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# $k$-edge connected polyhedra on series–parallel graphs

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**Abstract**

We give a complete description of the $k$-edge connected spanning subgraph polytope (for all $k$) on series–parallel graphs.

*Keywords:* $k$-edge connected subgraphs; Polytopes; Series–parallel graphs

## 1. Introduction

The graphs we consider are finite, undirected, loopless and may have multiple edges. A graph is denoted by $G=(V,E)$ where $V$ is the set of *nodes* and $E$ is the set of *edges* of $G$.

A graph $G=(V,E)$ is called *k-edge connected* (where $k$ is a positive integer) if for any pair of nodes $i, j \in V$, there are at least $k$ edge-disjoint paths from $i$ to $j$.

Given a graph $G=(V,E)$ and a weight function $\omega$ on $E$ that associates with an edge $e \in E$, the weight $\omega(e) \in \mathbb{R}$, the *k-edge connected spanning subgraph problem* (*kECSP* for short) is to find a $k$-edge connected subgraph $H=(V,F)$ of $G$, spanning all the nodes in $V$, such that $\sum_{e \in F} \omega(e)$ is minimum.

The *kECSP* arises in the design of communication and transportation networks [4,7,26,27]. It is NP-hard for $k \geqslant 2$. For $k=1$, the problem reduces to the minimum spanning tree problem and thus can be solved in polynomial time.

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If $G=(V,E)$ is a graph and $F \subseteq E$, the incidence vector of $F$ will be denoted by $x^F$. The convex hull of the incidence vectors of all edge sets of $k$-edge connected spanning subgraphs of $G$ is called the *k-edge connected spanning subgraph polytope* and denoted by *kECP*$(G)$, that is,

$$kECP(G) = \mathrm{conv}\{x^F \in \mathbb{R}^E \mid (V,F) \text{ is } k\text{-edge connected spanning subgraph of } G\}.$$

In this paper we discuss the polytope *kECP*$(G)$, we give a complete description of this polytope on series–parallel graphs.

The *kECSP* and the polytope *kECP*$(G)$ have received special attention in the past few years. Grötschel and Monma [19] and Grötschel et al. [20–22] studied the *kECSP* within the framework of a more general model related to the design of minimum cost survivable networks. In [19] basic facets of *kECP*$(G)$ are described. In [20–22], further classes of valid inequalities and facets are characterized, and a cutting plane algorithm along with computational results are discussed. A complete survey of that model can be found in [27].


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In [5], Chopra considered a relaxation of *kECSP*, namely when multiple copies of an edge are allowed. The problem here is to determine an integer vector $x \in \mathbb{N}^E$ such that the graph $(V, E(x))$, where $E(x)$ is the set of edges obtained by replacing each edge $e$ of $E$ by $x(e)$ edges, is $k$-edge connected and $\sum_{e\in E}\omega(e)x(e)$ is minimum. Chopra [5] studied the polyhedron $P_k(G)$, associated with the solutions to that problem, that is,

$$P_k(G) = \mathrm{conv}\{x \in \mathbb{N}^E | (V, E(x)) \text{ is } k\text{-edge connected}\}.$$

He described various classes of facets of $P_k(G)$ and gave a complete description of $P_k(G)$ when $G$ is outerplanar and $k$ is odd. (A graph is outerplanar if it consists of a cycle with noncrossing chords.) The polyhedron $P_k(G)$ has previously been studied by Cornuéjols et al. [8]. They showed that when $k = 2$ and the graph is series–parallel, $P_k(G)$ is completely characterized by the non-negativity inequalities and the cut inequalities. Their proof can easily be generalized to show that these inequalities also suffice to characterize $P_k(G)$ for all even $k$, when $G$ is series–parallel.

The polytope $kECP(G)$ has been characterized for $k=1$, for general graphs and for $k=2$, for some classes of graphs. Using Edmonds' characterization of matroid polytopes [13, 14], Grötschel and Monma [19] (see also [8]) showed that the so-called partition inequalities together with the trivial inequalities suffice to describe the $1ECP(G)$. Fonlupt and Naddef [16] characterized the class of graphs $G$ for which the polyhedron described by the non-negativity inequalities and the cut inequalities, when $k = 2$, is the convex hull of the tours of G (a tour is a cycle going at least once through each node of $G$). Mahjoub [23] showed that when $G$ is series–parallel, $2ECP(G)$ is completely described by the trivial inequalities and the cut inequalities. Barahona and Mahjoub [3] characterized $2ECP(G)$ for the class of Halin graphs. Baïou and Mahjoub [1] characterized the Steiner 2-edge connected subgraph polytope for series–parallel graphs.

Related work can also be found in [9, 10, 17, 24]. In [9, 10], Coullard et al. studied the steiner 2-node connected subgraph polytope. In [24] Margot et al. (see also [17] ) gave an extended formulation for the Steiner tree problem and showed that it is a complete linear description of the associated polytope when the graph is a 2-tree (a maximal series–parallel graph). In [17] Goemans also gave large classes of facets for the Steiner tree polytope.

In this paper we are going to study $kECP(G)$ for all $k$, when $G$ is series–parallel. In the next section we discuss some valid inequalities for $kECP(G)$. In Section 3 we give a complete description of this polytope for that class of graphs.

The remainder of this section is devoted to more definitions and notations.

A *homeomorph* of $K_4$ (the complete graph on 4 nodes) is a graph obtained from $K_4$ where its edges are subdivided into paths by inserting new nodes of degree two. A graph $G = (V, E)$ is called *series–parallel* if it does not contain a homeomorph of $K_4$ as a subgraph. Connected series–parallel graphs have the following property [12].

**Lemma 1.1.** *If $G = (V,E)$ is a connected series–parallel graph with $|V| \geqslant 3$, then $G$ contains a node that is adjacent to exactly two nodes in $G$.*

Given a graph $G=(V,E)$ and $W \subseteq V$, the set of edges having one node in $W$ and the other one in $V \setminus W$ is called a *cut* and is denoted by $\delta(W)$. If $W = \{v\}$ for some $v \in V$, then we write $\delta(v)$ for $\delta(W)$. If $G = (V,E)$ is a graph and $e \in E$, then $G - e$ will denote the graph obtained from $G$ by removing $e$. If $W \subseteq V$, then $G(W)$ denotes the induced subgraph of $G$ on $W$. If $W_1$, $W_2$ are disjoint subsets of $V$, then $[W_1, W_2]$ denotes the set of edges of $G$ which have one node in $W_1$ and the other one in $W_2$.

Given a constraint $ax \geqslant \alpha$, $a \in \mathbb{R}^E$, and a solution $x^*$, we will say that $ax \geqslant \alpha$ is tight for $x^*$ if $ax^* = \alpha$.

In the sequel we consider $k$-edge connected graphs. A basic knowledge of polyhedral combinatorics is assumed. Undefined polyhedral terminology and notation are consistent with that of Pulleyblank [25].

## 2. Valid inequalities for kECP($G$)

Let $G=(V,E)$ be a graph. Given a function $b : E \to \mathbb{R}$ and $F \subseteq E$, $b(F)$ will be used to denote $\sum_{e\in F} b(e)$. If $(V, F)$ is a $k$-edge connected spanning subgraph of $G$, then $x^F$, the incidence vector of $F$, must satisfy the following inequalities:

$$x(e)\geqslant 0 \quad \text{for all } e \in E, \tag{2.1}$$

$$x(e)\leqslant 1 \quad \text{for all } e \in E, \tag{2.2}$$

$$x(\delta(W))\geqslant k \quad \text{for all } W \subset V,\ W \neq \emptyset. \tag{2.3}$$

The inequalities (2.1) and (2.2) are called *trivial inequalities* and the inequalities (2.3) are called *cut inequalities*. In [19] Grötschel and Monma gave necessary and sufficient conditions for inequalities (2.1)–(2.3) to define facets for *kECP*($G$).

If $(V_1,\ldots,V_p)$ $(p\geqslant 2)$ is a partition of $V$, then $\delta(V_1,\ldots,V_p)$ will denote the set of edges having nodes in different members of the partition. The following inequalities have been introduced by Chopra [5] and shown to be valid for the polyhedron $P_k(G)$ when $G$ is outerplanar and $k$ is odd:

$$x(\delta(V_1,\ldots,V_p))\geqslant \lceil \tfrac{k}{2}\rceil p - 1,$$
$$\text{for all partition } (V_1,\ldots,V_p) \text{ of } V,$$
$$\text{such that } G(V_i) \text{ is connected for } i = 1,\ldots,p. \tag{2.4}$$

Inequalities (2.4) are called *outerplanar partition inequalities*. Chopra [5] showed that when $G$ is outerplanar and $k$ is odd, $P_k(G)$ is completely described by inequalities (2.1) and (2.4). Also he conjectured that this remains true even when the graph $G$ is series-parallel. In Section 3 we obtain this as a consequence of a more general result. We will show that when $G$ is series-parallel and $k$ is odd, the polytope *kECP*($G$) is completely described by inequalities (2.1), (2.2), (2.4). To this end, let us first show that inequalities (2.4) remain valid for *kECP*($G$) when $G$ is series-parallel.

**Theorem 2.1.** *If $G=(V,E)$ is series–parallel and $k$ is odd, then inequalities* (2.4) *are valid for kECP*($G$).

**Proof.** It suffices to show that, for every $k$-edge connected series parallel graph $G=(V,E)$, we have

$$|E|\geqslant \lceil \tfrac{k}{2}\rceil |V| - 1.$$

The proof is by induction on $|V|$. If $|V| = 2$, constraint (2.4) is a cut constraint and so it is valid for *kECP*($G$). So let us assume that (2.4) is valid for every $k$-edge connected series-parallel graph with no more than $n$ nodes and suppose that $G$ is $k$-edge connected with $n+1$ nodes. Since $G$ is series-parallel, by Lemma 1.1 there is a node $v \in V$ that is adjacent to exactly two nodes, say $v_1$, $v_2$ in $V$. Let us denote by $F_1$ ($F_2$) the set of edges between $v$ and $v_1$ ($v_2$). W.l.o.g we may assume that $|F_1|\geqslant|F_2|$. Since $G$ is $k$-edge connected, we have $|F_1|\geqslant\lceil k/2\rceil$ (otherwise, the cut $\delta(v)$ would have less than $k$ edges). Let $G^* = (V^*,E^*)$ be the graph obtained from $G$ by contracting the edges of $F_1$. Obviously, the graph $G^*$ is series-parallel and $k$-edge connected. Furthermore, since $|V^*|=n$, by the induction hypothesis it follows that

$$|E^*|\geqslant \lceil \tfrac{k}{2}\rceil |V^*| - 1.$$

We have

$$\begin{aligned}|E| &= |E^*| + |F_1|\\ &\geqslant \lceil \tfrac{k}{2}\rceil |V^*| - 1 + \lceil \tfrac{k}{2}\rceil\\ &= \lceil \tfrac{k}{2}\rceil (|V^*| + 1) - 1\\ &= \lceil \tfrac{k}{2}\rceil |V| - 1. \quad \square\end{aligned}$$

Given a graph $G=(V,E)$ and a partition $\pi = (V_1,\ldots,V_p)$ of $V$, we let $G_\pi=(V_\pi, E_\pi)$ be the graph obtained from $G$ by contracting the sets $V_i$. Theorem 2.1 implies that, if $G=(V,E)$ is a graph (not necessarily series–parallel) and $\pi = (V_1,\ldots,V_p)$ is a partition of $V$ such that $G_\pi$ is series–parallel, then the associated constraint (2.4) is valid for *kECP*($G$). Unfortunately, this is no longer true if $G_\pi$ is not series–parallel. In fact, consider, for instance, the graph $K_4=(V,E)$ and let $k=3$. Also consider the inequality (2.4) where $p = 4$ and $|V_i|=1$ for $i = 1,\ldots,4$. The right-hand side of inequality (2.4), in this case, is 7 whereas the graph $K_4$, which is 3-edge connected, contains only 6 edges.

**Remark 2.2.** For $k$ even, inequalities (2.4) are implied by the cut inequalities, and for $k$ odd the former inequalities, include the latter ones.

We will refer to inequalites (2.4) as *series–parallel partition inequalities* (SP-partition inequalities).

Grötschel et al. [20] introduced a general class of partition inequalities which are valid for *kECP*($G$). If $G = (V,E)$ is a graph, and $V_1,\ldots,V_p$ is a partition of $V$, then the associated partition inequality is defined as

$$x(\delta(V_1,\ldots,V_p)) \geqslant \begin{cases} p-1 & \text{if } k=1, \\ \lceil \frac{kp}{2} \rceil & \text{if } k \neq 1. \end{cases} \quad (2.5)$$

In [20], necessary and sufficient conditions are given for inequalities (2.5) to define facets for *kECP*($G$).

Notice that, in the case where a partition $V_1,\ldots,V_p$ of $V$ yields a series–parallel graph, the associated partition inequality (2.5) coincides with the corresponding SP-partition inequality, if $k=1$. For $k \geqslant 2$, we have that the former inequality is implied by the latter one.

The separation problem for the SP-partition inequalities (and the partition inequalities (2.5)) can be solved as a sequence of $|E|$ max-flow problems using an algorithm of Cunningham [11] or as $|V|$ max-flow problems using an algorithm of Barahona [2]. From the ellipsoid method it follows that the *kECSP* can be solved in polynomial time in the graphs for which the polytope *kECP*($G$) can be described by the trivial inequalities and the SP-partition inequalities. In the next section we shall show that series–parallel graphs are among those graphs.

In what follows we give some necessary conditions for SP-partition inequalities to define facets for *kECP*($G$).

**Theorem 2.3.** *Let $G=(V,E)$ be a graph and $\pi = (V_1,\ldots,V_p)$ a partition of $V$. Then the inequality* (2.4) *defines a facet for kECP($G$), different from a face defined by a bound inequality* (2.2) *when $k$ is odd, only if*

(i) *$G(V_i)$ is connected, for $i=1,\ldots,p$,*

(ii) *the graph $G_\pi = (V_\pi, E_\pi)$ is 2-node connected,*

(iii) *for every edge $e \in E_\pi$ such that $G-e$ is $k$-edge connected, $G_\pi - e$ is 2-node connected,*

(iv) *if $e_0$ is an edge of $G(V_i)$, for some $i \in \{1,\ldots,p\}$, such that $G-e_0$ is $k$-edge connected, and $(V_i^1, V_i^2)$ is a partition of $V_i$ such that $e_0 \in [V_i^1, V_i^2]$, then $|[V_i^1, V_i^2]| \geqslant \lceil k/2 \rceil + 1$.*

**Proof.** (i)–(iii) are easily seen to be true.

(iv) Assume the contrary. Then

$$|[V_i^1, V_i^2]| \leqslant \lceil \tfrac{k}{2} \rceil. \quad (2.6)$$

Let $F \subseteq E$ be an edge subset inducing a $k$-edge connected subgraph such that $x^F$ satisfies (2.4) with equality. We claim that $e_0 \in F$. Indeed, if this is not the case, by (2.6) together with the bound inequalities (2.2) it follows that

$$x^F([V_i^1, V_i^2]) < \lceil \tfrac{k}{2} \rceil.$$

Let $\pi' = (V_1',\ldots,V_{p+1}')$ be the partition such that

$$V_t' = V_t \quad \text{for } t = 1,\ldots,i-1,$$
$$V_i' = V_i^1,$$
$$V_{i+1}' = V_i^2,$$
$$V_t' = V_{t-1} \quad \text{for } t = i+2,\ldots,p+1.$$

We have

$$\begin{aligned} x^F(\delta(V_1',\ldots,V_{p+1}')) &= x^F(\delta(V_1,\ldots,V_p)) \\ &\quad + x^F([V_i^1, V_i^2]) \\ &< \lceil \tfrac{k}{2} \rceil |V_\pi| - 1 + \lceil \tfrac{k}{2} \rceil \\ &= \lceil \tfrac{k}{2} \rceil |V_{\pi'}| - 1, \end{aligned}$$

a contradiction. Consequently, for any edge subset $F$ inducing a $k$-edge connected spanning subgraph of $G$, such that $x^F$ satisfies (2.4) with equality, we have $e_0 \in F$. This implies that the face defined by inequality (2.4) is the same as the one defined by the inequality $x(e_0) \geqslant 1$. But this is a contradiction. □

## 3. The kECP($G$) of a series–parallel graph

In this section we are going to show that the SP-partition inequalities together with the trivial inequalities completely describe *kECP*($G$) when $G$ is series–parallel and $k$ is odd. When $k$ is even, we shall show that the cut inequalities together with the trivial inequalities completely describe the polytope for that class of graphs. This generalizes the results known for $k=1$ [19] and $k=2$ [23].

Let $G=(V,E)$ be a graph. Let $P(G,k)$ be the polyhedron defined by the inequalities (2.1)–(2.4). If $x$ is a solution of $P(G,k)$ and $\delta(W)$ (($V_1,\ldots,V_p$)) is a cut (partition) of $G$ whose the associated inequality (2.3) ((2.4)) is satisfied with equality by $x$, then we say that $\delta(W)$ (($V_1,\ldots,V_p$)) is *tight* for $x$.

In what follows we give a technical lemma which will be useful in the proof of the main result.

**Lemma 3.1.** *Let $x$ be a solution of $P(G,k)$.*

(i) *If* $\pi = (V_1,\ldots,V_p)$ *is a partition of* $V$ *tight for* $x$ *then*

$$x([V_i,\ V_j]) \leqslant \lceil \tfrac{k}{2} \rceil \quad \text{for all } i,j \in \{1,\ldots,p\}. \tag{3.1}$$

*Moreover, if* (3.1) *is satisfied with equality for* $i,j$, $i<j$, *then the partition* $\pi' = (V'_i,\ldots,V'_{p-1})$ *such that*

$$V'_t = V_t \quad \text{for } t = 1,\ldots,i-1,\ i+1,\ldots,j-1,$$
$$V'_i = V_i \cup V_j,$$
$$V'_t = V_{t+1} \quad \text{for } t = j,\ldots,p-1$$

*is also tight for* $x$.

(ii) *If* $\pi = (V_1,\ldots,V_p)$ $(\delta(W))$ *is a partition (a cut) tight for* $x$ *and* $V'$, $V''$ *is a partition of* $V_i$ *for* $i \in \{1,\ldots,p\}$, *then* $x([V',\ V'']) \geqslant \lceil k/2 \rceil$.

(iii) *If* $\delta(W_1)$ *and* $\delta(W_2)$ *are two cuts tight for* $x$, *then* $x(\delta(W_1) \cap \delta(W_2)) \leqslant \lceil k/2 \rceil$.

**Proof.** We will show (i) (the proof for (ii) and (iii) is similar). Consider the partition $\pi' = (V'_1,\ldots,V'_{p-1})$ defined in the second part of the statement. We have

$$\begin{aligned} x(V'_1,\ldots,V'_{p-1}) &= x(V_1,\ldots,V_p) - x[V_i,\ V_j] \\ &= \lceil \tfrac{k}{2} \rceil p - 1 - x[V_i,\ V_j] \\ &\geqslant \lceil \tfrac{k}{2} \rceil (p-1) - 1, \end{aligned}$$

which implies that $x[V_i,\ V_j] \leqslant \lceil k/2 \rceil$.

Furthermore, if $x[V_i, V_j] = \lceil k/2 \rceil$, then the above inequality is satisfied as equation and thus $(V'_1,\ldots,V'_{p-1})$ is tight for $x$. □

**Theorem 3.2.** *If* $G=(V,E)$ *is a series–parallel graph and* $k$ *is even (odd), then kECP(G) is completely described by inequalities* (2.1), (2.2), (2.3) ((2.1), (2.2), (2.4)).

**Proof.** We will show that $kECP(G) = P(G,k)$, the theorem thus follows from Remark 2.2. The proof is by induction on $|E|$. If $G$ consists of two nodes joined by $k$ edges, then clearly $kECP(G) = P(G,k)$. So suppose that the claim holds for every series–parallel graph with no more than $m$ edges and suppose that $G$ has $m+1$ edges. Since inequalities (2.1)–(2.4) are valid for $kECP(G)$, we have $kECP(G) \subseteq P(G,k)$. Also, any integer solution of $P(G,k)$ belongs to $kECP(G)$. If $kECP(G) \neq P(G,k)$, then there must exist a fractional extreme point $x$ of $P(G,k)$, which by the induction hypothesis, must satisfy

$$x(e) > 0 \quad \text{for all } e \in E. \tag{3.2}$$

Let $E_1$ be the set of edges $e$ such that $x(e) = 1$. Since $x$ is an extreme point of $P(G,k)$, there must exist a set $\{\delta(W_i),\ i = 1,\ldots,r\}$ of tight cuts and a set $\{\pi_1,\ldots,\pi_s\}$ of tight partitions of $V$ with $|V_{\pi_j}| \geqslant 3$ for $j = 1,\ldots,s$, such that $x$ is the unique solution of the linear system

$$\begin{cases} x(e) = 1 & \text{for all } e \in E_1, \\ x(\delta(W_i)) = k & \text{for } i = 1,\ldots,r, \\ x(E_{\pi_j}) = \lceil \tfrac{k}{2} \rceil |V_{\pi_j}| - 1, & \text{for } j = 1,\ldots,s, \end{cases} \tag{3.3}$$

where $r + s + |E_1| = |E|$.

**Claim 1.** *Each variable* $x(e)$ *has a nonzero coefficient in at least two equations of* (3.3).

**Proof.** It is clear that each variable $x(e)$ must have a nonzero coefficient in at least one of the equations of (3.3). Suppose that for an edge $e_0 = uv \in E$, $x(e_0)$ has a nonzero coefficient in exactly one equation of (3.3). Let $F$ be the set of edges between $u$ and $v$ and let $G' = (V', E')$ be the graph obtained by contracting the edges of $F$. Let $x'$ be the restriction of $x$ on $E'$. Clearly, $x' \in P(G',k)$. We claim that $x'$ is an extreme point of $P(G',k)$. Indeed, since any cut (partition) either contains all the edges of $F$ or does not intersect this set, it follows that at most one of the edges of $F$ may have a fractional value. We claim that there are exactly $|F|$ equations of the system (3.3) that involve variables $x(e)$ with $e \in F$. In fact, if $x(e_0) = 1$, then $x(e) = 1$ for all $e \in F$. Otherwise, there exists an edge $e' \in F \setminus \{e_0\}$ with $0 < x(e') < 1$. And thus at least one of the equations of the system (3.3) different from $x(e_0) = 1$ must contain $x(e')$. Since this equation also contains $x(e_0)$, it follows that $x(e_0)$ has a nonzero coefficient in at least two equations of the system (3.3), a contradiction. Consequently, $x(e) = 1$ for all $e \in F$. Moreover, these equations are the only equations involving the edges of $F$. If $0 < x(e_0) < 1$, then, by the remark above, $x(e) = 1$ for all $e \in F \setminus \{e_0\}$. Moreover, any edge $e \in F \setminus \{e_0\}$ cannot be in a cut or a partition among those defining system (3.3) other than that containing $e_0$. Otherwise, $x(e_0)$ would have nonzero coefficients in more than one equation of (3.3). Thus, the only equations of (3.3) that involve edges of $F$ are $x(e) = 1$, for all $e \in F \setminus \{e_0\}$ and the (nontrivial) equation that contains $x(e_0)$. Therefore, there are exactly $|F|$ equations of (3.3) that contain variables $x(e)$, $e \in F$. Let (3.3)$'$ be the system obtained from (3.3) by deleting the equations involving a variable $x(e)$, $e \in F$. We have that $x'$ is the unique solution for the system (3.3)$'$. Thus, $x'$ is an extreme point of $P(G', k)$. Moreover, $x'$ is fractional. This is clear if $x(e) = 1$ for all $e \in F$. If this is not the case, since system (3.3) has integer right-hand side and 0–1 coefficients, there must exist at least one more variable $x(e)$, $e \neq e_0$, that has a fractional value, and hence $x'$ is fractional. Since $|E'| \leqslant m$, this contradicts the induction hypothesis. □

Since $G$ is series–parallel by Lemma 1.1, there is a node $v$ that is adjacent to exactly two nodes $v_1$ and $v_2$ in $G$. As we did before, let us denote by $F_1$ ($F_2$) the set of the edges between $v$ and $v_1$ ($v_2$). W.l.o.g we assume that $|F_1| \geqslant |F_2|$. Since $G$ is $k$-edge connected, it then follows that

$$|F_1| \geqslant \lceil \tfrac{k}{2} \rceil. \tag{3.4}$$

**Claim 2.** *There is an edge $f \in F_1$, such that $0 < x(f) < 1$.*

**Proof.** Suppose $x(e) = 1$ for all $e \in F_1$ (i.e $F_1 \subseteq E_1$). We claim that the set $\{\delta(W_1), \ldots, \delta(W_r)\}$ and the set $\{\pi_1, \ldots, \pi_s\}$ of tight cuts and tight partitions, respectively, of the system (3.3) can be chosen so that $\delta(W_i) \cap F_1 = \emptyset$ for $i = 1, \ldots, r$ and $E_{\pi_j} \cap F_1 = \emptyset$ for $j = 1, \ldots, s$. In fact, let us examine first the partitions $\{\pi_1, \ldots, \pi_s\}$. Suppose that for some partition $\pi_j = (V_1, \ldots, V_p)$ where $p \geqslant 3$, we have $E_{\pi_j} \cap F_1 \neq \emptyset$. Then $F_1 \subset E_{\pi_j}$. W.l.o.g we may assume that $F_1 \subseteq [V_1, V_2]$. As $x(e) = 1$ for all $e \in F_1$, we have $x[V_1, V_2] \geqslant \lceil k/2 \rceil$. Since by Lemma 3.1 (i) one should have $x[V_1, V_2] \leqslant \lceil k/2 \rceil$, it then follows that $x[V_1, V_2] = \lceil k/2 \rceil$, and thus the partition $\pi'_j = (V'_1, \ldots, V'_{p-1})$ such that

$$V'_1 = V_1 \cup V_2,$$

$$V'_i = V_{i+1} \quad \text{for } i = 2, \ldots, p-1,$$

is tight for $x$. Since the equation defined by $\pi'_j$ is implied by the one defined by $\pi_j$ together with $x(e) = 1$, for all $e \in F_1$, it cannot be among the equations of the system (3.3). Now consider the system obtained from (3.3) by replacing the equation defined by $\pi_j$ by the one defined by $\pi'_j$. Clearly, $x$ is also the unique solution of the new system.

Now suppose that $\delta(W_i) \cap F_1 \neq \emptyset$ for some $i \in \{1, \ldots, r\}$. Then $F_1 \subseteq \delta(W_i)$. W.l.o.g we may suppose that $v \in W_i$ (and $v_1 \in V \setminus W_i$). Suppose that $v_2 \in V \setminus W_i$, then $W_i = \{v\}$. We claim that $x(e) = 1$ for all $e \in F_2$. Indeed, suppose this is not the case. Since $x(e) = 1$, for all $e \in F_1$ and $x(\delta(v)) = k$, $F_2$ would contain at least two edges with fractional values. Let $f$ and $g$ be two such edges. Let $x'$ be defined as

$$x'(e) = \begin{cases} x(e) + \omega & \text{if } e = f, \\ x(e) - \omega & \text{if } e = g, \\ x(e) & \text{if } e \in E \setminus \{f, g\}, \end{cases}$$

where $\omega$ is a scalar sufficiently small. Clearly, $x'$ is a solution of (3.3). Since $x \neq x'$, this contradicts the extremality of $x$ and our claim is proved. Consequently, $x(e) = 1$ for all $e \in \delta(v)$. This implies that $x(\delta(v)) = k$ is redundant in the system (3.3), a contradiction.

Thus, $v_2 \in W_i$. Let $W'_i = W_i \setminus \{v\}$. Since

$$\begin{aligned} x(\delta(W'_i)) &= x(\delta(W_i)) - x(F_1) + x(F_2), \\ &= k - x(F_1) + x(F_2) \\ &\geqslant k, \end{aligned}$$

it follows that $x(F_2) \geqslant x(F_1)$. Since $x(e) = 1$ for all $e \in F_1$ and $|F_2| \leqslant |F_1|$, we have $x(F_2) \leqslant x(F_1)$. Hence, $x(F_2) = x(F_1)$ and $\delta(W'_i)$ is tight for $x$. Now by replacing the cut $\delta(W_i)$ in the system (3.3) by the cut $\delta(W'_i)$, we obtain a nonredundant system whose $x$ is still a unique solution. Thus, system (3.3) can be chosen so that $F_1$ does not intersect any of the cuts $\delta(W_i), \ldots, \delta(W_r)$ and the edge sets $E_{\pi_1}, \ldots, E_{\pi_s}$. Therefore, for every edge $e \in F_1$, the variable $x(e)$ belongs to exactly one equality of (3.3), namely $x(e) = 1$. But this contradicts Claim 1. □

Now, to complete the proof of Theorem 3.2 we distinguish two cases.

*Case 1:* $|F_1| = \lceil k/2 \rceil$. Then $|F_2| = \lceil k/2 \rceil$. Otherwise, we would have $|\delta(v)| = k$ and thus $x(e) = 1$ for all $e \in F_1$, contradicting Claim 2.

Also, by symmetry, from Claim 2, there must exist an edge $g \in F_2$ such that $0 < x(g) < 1$. We claim that every cut $\delta(W_i)$, $i = 1,\ldots,r$, (partition $\pi_j$, $j = 1,\ldots,s$), either contains all the edges of both $F_1$ and $F_2$ or does not intersect any one of these sets. In fact, suppose, for instance, that for some partition $\pi_j = (V_1,\ldots,V_p)$ we have, say, $F_1 \subseteq E_{\pi_j}$ and $F_2 \cap E_{\pi_j} = \emptyset$. (The proof is similar for a cut.) W.l.o.g we may suppose $v \in V_1$. Thus $x[v, V_1 \setminus \{v\}] = x(F_2) \leqslant \lceil k/2 \rceil$. By Lemma 3.1 (ii), it follows that $x[v, V_1 \setminus \{v\}] = \lceil k/2 \rceil$ and thus $x(e) = 1$ for all $e \in F_2$, a contradiction. Let $\bar{x}$ be the solution given by

$$\bar{x}(e) = \begin{cases} x(e) + \omega & \text{if } e = f, \\ x(e) - \omega & \text{if } e = g, \\ x(e) & \text{if } e \in E \setminus \{f, g\}, \end{cases}$$

where $\omega$ is a scalar sufficiently small. From the above claim it follows that $\bar{x}$ is also a solution of the system (3.3). Since $\bar{x} \neq x$, this contradicts the extremality of $x$.

*Case 2:* $|F_1| > \lceil k/2 \rceil$. First of all, it is clear that $f$ is the only edge of $F_1$ with a fractional value. If not, one can construct as above a solution different from $x$ satisfying with equation the constraints tight for $x$, a contradiction. Consequently, $x(e) = 1$ for all $e \in F_1 \setminus \{f\}$ and by (3.2), it follows that

$$x(F_1) > \lceil \tfrac{k}{2} \rceil. \tag{3.5}$$

By Lemma 3.1(i), $F_1$ cannot be in the edge set of any of the partitions $\pi_j = 1,\ldots,s$. Thus, by Claim 1 there must exist two cuts $\delta(W_i)$ and $\delta(W_j)$, $i, j \in \{1,\ldots,r\}$ containing $f$. Hence, $F_1 \subseteq \delta(W_i) \cap \delta(W_j)$, and by (3.2) and (3.5), $x(\delta(W_i) \cap \delta(W_j)) > \lceil k/2 \rceil$, contradicting Lemma 3.1 (iii), which finishes the proof of our theorem. □

As a consequence of Theorem 3.2 we obtain the following result which has been conjectured by Chopra [5] and, independently, proved by Chopra and Stoer [6].

**Corollary 3.3.** *If $G = (V,E)$ is a connected series–parallel graph (not necessarily $k$-edge connected) and $k$ is odd, then $P_k(G)$ is completly described by inequalities* (2.1) *and* (2.4).

**Proof.** Let $P_k^*(G)$ be the polyhedron described by inequalities (2.1) and (2.4). It suffices to show that the extreme points of $P_k^*(G)$ are integral. Suppose, on the contrary, that there exists a fractional extreme point $x \in \mathbb{R}^E$ of $P_k^*(G)$. Let $G' = (V',E')$ be the graph obtained from $G$ by replacing each edge $e$ of $E$ by $\lceil x(e) \rceil$ edges $e_1,\ldots,e_{\lceil x(e) \rceil}$. Let $x' \in \mathbb{R}^{E'}$ be the solution given by

$$\left\{ \begin{array}{l} x'(e_i) = 1 \quad \text{for } i = 1,\ldots,\lceil x(e) \rceil - 1, \\ x'(e_i) = x(e) - \lceil x(e) - 1 \rceil \\ \qquad\qquad \text{for } i = \lceil x(e) \rceil, \end{array} \right\} \quad \text{if } x(e) \neq 0.$$

It is easily seen that $x'$ satisfies inequalities (2.1), (2.2) and (2.4). Moreover, $x'$ is an extreme point of the polytope described by these inequalities. In fact, if this is not the case, there must exist two integer solutions $y_1'$, $y_2'$, $y_1' \neq y_2'$, that satisfy inequalities (2.1), (2.2), (2.4), and such that $x' = \frac{1}{2}(y_1' + y_2')$. Now let $y_1$, $y_2 \in \mathbb{R}^E$ be the solutions such that

$$y_i(e) = \sum_{j=1}^{\lceil x(e) \rceil} y_i'(e_j),$$

for $e \in E$ and $i = 1, 2$. It is clear that $y_1$, $y_2 \in P_k^*(G)$. Morever, we have that $x = \frac{1}{2}(y_1 + y_2)$. Since $y_1 \neq y_2$, this contradicts the fact that $x$ is an extreme point of $P_k^*(G)$.

Consequently, $x'$ is an extreme point of the polytope given by inequalities (2.1), (2.2) and (2.4). Since $x'$ is fractional and $G^*$ is series–parallel, this contradicts Lemma 3.1. □

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