', A')$ as follows:

(a) If $w(i,j) > w(j,i) \geq 0$, then $(i,j) \in A'$ and $w'(i,j) = w(i,j) - w(j,i)$;

(b) If $w(i,j) = w(j,i) > 0$, then $(i,j) \in A'$ or $(j,i) \in A'$ but not both, say $(i,j) \in A'$ with $w'(i,j) = 0$;

(c) If $w(i,j) \geq 0 > w(j,i)$, then $(i,j) \in A'$ and $w'(i,j) = w(i,j)$;

(d) If $w(i,j) < 0$ and $w(j,i) < 0$, we do not put any arc between i and j.

Denote by $\mathscr{P}'$ and $\mathscr{D}'$ problems (6.1) and (6.2) when they are associated with D'. Let $\bar{x}$ and $(\bar{y}, \bar{\gamma})$ be optimal solutions of $\mathscr{P}'$ and $\mathscr{D}'$, respectively; we can construct optimal solutions of $\mathscr{P}$ and $\mathscr{D}$ as shown below.

Set $\bar{\bar{x}}(i,j) = \bar{x}(i,j)$, $\bar{\bar{x}}(j,i) = 1 - \bar{x}(i,j)$ if (a) or (b) holds; $\bar{\bar{x}}(i,j) = \bar{x}(i,j)$, $\bar{\bar{x}}(j,i) = 0$ if (d) holds; $\bar{\bar{x}}(i,j) = \bar{\bar{x}}(j,i) = 0$ if (d) holds; $\bar{\bar{y}}_C = \bar{y}_C$ if C is a cycle of D'; $\bar{\bar{y}}_C = \bar{\gamma}(i,j) + w(j,i)$ if $C = \{(i,j),(j,i)\}$ and (a) or (b) holds; $\bar{\bar{y}}_C = \bar{\gamma}(i,j)$ if $C = \{(i,j),(j,i)\}$ and (c) holds; $\bar{\bar{y}}_C = 0$ otherwise; and $\bar{\bar{\gamma}}_{(i,j)} = 0$ for all (i,j). Therefore, instead of studying $D(K_5)$, we study the different orientations of K_5.

In the remainder of this section, $D = (V, A)$ denotes an orientation of K_5. We first prove that (6.1) defines a polytope with integral extreme points. Let $\bar{x}$ be an extreme point; the following remarks allow us to rule out many cases.

Remark 6.1. If there is an arc (i,j) that does not belong to any cycle, then $\bar{x}$ is integer-valued.

Proof. If (i,j) does not belong to any cycle, then the nontrivial inequalities of (6.1) are associated with $D\backslash(i,j)$, and this is a planar graph. □

Remark 6.2. If $\bar{x}(i,j) = 0$ for an arc (i,j), then $\bar{x}$ is integer-valued.

Proof. Consider $D' = D\backslash(i,j)$ and let x' be $\bar{x}$ without the component associated with (i,j). Since D' is planar, we have that $x' \in P(D')$ and x' is a convex combination of a set of vectors $\{x_j\}$ incidence vectors of acyclic subgraphs of D'. For each vector x_j, we can add a zero component and obtain the incidence vector of an acyclic subgraph of D. We have then that $\bar{x}$ is a convex combination of them. □

Remark 6.3. For any set S, $\varnothing \neq S \subset V$, there is at least one arc $(i,j) \in \delta^+(S) \cup \delta^-(S)$ with $\bar{x}(i,j) = 1$.

Proof. If $0 < \bar{x}(i,j) < 1$ for every arc $(i,j) \in \delta^+(S) \cup \delta^-(S)$, define x' as

$$x'(i,j) = \begin{cases} \bar{x}(i,j) + \varepsilon & \text{if } (i,j) \in \delta^+(S), \\ \bar{x}(i,j) - \varepsilon & \text{if } (i,j) \in \delta^-(S), \\ \bar{x}(i,j) & \text{otherwise.} \end{cases}$$

For ε sufficiently small, x' satisfies (6.1), and, if some of these inequalities hold as equation for $\bar{x}$, they also do for x'. This contradicts the assumption that $\bar{x}$ is an extreme point. □

Remark 6.4. It follows from Remark 6.1 that we can assume that D is strongly connected.

Remark 6.5. It follows from Remark 6.3 that we can assume that there is a tree $\mathscr{T}$ of arcs (i, j) with $\bar{x}(i, j) = 1$.

Remark 6.6. For every variable $x(i, j)$ with $0 < \bar{x}(i, j) < 1$, we can assume that it appears in at least two tight cycle constraints.

Proof. If $0 < \bar{x}(i, j) < 1$ and $x(i, j)$ appears in only one tight constraint, we can set $x(i, j) = 0$; this new vector is also an extreme point, and, from Remark 6.2, we can conclude that it is integral. This gives a contradiction. □

Remark 6.7. If there is a node v that covers every cycle, then $\bar{x}$ should be integer-valued.

Proof. Consider $D' = (V', A')$, where $V' = V\backslash\{v\} \cup \{s, t\}$ and A' is defined below:

$$(i, j) \in A, \quad i \neq v, \quad j \neq v \Rightarrow (i, j) \in A',$$

$$(v, i) \in A \Rightarrow (s, i) \in A',$$

$$(i, v) \in A \Rightarrow (i, t) \in A'.$$

Let M be the incidence matrix of all directed paths from s to t in D'; it is well known that, for any $w \geq 0$, the problem

$$\text{minimize } wx \quad \text{s.t. } Mx \geq 1, x \geq 0$$

has an integer-valued optimal solution $\tilde{x}$. Since M is also the incidence matrix of the cycles in D, the vector $\hat{x}$ defined by

$$\hat{x}(i, j) = 1 - \tilde{x}(i, j) \quad \text{for } (i, j) \in A$$

is an optimal solution of (6.1). □

Now we can prove the following result.

THEOREM 6.8. *All the extreme points of the polyhedron defined by* (6.1) *are integral.*

Proof. There are three cases to study.

Case 1. The tree $\mathscr{T}$ contains a directed path with two arcs; i.e., suppose that $\bar{x}(1, 2) = \bar{x}(2, 3) = 1$. We have that $(1, 3) \in A$. Consider the cycle inequalities that are tight for $\bar{x}$; if there is a cycle that contains (1, 3) and goes through 2, it should be $C = (1, 3, 4, 2, 5, 1)$. Since

$$\bar{x}(2, 3) + \bar{x}(3, 4) + \bar{x}(4, 2) \leq 2,$$

$$\bar{x}(2, 5) + \bar{x}(5, 1) + \bar{x}(1, 2) \leq 2,$$

$$\bar{x}(1, 3) \leq 1,$$

we have that

$$\bar{x}(1, 3) + \bar{x}(3, 4) + \bar{x}(4, 2) + \bar{x}(2, 5) + \bar{x}(5, 1) \leq 3,$$

a contradiction.

Therefore we should assume that every cycle containing (1, 3) does not go through 2.

Let C be a cycle that contains (1, 3); if the constraint associated with C is tight, then $\bar{x}(1, 3) = 1$; otherwise, the constraint

$$\sum_{(i,j) \in C'} x(i, j) \leq |C'| - 1$$

would be violated, where $C' = C\backslash\{(1, 3)\} \cup \{(1, 2), (2, 3)\}$. Then we must only study the case where $\bar{x}(1, 3) = 1$.

Now consider the graph $D' = D\backslash(1, 3)$. Let $\bar{x}'$ be the restriction of $\bar{x}$ to the arc set of D'. We have that $\bar{x}' = \sum \lambda_i y_i$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, where the vectors $\{y_i\}$ are incidence vectors of acyclic subgraphs of D'.

Now define $\bar{y}_i$ as follows:

$$\bar{y}_i(k, l) = y_i(k, l) \quad \text{if } (k, l) \neq (1, 3),$$

$$\bar{y}_i(1, 3) = 1 \quad \text{for all } i.$$

The vectors $\{\bar{y}_i\}$ are incidence vectors of acyclic subgraphs of D, and $\bar{x} = \sum \lambda_i \bar{y}_i$; then $\bar{x}$ should be an integer vector.

Case 2. The tree $\mathscr{T}$ is $\{(1, 2), (3, 2), (3, 4), (3, 5)\}$. We can assume that $(4, 5) \in A$. We should assume that $(1, 3) \in A$; otherwise, there is no cycle going through 3. We also assume that $(5, 1) \in A$; otherwise, 4 covers every cycle. For the same reasons, we assume that $(4, 1) \in A$. Also, we should have that $(5, 2)$ and $(2, 4)$ are in A; otherwise, 1 would cover every cycle. See Fig. 6.1.

We have that

$$\begin{aligned} &\bar{x}(2, 4) + \bar{x}(4, 1) + \bar{x}(1, 3) \leq 2, \\ &\bar{x}(2, 4) + \bar{x}(4, 1) + \bar{x}(1, 3) + \bar{x}(5, 2) \leq 3. \end{aligned} \tag{6.3}$$

Thus (6.3) holds as equation only if $\bar{x}(5, 2) = 1$; then, however, we would be in Case 1. If (6.3) does not hold, we would have that $(5, 2)$ only appears in one tight cycle inequality; then we can apply Remark 6.6.

Case 3. The tree $\mathscr{T}$ is $\{(1, 2), (3, 2), (3, 4), (5, 4)\}$. We can assume that $(2, 4) \in A$. Therefore $(4, 1) \in A$; otherwise, there is no cycle going through 4. Then every cycle containing $(2, 4)$ also contains $(4, 1)$. Consider the cycle $C = \{(2, 4), (4, 1), (1, 2)\}$; we have that

$$\bar{x}(2, 4) + \bar{x}(4, 1) \leq 1.$$

Thus, if there is any other tight cycle inequality containing $(2, 4)$, all its variables different from $\bar{x}(2, 4)$ and $\bar{x}(4, 1)$ should take the value 1; this is Case 1. This concludes Case 3 and the proof of the theorem. □

It remains to prove that (6.1) defines a TDI system. This has been proved by Applegate, Cook, and McCormick [1] using an algorithm that tests whether a system is TDI. We present here a proof that does not involve computer calculations.

Suppose that, for every integer vector $w \leq \mu$, $w \neq \mu$, problem (6.1) has an integer dual solution and let $z(w)$ be its value. Now we study the weights $w = \mu$; we should

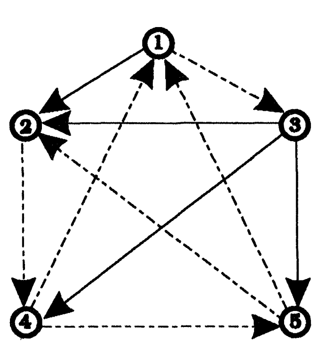

FIG. 6.1

assume that $\mu > 0$. If one component of μ is zero, then the associated arc can be removed yielding to a planar graph. Consider the set of inequalities that are tight for every optimal solution of (6.1).

Case 1. Assume that

$$x(i, j) \leq 1 \tag{6.4}$$

is tight; then we can define w' as

$$w'(k, l) = \begin{cases} w(k, l) & \text{if } (k, l) \neq (i, j), \\ w(i, j) - 1 & \text{if } (k, l) = (i, j). \end{cases}$$

We have that $z(w') = z(w) - 1$, and there is an integer dual solution for the objective function w'. We increase by 1 the value of the dual variable associated with (6.4) and we have a dual integer solution for the vector w.

Case 2. Consider now a cycle C of length 3, $C = \{(1, 2), (2, 3), (3, 1)\}$ say. Assume that the constraint associated with C is tight. Define w' by subtracting 1 from the costs coefficients of the arcs in C.

LEMMA 6.9. *The value of the new optimum is* $z(w') = z(w) - 2$.

Proof. If $z(w') = z(w)$, there is an optimal solution for the new objective function that does not contain any arc of C. However, we can always add one of the arcs of C to that solution without creating a cycle. This gives a solution for the original problem with value $z(w) + 1$, which is a contradiction.

If $z(w') = z(w) - 1$, there is an optimal solution for the new problem that contains one of the arcs of C. This is also an optimal solution for the objective function w, which is impossible because the constraint associated with C is tight for every optimum of the original problem. $\square$

Given an integer optimal dual vector for the objective function w', we increase by 1 the value of the dual variable associated with C and we obtain a dual optimal solution for w.

Case 3. A cycle of length 4 is tight. In this case, it is easy to see that there is always a cycle of length 3 that is tight, and we are in Case 2.

Case 4. A cycle of length 5 is tight. In this case, there is also a cycle of length 3 or 4 that is tight.

We can then state the main result of this section.

THEOREM 6.10. *For* $D(K_5)$, *system* (6.1) *is* TDI.

7. Graphs with no $K_{3,3}$ minor. Grötschel, Jünger, and Reinelt [7] proved that, if D is a subdivision of the graph of Fig. 3.1(a), then an inequality analogous to (3.3) defines a facet of $P(D)$. Therefore, if G contains $K_{3,3}$ as a minor, then system (1.1), (1.2) is not sufficient to define $P(D(G))$. Wagner [10] proved that, if an undirected graph G has no $K_{3,3}$ minor, then either it is planar, it is K_5, or it has a two-vertex cut. We know that, if G is planar or it is K_5 then $P(D(G))$ is defined by (1.1), (1.2), and this is a TDI system. This and the result of §5 imply the following result.

THEOREM 7.1. *Given a graph G, system* (1.1), (1.2) *defines* $P(D(G))$ *and is* TDI *if and only if G does not contain* $K_{3,3}$ as a minor.

An immediate consequence is the result below.

COROLLARY 7.2. *If D is a directed graph that does not contain a subdivision of* $K_{3,3}$ *then the cardinality of a minimum feedback set is equal to the maximum number of arc-disjoint cycles.*

For planar graphs, this follows from the theorem of Lucchesi and Younger [9].

Let A be a matrix with nonnegative entries. Let $P = \{x \mid Ax \geq 1, x \geq 0\}$ and let us assume that this system is not redundant. If B is the matrix whose rows are the extreme points of P, then the pair (A, B) is called a blocking pair; see Fulkerson [4]. Actually, extreme points of $Q = \{x \mid Bx \geq 1, x \geq 0\}$ are the rows of A.

Theorem 7.1 implies the next corollary.

COROLLARY 7.3. *Given an undirected graph G, let A be the incidence matrix of the directed cycles of $D(G)$ and let B be the incidence matrix of all minimal feedback sets of $D(G)$. We have that (A, B) is a blocking pair if and only if G has no $K_{3,3}$ minor.*

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