SIAM J. DISCRETE MATH.
Vol. 10, No. 3, pp. 505–514, August 1997

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STEINER 2-EDGE CONNECTED SUBGRAPH POLYTOPES ON SERIES-PARALLEL GRAPHS*

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Abstract. Given a graph $G = (V, E)$ with weights on its edges and a set of specified nodes $S \subseteq V$, the Steiner 2-edge survivable network problem is to find a minimum weight subgraph of G such that between every two nodes of S there are at least two edge-disjoint paths. This problem has applications to the design of reliable communication and transportation networks. In this paper, we give a complete linear description of the polytope associated with the solutions to this problem when the underlying graph is series-parallel. We also discuss related polyhedra.

Key words. Steiner 2-edge connected subgraphs, polytopes, series-parallel graphs

AMS subject classifications. 05C85, 90C27

PII. S0895480193259813

1. Introduction. A graph $G = (V, E)$ is said to be *k-edge* (*respectively, k-node*) *connected* ($1 \leq k \leq |V| - 1$) if for any pair of nodes $i,\ j \in V$ there are at least k edge-disjoint (respectively, node-disjoint) paths from i to j. Let $G = (V, E)$ be a graph and $w \in R^E$ a weight vector associated with the edges of G. The weight of a subgraph of G is the sum of the weights of its edges. Given a subset of distinguished nodes $S \subseteq V$, called *terminals*, *the Steiner 2-edge survivable network problem* (STESNP) is the problem of finding a minimum weight subgraph of G spanning S such that between every two nodes i, j of S there are at least two edge-disjoint paths between i and j. The STESNP has applications to the design of reliable communication and transportation networks [5], [25], [26].

In this paper, we discuss the polytope associated with the solutions to this problem. We give a complete linear description of this polytope when the graph is series-parallel.

The STESNP is NP-hard in general. It has been shown to be polynomially solvable in some special cases of graphs. In [28], [29], Winter devised linear time algorithms to solve the STESNP in Halin graphs [28] and series-parallel graphs [29]. Actually, Winter considers the following more general problem called the *general Steiner problem:* Given a set $S \subseteq V$ and an integer $(|S|, |S|)$-matrix $R = (R_{ij})$ (defining certain pairwise connectivity requirements), find a minimum weight subgraph spanning S such that between every pair (i, j) of nodes in S there are at least R_{ij} edge (node)-disjoint paths. He showed that this problem can be solved in linear time if the graph is series-parallel or a Halin graph. This problem has been considered later by Grötschel and Monma [18] and Grötschel, Monma, and Stoer [19], [20], [21] in the framework of a more general model. In particular, Grötschel, Monma, and Stoer studied polyhedral aspects of that model and devised cutting plane algorithms.

Given a graph $G = (V, E)$ and a node subset $W \subseteq V$ of G, the set of edges having one endnode in W and the other in $V \setminus W$ is called a *cut* of G and denoted by $\delta(W)$.

* Received by the editors November 10, 1993; accepted for publication (in revised form) July 19, 1996.
http://www.siam.org/journals/sidma/10-3/25981.html

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If $v \in V$ is a node of G, then we write $\delta(v)$ for the cut $\delta(\{v\})$ and we say that $\delta(v)$ is defined by v. If a cut contains k edges, it is also called a *k-edge cut set*.

Let $G = (V, E)$ be a graph. Let $x(e)$ be a variable associated with each edge e and for an edge subset $F \subseteq E$, the 0-1 vector $x^F \in R^E$ with $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ otherwise is called the *incidence vector* of F. For any subset of edges $F \subseteq E$, we define $x(F) = \sum_{e \in F} x(e)$. If $W \subseteq V$, then we denote by $E(W)$ the set of edges having both endnodes in W. The STESNP can be formulated as the following integer linear program.

$$\begin{array}{lll} & \text{Min } wx & \\ & \text{subject to} & \\ (1.1) & 0 \le x(e) \le 1 & \text{for all } e \in E, \\ (1.2) & x(\delta(W)) \ge 2 & \text{for all } W \subseteq V,\ S \ne W \bigcap S \ne \emptyset, \\ (1.3) & x(e) \in \{0,1\} & \text{for all } e \in E. \end{array}$$

Let

$$\text{STESNP}(G,S) = \text{conv}\{x \in R^E \ \mid x \text{ satisfies (1.1), (1.2), and (1.3)}\}$$

be the polytope associated with the STESNP.

Using a polynomial time algorithm for the maximum flow problem [10], [12] and the famous maximum flow-minimum cut theorem (cf. Ford and Fulkerson [14]), one can solve in polynomial time the separation problem for inequalities (1.2) (the problem that consists to determine whether a given solution x satisfies the inequalities (1.2), and if not, to find an inequality among (1.2) which is violated by x). This implies, from the ellipsoid method [17], that there is a polynomial time algorithm for solving STESNP whenever STESNP(G,S) is completely described by the inequalities (1.1) and (1.2). Also one can obtain an equivalent extended compact formulation for the system given by (1.1) and (1.2) using the max flow-min cut theorem. This yields a further polynomial time algorithm for solving the STESNP when STESNP(G,S) is described by these inequalities.

In this paper, we will show that if the graph is series-parallel, then the polytope STESNP(G,S) is given by inequalities (1.1) and (1.2).

To the best of our knowledge, the STESNP(G,S) has not been considered in the literature. However some special cases received much attention. In particular, the case where $S = V$ has been extensively investigated.

For $S = V$, Mahjoub [22] gave a complete description of STESNP(G,S) in the case where the graph is series-parallel and he introduced a large class of facet defining inequalities for the polytope STESNP(G,S) called the odd-wheel inequalities. This class of facet defining inequalities has been generalized by Grötschel, Monma, and Stoer [19] for more general polyhedra. In [2] Barahona and Mahjoub characterized the polytope STESNP(G,S) for Halin graphs. In [18] Grötschel and Monma discuss a general model related to the design of minimum-cost survivable networks. They discuss polyhedral aspects of this model and identify basic facets of the associated polyhedra. Grötschel, Monma, and Stoer [19], [20], [21] describe further classes of facets of these polyhedra, develop a cutting plane algorithm for the associated problem and present computational results. A complete survey of that model and related work is given in Stoer [26].

Coullard, Rais, Rardin, and Wagner [7], [8], [9] consider the Steiner 2-node connected subgraph polytope, that is the polytope, the extreme points of which are the incidence vectors of the edge sets of the 2-node connected subgraphs of G, spanning S. They give a complete description of that polytope when the graph is series-parallel [7]. In [9] they characterize the dominant of that polytope for the graphs noncontractible to W_4 (the wheel on five nodes). In [8] they devise linear time algorithms for the Steiner 2-node connected subgraph problem in the graphs noncontractible to W_4 and Halin graphs.

Related work can also be found in [4], [6], [13]. In [6] Cornuéjols, Fonlupt, and Naddef studied some related polyhedra to STESNP(G, S). They showed that when $S = V$ and G is series-parallel, the polyhedron given by the nonnegativity inequalities and the cut-inequalities is integral. Fonlupt and Naddef [13] characterized the class of graphs for which the system given by these inequalities defines the convex hull of the incidence vectors of the tours of G (a tour is a cycle going at least once through each node). Chopra [4] considers the polyhedron, the extreme points of which are the incidence vectors of the edge sets of the k-edge connected spanning subgraphs of G, when multiple copies of an edge may be considered. He characterized this polyhedron for the class of outerplanar graphs when k is odd.

In the next section, we describe the polytope STESNP(G, S) for series-parallel graphs, and we give some structural properties for the system of inequalities defining that polytope. In section 3 we prove our main result. Concluding remarks are given in section 4. The remainder of this section is devoted to more definitions and notations.

The graphs we consider are finite, undirected, connected, and may have multiple edges and loops. We denote a graph by $G = (V, E)$ where V is the *node set* and E is the *edge set* of G. If e is an edge with endnodes u and v, then we write $e = uv$.

A graph G is said to be *contractible* to a graph H if H may be obtained from G by a sequence of elementary removal and contractions of edges. A contraction consists of identifying a pair of adjacent vertices and of preserving all other vertices and of preserving all other adjacencies between vertices. A graph is called *series-parallel* [11] if it is not contractible to K_4 (the complete graph on four nodes). Note that if G is a series-parallel graph and G is contractible to a graph H, then H is series-parallel. It is easily seen that series-parallel graphs have the following property.

LEMMA 1. *Any connected series-parallel graph with more than two nodes and without nodes defining* 2*-edge cut sets contains multiple edges.*

If $G = (V, E)$ is a graph and $W \subseteq V$ is a subset of nodes, we denote by $G(W)$ the subgraph of G induced by W. For $W,\ W' \subseteq V,\ (W,\ W')$ denotes the set of edges having one endnode in W and the other in W'. If $F \subseteq E$, $V(F)$ will denote the set of the nodes of the edges of F. If $W \subseteq V$, we let $\overline{W} = V \setminus W$. Given a constraint $ax \geq \alpha,\ a \in R^E$ and a solution x^*, we will say that $ax \geq \alpha$ is *tight* for x^* if $ax^* = \alpha$. If $G = (V, E)$ is a graph and $e \in E$, $G - e$ will denote the graph obtained from G by removing e.

2. The polytope STESNP(G,S) of a series-parallel graph. Let $G = (V, E)$ be a graph and $S \subseteq V$ a set of terminals. We will suppose $|S| \geq 2$, (if $|S| = 1$, then an optimal solution to the problem STESNP would consist of the edges of negative weights). Let $P(G, S)$ denote the polytope given by inequalities (1.1) and (1.2). These inequalities will be called, respectively, *trivial* and *Steiner-cut inequalities*. A cut corresponding to a Steiner-cut inequality will be called *Steiner-cut.* Given a Steiner-cut $\delta(W)$ and a solution x for which the corresponding Steiner-cut inequality is tight, we will also say that $\delta(W)$ is a Steiner-cut tight for x.

Our main result is the following.

THEOREM 2. *If* $G = (V, E)$ *is a series-parallel graph and* $S \subseteq V$ *a set of terminals,* *then STESNP*$(G,S) = P(G,S)$.

The proof of this theorem will be given in the following section. In what follows, we are going to discuss some properties of the extreme points of the polytope $P(G,S)$. These properties will be useful in the sequel. First we give a technical lemma.

LEMMA 3. *Let x be a solution of $P(G,S)$. If $\delta(W_1)$ and $\delta(W_2)$ are two Steiner-cuts tight for x and $(W_1 \cap W_2) \cap S \neq \emptyset$ and $(\overline{W_1 \cup W_2}) \cap S \neq \emptyset$ (respectively, $(W_1 \setminus W_2) \cap S \neq \emptyset$ and $(W_2 \setminus W_1) \cap S \neq \emptyset$), then $\delta(W_1 \cap W_2)$ and $\delta(\overline{W_1 \cup W_2})$ (respectively, $\delta(W_1 \setminus W_2)$ and $\delta(W_2 \setminus W_1)$) are two Steiner-cuts tight for x, and $x(W_1 \setminus W_2,\ W_2 \setminus W_1) = 0$ (respectively, $x(W_1 \cap W_2, \overline{W_1 \cup W_2}) = 0$.*

Proof. Since $\delta(W_1)$ and $\delta(W_2)$ are tight for x we have

$$\begin{aligned} 4 &= x(\delta(W_1)) + x(\delta(W_2)) \\ &= x(\delta(W_1 \cap W_2)) + x(\delta(W_1 \cup W_2)) + 2x(W_1 \setminus W_2, W_2 \setminus W_1) \\ &\geq x(\delta(W_1 \cap W_2)) + x(\delta(W_1 \cup W_2)) \\ &\geq 4. \end{aligned}$$

The two last inequalities follow from the fact that $x(e) \geq 0$ for all $e \in E$ and $\delta(W_1 \cap W_2)$ and $\delta(W_1 \cup W_2)$ are both Steiner-cuts. This implies that all the above inequalities are satisfied at equality. Consequently, $x(\delta(W_1 \cap W_2)) = x(\delta(W_1 \cup W_2)) = 2$ and $x(W_1 \setminus W_2, W_2 \setminus W_1) = 0$.

If $(W_1 \setminus W_2) \cap S \neq \emptyset$ and $(W_2 \setminus W_1) \cap S \neq \emptyset$, then the cuts $\delta(W_1 \setminus W_2)$ and $\delta(W_2 \setminus W_1)$ are Steiner-cuts and in a similar way, we obtain that these cuts are tight for x and $x(W_1 \cap W_2, \overline{W_1 \cup W_2}) = 0$. ◻

If x is an extreme point of $P(G,S)$, then there exist two edge subsets, $E^0,\ E^1 \subseteq E$ of G and a family of Steiner-cuts $\{\delta(W_i),\ i = 1, ..., l\}$ such that x is the unique solution of the system

$$\text{(2.1)} \qquad \begin{cases} x(e) = 0 & \text{for all } e \in E^0, \\ x(e) = 1 & \text{for all } e \in E^1, \\ x(\delta(W_i)) = 2 & \text{for } i = 1, \ldots, l, \end{cases}$$

where $|E^0| + |E^1| + l = |E|$.

LEMMA 4. *Let $x \in R^E$ be a solution of $P(G,S)$ such that $x(e) > 0$ for all $e \in E$. If $\delta(W)$ is a Steiner-cut tight for x, then $G(W)$ and $G(\overline{W})$ are both connected.*

Proof. Suppose, for instance, that $G(\overline{W})$ is not connected. Let $\overline{W}^1$, $\overline{W}^2$ be a partition of $\overline{W}$ such that $(\overline{W}^1, \overline{W}^2) = \emptyset$. Since G is connected, it follows that $(W, \overline{W}^1) \neq \emptyset \neq (W, \overline{W}^2)$. From the hypothesis we then have

$$\text{(2.2)} \qquad x(W, \overline{W}^1) > 0, x(W, \overline{W}^2) > 0.$$

In addition, since $\overline{W} \cap S \neq \emptyset$, we may, without loss of generality (w.l.o.g.), assume that $\overline{W}^1 \cap S \neq \emptyset$. Hence $\delta(\overline{W}^1)$ is a Steiner-cut of G. However, as

$$x(\delta(W)) = x(W, \overline{W}^1) + x(W, \overline{W}^2) = 2,$$

it follows by (2.2) that $x(\delta(\overline{W}^1)) = x(W, \overline{W}^1) < 2$, a contradiction. ◻

The following remark will be used frequently in the next section.

Remark 5. Let $G' = (V', E')$ be a graph obtained from G by contracting a connected edge subset $F \subseteq E$. Let $S' = (S \setminus V(F)) \cup \{s'\}$ if $S \cap V(F) \neq \emptyset$ and

$S' = S$ if not, where s' is the node that arises in the contraction of F. Let x' be the restriction of x on G'. Then x' is a solution of $P(G', S')$.

Proof. Obviously, x' satisfies the inequalities (1.1). Furthermore, since any Steiner-cut $\delta(W)$ of G' with respect to S' is a Steiner-cut of G with respect to S, it follows that inequalities (1.2) are also satisfied by x'. $\square$

3. Proof of Theorem 2. The proof is by induction on the number of edges. The theorem is trivially true for a graph with no more than two edges. Suppose it is true for any series-parallel graph with no more than m edges and suppose G contains exactly $m+1$ edges. Let us assume that, on the contrary, $\text{STESNP}(G, S) \neq P(G, S)$. And let x be a fractional extreme point of $P(G, S)$. Also let us assume that, under the induction hypothesis, $|S|$ is maximum. That is, for any series-parallel graph $G' = (V', E')$ with $|E'| = m + 1$ and a set of terminals S' such that $|S'| > |S|$, we have $\text{STESNP}(G', S') = P(G', S')$. We have the following lemmas:

LEMMA 6. *$x(e) > 0$ for all $e \in E$.*

Proof. If e_0 is an edge such that $x(e_0) = 0$, then let x' be the point given by $x'(e) = x(e)$ for all $e \in E \setminus \{e_0\}$. Clearly, x' belongs to $P(G - e_0)$. Moreover x' is an extreme point of $P(G - e_0)$. Since x' is fractional, we have a contradiction. $\square$

LEMMA 7. *Let x be an extreme point of $P(G, S)$ and $g = uv$ an edge of G such that $x(g) > 0$. Then there exist at least two constraints containing g in the system* (2.1) *defining x.*

Proof. System (2.1) must be of full rank and therefore there must exist at least one constraint containing g in the system (2.1). So, let us assume that there exists exactly one constraint of system (2.1) that contains g. Let $(2.1)'$ be the system obtained from (2.1) by deleting this constraint. Let $x' \in R^m$ be the solution given by $x'(e) = x(e)$ for all $e \in E \setminus \{g\}$. We claim that x' is fractional. In fact, this is clear if $x(g) = 1$. If not, then g belongs to a tight Steiner-cut and thus there must exist at least one more edge in G with a fractional value, which implies that x' is fractional. Moreover, x' is the unique solution of the system $(2.1)'$. Now let G' be the graph obtained from G by contracting g. Let $S' = (S - \{u, v\}) \bigcup \{w\}$ if $g \in E(S)$ and $S' = S$ if not, where w is the node arising from the contraction of g. By Remark 5 we have $x' \in P(G', S')$. Furthermore, note that the system $(2.1)'$ is included in $P(G', S')$. This implies that x' is an extreme point of $P(G', S')$. Since G' is series-parallel and has less edges than G this contradicts the induction hypothesis and thus our lemma is proved. $\square$

LEMMA 8. *G does not contain a node defining a 2-edge cut set.*

Proof. Suppose that G contains a node v such that $\delta(v) = \{e_1, e_2\}$ where $e_1 = vw_1$ and $e_2 = vw_2$. We will distinguish two cases.

Case 1. $x(e_1) = x(e_2)$.

Let G' be the graph obtained from G by contracting e_1. Clearly, G' is series-parallel. Let x' be the restriction of x on G' and let $S' = (S \setminus \{v, w_1\}) \cup \{v'\}$ if $\{v, w_1\} \cap S \neq \emptyset$ and $S' = S$ if not, where v' is the node that arises in the contraction of e_1. By Remark 5, we have that x' belongs to $P(G', S')$. We claim that x' is an extreme point of $P(G', S')$. In fact, if this is not the case, then there are two solutions y^1 and y^2 of $P(G', S')$, $y^1 \neq y^2$ such that $x' = \frac{1}{2}(y^1 + y^2)$. Let x^1 and x^2 be the solutions given by

$$x^1(e) = \begin{cases} y^1(e) & \text{if } e \in E \setminus \{e_1\}, \\ y^1(e_2) & \text{if } e = e_1, \end{cases}$$

and

$$x^2(e) = \begin{cases} y^2(e) & \text{if } e \in E \setminus \{e_1\}, \\ y^2(e_2) & \text{if } e = e_1. \end{cases}$$

We claim that x^1 and x^2 both belong to $P(G,S)$. Clearly, both x^1 and x^2 satisfy the trivial inequalities. In what follows we show that they also satisfy inequalities (1.2). We show this for x^1, the proof for x^2 is identical.

Let $\delta(W)$ be a Steiner-cut of G. If $e_1 \notin \delta(W)$, then $\delta(W)$ is a Steiner-cut of G' with respect to S' and then $x^1(\delta(W)) = y^1(\delta(W)) \geq 2$. So suppose that $e_1 \in \delta(W)$. Also, suppose, w.l.o.g, that $v \in \overline{W}$. Hence $w_1 \in W$. We consider two cases.

Case 1.1. $v \in S$.

Since $\delta(v)$ is a Steiner-cut, it follows from inequalities (1.1) and (1.2) that $x(e_1) = x(e_2) = 1$. Hence

$$x^1(e_1) = x^1(e_2) = 1. \tag{3.1}$$

We claim that $w_2 \in S$. In fact, first remark that $x(\delta(Z)) \geq 2$ holds for every cut $\delta(Z)$ such that $S \subseteq Z$ and $w_2 \in \overline{Z}$. This is clear if w_1 (and w_2) belong to $\overline{Z}$. Now suppose that $w_1 \in Z$. Let $Z' = Z \setminus \{v\}$. Since $|S| \geq 2$ and $v \in S$, we have that $Z' \cap S \neq \emptyset$ and $\overline{Z'} \cap S \neq \emptyset$. Thus $\delta(Z')$ is a Steiner-cut of G. Moreover, we have $\delta(Z') = (\delta(Z) \setminus \{e_2\}) \cup \{e_1\}$. Since $x(e_1) = x(e_2)$ it then follows that $x(\delta(Z)) = x(\delta(Z')) \geq 2$.

Now if $w_2 \notin S$, then let $\overline{S} = S \cup \{w_2\}$. From the above remark, we have that x is an extreme point of $P(G,\overline{S})$. Since x is fractional and $|S| < |\overline{S}|$, this contradicts the maximality of S.

Thus $w_2 \in S$. Now if $e_1, e_2 \in \delta(W)$ then by (3.1), we have $x^1(\delta(W)) \geq 2$. If not, since $e_1 \in \delta(W)$, we have $\{e_1, e_2\} \cap \delta(W) = \{e_1\}$, and thus $w_2 \in \overline{W}$. Let $W' = (W \setminus \{w_1\}) \cup \{v'\}$. As v' and w_2 belong to S, $\delta(W')$ is a Steiner-cut of G'. Since $\delta(W') = (\delta(W) \setminus \{e_1\}) \cup \{e_2\}$, it follows by (3.1) that

$$x^1(\delta(W)) = x^1(\delta(W')) = y^1(\delta(W')) \geq 2. \tag{3.2}$$

Case 1.2. $v \notin S$.

First of all note that every constraint of type (1.2), with $e_1, e_2 \in \delta(W)$ is redundant in $P(G,S)$. Since $w_1 \in W$ and $v \in \overline{W}$ we may then suppose that $\{e_1, e_2\} \cap \delta(W) = \{e_1\}$. By setting $W' = (W \setminus \{w_1\}) \cup \{v'\}$, we obtain that $\delta(W')$ is a Steiner-cut in G' and that (3.2) holds.

In both cases, x^1 satisfies the inequality associated with $\delta(W)$, and thus $x^1 \in P(G,S)$. Consequently, $x^1, x^2 \in P(G,S)$. But $x = \frac{1}{2}(x^1 + x^2)$. Since $x^1 \neq x^2$, this contradicts the extremality of x.

Case 2. $x(e_1) \neq x(e_2)$.

Without loss of generality, we may suppose that $x(e_1) > x(e_2)$. Thus e_1 cannot belong to any Steiner-cut tight for x. In fact, first note that v cannot be in S. Otherwise $\delta(v)$ would be a Steiner-cut not satisfied by x which is impossible. Now suppose that there is a Steiner-cut $\delta(W)$ tight for x with $e_1 \in \delta(W)$. W.L.O.G., we may suppose $v \in W$. Then $\delta(W')$ where $W' = W \setminus \{v\}$ is a Steiner-cut of G. Moreover, $x(\delta(W')) = x((\delta(W) \setminus \{e_1\}) \cup \{e_2\}) = 2 - x(e_1) + x(e_2) < 2$, a contradiction. As a consequence, e_1 belongs to only one constraint of system (2.1), namely $x(e_1) = 1$. But this contradicts Lemma 7 and our lemma is proved. □

LEMMA 9. *G cannot contain two multiple edges f and g such that $x(f) = x(g) = 1$.*

Proof. Suppose the contrary. Let $G' = (V', E')$ be the graph obtained from G by contracting the edges f and g. Clearly, G' is series-parallel. Let $S' = (S \setminus \{u, v\}) \cup \{w\}$, if $S \cap \{u, v\} \neq \emptyset$ and $S' = S$ if not, where u and v are the endnodes of f and g and w is the node arising from the contraction of f and g. Let $x' \in R^{m-1}$ be the solution given by $x' = x(e)$ for all $e \in E \setminus \{f, g\}$. By Remark 5, x' is a solution of $P(G', S')$. Moreover x' is an extreme point of $P(G', S')$. Indeed, if this is not the case, then there must exist two solutions y^1 and y^2, $y^1 \neq y^2$, of $P(G', S')$ such that $x' = \frac{1}{2}(y^1 + y^2)$. Now consider the solutions $y^{1'}$, $y^{2'} \in R^{m+1}$ given by

$$y^{1'}(e) = \begin{cases} y^1(e) & \text{if } e \in E \setminus \{f, g\}, \\ 1 & \text{if } e \in \{f, g\}, \end{cases}$$

and

$$y^{2'}(e) = \begin{cases} y^2(e) & \text{if } e \in E \setminus \{f, g\}, \\ 1 & \text{if } e \in \{f, g\}. \end{cases}$$

It is clear that $y^{1'}$ and $y^{2'}$ both belong to $P(G, S)$. Also we have that $x = \frac{1}{2}(y^{1'} + y^{2'})$, a contradiction. Consequently, x' is an extreme point of $P(G', S')$. Since x' is fractional and $|E'| < |E|$, this contradicts the induction hypothesis. ∎

LEMMA 10. *G does not contain two multiple edges f and g such that $x(f) = 1$ and $0 < x(g) < 1$.*

Proof. Let us suppose the contrary. Let u and v be the endnodes of f and g. Since $x(g)$ is fractional, there must exist a Steiner-cut $\delta(W_1)$, $W_1 \subset V$, tight for x, and containing g (and f). From Lemma 4, it follows that $G(W_1)$ and $G(\overline{W}_1)$ are both connected. We consider two cases.

Case 1. $|W_1| \geq 2$, $|\overline{W}_1| \geq 2$.

Let G^1 and G^2 be the graphs obtained from G by contracting W_1 and $\overline{W}_1$, respectively. Since $G(W_1)$ and $G(\overline{W}_1)$ are connected, both graphs G^1 and G^2 are series-parallel. Let $S^1 = (S \cap \overline{W}_1) \cup \{s_1\}$ and $S^2 = (S \cap W_1) \cup \{s_2\}$ where s_1 and s_2 are the nodes arising from the contractions of W_1 and $\overline{W}_1$, respectively. Since G^1 and G^2 contain less edges than G, by the induction hypothesis, $P(G^1, S^1)$ and $P(G^2, S^2)$ are both integral. Let x^1 and x^2 be the restrictions of x on G^1 and G^2, respectively. By Remark 5, x^1 and x^2 are solutions of $P(G^1, S^1)$ and $P(G^2, S^2)$, respectively. Hence there must exist two integral solutions y^1 and y^2 of $P(G^1, S^1)$ and $P(G^2, S^2)$ such that every constraint of $P(G^1, S^1)$ (respectively, $P(G^2, S^2)$) that is tight for x^1 (respectively, x^2) is also tight for y^1 (respectively, y^2). In particular we have $y^1(\delta(W_1)) = y^2(\delta(W_1)) = 2$ and $y^1(f) = y^2(f) = 1$. Moreover, since $0 < x^1(g) = x^2(g) = x(g) < 1$, y^1 and y^2 can be chosen so that $y^1(g) = y^2(g) = 1$. Consequently, $y^1(\delta(W_1) \setminus \{f, g\}) = y^2(\delta(W_1) \setminus \{f, g\}) = 0$. Now consider the solution $x^* \in R^{m+1}$ given by

$$x^*(e) = \begin{cases} y^1(e) & \text{if } e \in E(\overline{W}_1), \\ y^2(e) & \text{if } e \in E(W_1), \\ 1 & \text{if } e \in \{f, g\}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that every constraint of $P(G, S)$ that is tight for x is also tight for x^*.

Let $e \in E$ such that $x(e) = 1$. Then e belongs either to $E(W_1)$ or $E(\overline{W}_1)$ or $e = f$. If $e \in E(W_1)$ (respectively, $e \in E(\overline{W}_1)$) then, $x^*(e) = y^2(e) = x^2(e) = 1$ (respectively, $x^*(e) = y^1(e) = x^1(e) = 1$). From Lemma 6 it then follows that every inequality of type (1.1) that is tight for x is also tight for x^*.

Consider now a Steiner-cut $\delta(W)$ tight for x.

(a) If $W \subseteq W_1$, then $x(\delta(W)) = x^2(\delta(W)) = y^2(\delta(W)) = 2$. Since $x^*(\delta(W)) = y^2(\delta(W))$, we obtain that $\delta(W)$ is tight for x^*.

(b) If $W \subseteq \overline{W}_1$, we obtain similarly that $\delta(W)$ is tight for x^*.

(c) Suppose that $W \not\subset W_1$, $W_1 \not\subset W$ and $W \cap W_1 \neq \emptyset$.

(c.1) Consider first the case where at least one of the sets $(W_1 \setminus W) \cap S$ and $(W \setminus W_1) \cap S$ is empty. Since both $\delta(W)$ and $\delta(W_1)$ are Steiner-cuts, it follows that $(W_1 \cap W) \cap S \neq \emptyset$ and $(\overline{W_1 \cup W}) \cap S \neq \emptyset$. Hence by Lemma 3, $\delta(W_1 \cap W)$ and $\delta(\overline{W_1 \cup W})$ are both Steiner-cuts tight for x and $x(W_1 \setminus W, W \setminus W_1) = 0$. By Lemma 6, this implies that $(W_1 \setminus W, W \setminus W_1) = \emptyset$. Furthermore, since $(W_1 \cap W) \subset W_1$, and $(\overline{W_1 \cup W}) \subset \overline{W}_1$, from Cases (a) and (b) above, it follows that $\delta(W_1 \cap W)$ and $\delta(\overline{W_1 \cup W})$ are both tight for x^*. Thus we have

$$x^*(\delta(W)) = x^*(\delta(W_1 \cap W)) + x^*(\delta(\overline{W_1 \cup W})) - x^*(\delta(W_1)) = 2 + 2 - 2 = 2.$$

And the constraint $x(\delta(W)) \geq 2$ is tight for x^*.

(c.2) If $(W_1 \setminus W) \cap S \neq \emptyset$ and $(W \setminus W_1) \cap S \neq \emptyset$, then by Lemma 3 we have that $\delta(W_1 \setminus W)$ and $\delta(W \setminus W_1)$ are both Steiner-cuts tight for x and $x(W_1 \cap W, \overline{W_1 \cup W}) = 0$. Using this, we obtain in a similar way as in c.1) that $x(\delta(W) \geq 2$ is also tight for x^*.

Consequently, every constraint of $P(G, S)$ that is tight for x is also tight for x^*. Since $x \neq x^*$, this contradicts the fact that x is an extreme point of $P(G, S)$.

Case 2. $|W_1| = 1$.

By Lemma 7, there must exist a further Steiner-cut $\delta(W_2)$ tight for x and containing g (and f). If $|W_2| \geq 2$, $|\overline{W}_2| \geq 2$ then Case 1 applies. Thus let us assume, for instance, that $|W_2| = 1$. Hence we may suppose that $W_1 = \{u\}$ and $W_2 = \{v\}$. This implies that $(V \setminus \{u, v\}) \cap S = \emptyset$. Otherwise $\delta(V \setminus \{u, v\})$ would be a Steiner-cut not satisfied by x, a contradiction. Hence any Steiner-cut of G contains both edges f and g. Furthermore, note that every Steiner-cut tight for x contains only one edge with integer value, namely f. Now consider the solution $\bar{x} \in R^E$ defined as

$$\bar{x}(e) = \begin{cases} 1 & \text{if } x(e) = 1 \text{ or } e = g, \\ 0 & \text{if not.} \end{cases}$$

We have that $\bar{x} \in P(G, S)$. Moreover any inequality of $P(G, S)$ which is tight for x is also tight for $\bar{x}$. Since $x \neq \bar{x}$, this contradicts the fact that x is an extreme point of $P(G, S)$, which achieves the proof of our lemma. ◻

From Lemmas 1, 8, 9, and 10 it follows that G contains two multiple edges f and g such that $0 < x(f) < 1$ and $0 < x(g) < 1$. Let x' be the solution such that

$$x'(e) = \begin{cases} x(e) + \epsilon & \text{if } e = g, \\ x(e) - \epsilon & \text{if } e = f, \\ x(e) & \text{otherwise,} \end{cases}$$

where ϵ is a scalar sufficiently small. Since any cut of G either contains both edges f and g or none of them, it follows that x' is also a solution of system (2.1). Since $x \neq x'$, we have a contradiction, and the proof of our theorem is complete. ◻

4. Concluding remarks. We have studied the Steiner 2-edge survivable network problem and have given a complete linear description of the associated polytope when the underlying graph is series-parallel. We have shown that in this case, the polytope is given by the trivial inequalities and the Steiner cut inequalities.

The following related problem, called the *Steiner 2-edge connected subgraph problem* (STECSP) has also been studied. Given a graph $G = (V, E)$ with weights on its edges and a set of terminals $S \subseteq V$, find a minimum 2-edge connected subgraph of G, spanning S. Note that any solution of STECSP is also a solution of STESNP. Moreover, if the weights are positive, then an optimal solution of STESNP is also an optimal solution of STECSP. And if $S = V$, then both problems coincide.

As the STESNP, the STECSP is NP-hard in general. Wald and Colbourn [27] showed that the STECSP can be solved in polynomial time in outerplanar graphs. Also from [24], [29] it can be shown that this problem is polynomially solvable in the more general class of series-parallel graphs.

The STECSP has also been studied by Monma, Munson, and Pulleyblank [23] in the metric case, that is when the underlying graph $G = (V, E)$ is complete and the weight function satisfies the triangle inequality (i.e., $w(e_1) \leq w(e_2) + w(e_3)$ for every three edges e_1, e_2, e_3 defining a triangle in G). In particular, Monma, Munson, and Pulleyblank showed that in this case the weight of a minimum 2-edge connected spanning subgraph in $(S, E(S))$ is at most $\frac{4}{3}$ times the weight of a minimum 2-edge connected subgraph of G, spanning S. Further structural properties and worst case analysis are given in Frederickson and Ja'Ja' [15], Bienstock, Brickell, and Monma [3] and Goemans and Bertsimas [16].

If (W, F), $W \subseteq V$, is a 2-edge connected subgraph of G, spanning S, then x^F, the incidence vector of F satisfies the following inequalities:

$$x(\delta(W)) - 2x(e) \geq 0 \qquad \text{for all } W \subseteq V, S \subseteq W, e \notin E(W). \tag{4.1}$$

Inequalities (4.1) express the fact that for a cut $\delta(W)$ that leaves S on one side, any 2-edges connected subgraph spanning S and containing an edge from $E \setminus E(W)$ must contain at least two edges from $\delta(W)$.

Let STECSP(G, S) be the polytope associated with the STECSP, that is, the convex hull of the incidence vectors of the edge sets of all the 2-edge connected subgraphs of G spanning S. Let Q(G, S) be the system given by inequalities (1.1), (1.2), and (4.1). We have the following result; its proof uses similar techniques as that of Theorem 2.

THEOREM 11. *If $G = (V, E)$ is a series-parallel graph and $S \subset V$ a set of terminals, then STECSP(G,S)=Q(G,S).*

Proof. For the proof, see [1].

Acknowledgments. We are grateful to the referees for their helpful comments. We thank M. Didi Biha for very stimulating suggestions.

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