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One-node cutsets and the dominating set polytope

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Abstract

In this paper we study a composition (decomposition) technique for the dominating set polytope in graphs which are decomposable by one-node cutsets. If G decomposes into G_1 and G_2, we show that the dominating set polytope of G can be described from two linear systems related to G_1 and G_2. This gives a way to characterize this polytope for classes of graphs that can be recursively decomposed. This also gives a procedure to describe facets for this polytope. Application of these techniques is discussed for the class of the cactus.

Keywords: Composition of polyhedra; One-node cutset; Dominating set polytope; Facets; Cactus

1. Introduction

Given a graph $G=(V,E)$ and two subgraphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ of G, G is called a *k-sum* of G_1 and G_2 if $V=V_1\cup V_2$, $|V_1\cap V_2|=k$, and the subgraph $(V_1\cap V_2, E_1\cap E_2)$ is complete. The set $V_1\cap V_2$ is called a *k-node cutset* of G.

In this paper we study a composition (decomposition) technique for the dominating set polytope in graphs which are decomposable by one-node cutsets. If G decomposes into G_1 and G_2, then we derive a system of inequalities that defines the dominating set polytope from systems related to G_1 and G_2. As a consequence, we obtain a procedure to construct this polytope in graphs that can be recursively decomposed. This technique also permits us to describe facets of the dominating set polytope by composition of facets from the pieces. We discuss applications of this technique for the class of the cactus.

Developing composition (decomposition) techniques for NP-hard combinatorial optimization problems has been a motivating subject for many researchers along the past decade [3, 4, 10, 14, 18, 22, 26]. Indeed, for an NP-hard combinatorial optimization

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problem, it is sometimes difficult to give a complete linear description of the associated polytope in some graph. However, if the graph decomposes into pieces (with respect to certain decomposition operations), it may be possible to give a complete description of the polytope from polytopes related to the pieces. This approach has been studied for different combinatorial optimization problems such as the max-cut problem [4, 18], the stable set problem [3, 10, 26], the acyclic subdigraph problem [3]. Margot [26] studied a general composition (decomposition) approach for combinatorial optimization polytopes using projection. This permitted him to generalize known results related to independence systems.

Given a graph $G=(V,E)$, a node subset $D \subseteq V$ of G is called *dominating set* if every node of $V \backslash D$ is adjacent to at least one node of D. Given a weight system $w(u)$, $u \in V$, associated with the nodes of G, the *minimum dominating set problem* (DSP) consists of finding a dominating set D of G such that $\sum_{u \in D} w(u)$ is minimum. This problem is a well-known intractable problem. Berge [5, 6] and Ore [27] are among the first who have discussed it.

The DSP arises in many applications [6, 11, 12], in particular those involving the strategic placement of men or pieces on the nodes of a network. As example, consider a computer network in which one wishes to choose a smallest set of computers that are able to transmit messages to all the remaining computers [9, 22]. Another example [27] is that of determining the minimum number of queen one wishes to place on a chess board so that every square of the board is dominated by at least one queen. (A square is dominated by a queen if it is placed in the same row, column or diagonal as the queen.) The DSP has also applications in matching theory [5].

The DSP has been extensively investigated from an algorithmic point of view [6,8, 11–13,15–17]. It is NP-hard in general. It has been shown to be polynomial in several classes of graphs such as the cactus [20] and the class of series–parallel graphs [21]. A complete survey of the algorithmic complexity of the DSP can be found in [13].

If $G=(V,E)$ is a graph and $S \subseteq V$ a node subset of G, then the 0–1 vector $x^S \in \mathbb{R}^V$ with $x^S(u)=1$ if $u \in S$ and $x^S(u)=0$ if not is called the *incidence vector* of S. The convex hull of the incidence vectors of all dominating sets of G, denoted by $P_D(G)$, is called the *dominating set polytope* of G, i.e.

$$P_D(G) = \operatorname{conv}\{x^S \in \mathbb{R}^V \mid S \subseteq V \text{ is a dominating set of } G\}.$$

Every optimal basic solution of the linear program

$$\min\{wx;\ x \in P_D(G)\}$$

is the incidence vector of a minimum dominating set of G.

Since the DSP is NP-hard, we cannot expect to find a complete characterization of $P_D(G)$ for all graphs. It may however be that for certain classes of graphs G, the polytope $P_D(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities, polynomial-time separation algorithm can be designed, so that the DSP on these graphs can be solved in polynomial time.

In contrast of many NP-hard combinatorial optimization problems, the polyhedral aspect of the DSP has not received much attention. To the best of our knowledge, the polytope $P_D(G)$ has been studied only in the class of threshold graphs [25] and the class of strongly chordal graphs [16] within the framework of totally balanced matrices. Our aim, in this paper, is indeed to study the DSP from a polyhedral point of view.

The concept of domination is closely related to that of independence. An *independent set* of $G=(V,E)$ is a node set $S \subseteq V$ such that there is no edge with both endnodes in S. The problem of finding a minimum independent dominating set has also been widely studied [1, 2, 16, 22]. Applications of this problem arise, in particular, in game theory [2].

The paper is organized as follows. In Section 2 we discuss basic facets and structural properties of $P_D(G)$. In Section 3 we study compositions of polyhedra. In Section 4 we discuss applications of these compositions for the cactus.

The rest of this section is devoted to more definitions and notations.

Let $G=(V,E)$ be a graph. If $e \in E$ is an edge whose endnodes are u and v, then we write $e=uv$. If $u \in V$ is a node that is not adjacent to any node of $V \setminus \{u\}$, then u is said to be *isolated*.

A *path* P of $G=(V,E)$ is a sequence of nodes $v_0, v_1, \ldots, v_k$, such that $v_i v_{i+1}$ is an edge for $i=0,\ldots,k-1$ and no node appears more than once in P. The nodes v_0 and v_k are the endnodes of P and we say that P links v_0 and v_k. If $v_0 v_k \in E$, then the sequence $v_0, v_1, \ldots, v_k$, is also called a *cycle*.

We use the standard notation of polyhedral theory. If $a \in \mathbb{R}^m - \{0\}$, $a_0 \in \mathbb{R}$ then the inequality $a^{\mathrm{T}}x \leqslant a_0$ is said to be *valid* with respect to a polyhedral $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^{\mathrm{T}}x \leqslant a_0\}$. We say that a valid inequality $a^{\mathrm{T}}x \leqslant a_0$ defines a *face* of P if $\emptyset \neq P \cap \{a^{\mathrm{T}}x = a_0\} \neq P$. In this case the polyhedron $P \cap \{a^{\mathrm{T}}x = a_0\}$ is called the face associated with $a^{\mathrm{T}}x \leqslant a_0$. A valid inequality $a^{\mathrm{T}}x \leqslant a_0$ defines a *facet* of P if it defines a face of P and if the dimension of $P \cap \{a^{\mathrm{T}}x = a_0\}$ is one less than the dimension of P.

2. On the facets of $P_D(G)$

In this section we shall describe basic facets of $P_D(G)$ and discuss some structural properties.

2.1. Basic facets of $P_D(G)$

Let $G=(V,E)$ be a graph. If $u \in V$ is a node of G, the *neighbourhood* of u in G, denoted by $N(u)$, is the node set consisting of u together with the nodes which are adjacent to u. If $u \in V$, we let $N^*(u) = N(u) \setminus \{u\}$. If $S \subseteq V$, $b : V \to \mathbb{R}$, $b(S)$ will denote $\sum_{u \in S} b(u)$.

If $S \subseteq V$ is a node set, then x^S, the incidence vector of S, satisfies the following inequalities

$$x(u) \geqslant 0 \quad \text{for all } u \in V, \tag{2.1}$$

$$x(u) \leqslant 1 \quad \text{for all } u \in V, \tag{2.2}$$

$$x(N(u)) \geqslant 1 \quad \text{for all } u \in V. \tag{2.3}$$

Inequalities (2.1) and (2.2) will be called *trivial inequalities* and inequalities (2.3) will be called *neighbourhood inequalities* .

In what follows, we shall study when the above inequalities define facets of $P_D(G)$. But first let us state the following lemmas. The first one is easily seen to be true.

Lemma 2.1. *If G does not contain isolated nodes, then $P_D(G)$ is full dimensional.*

Thus, if G is without isolated nodes, a linear system $Ax \geqslant b$ that defines $P_D(G)$ is minimal if and only if there is a bijection between the inequalities of the system and the facets of $P_D(G)$. Moreover, this system of inequalities is unique up to positive multiples.

In the rest of the paper we consider graphs that do not have isolated nodes.

Lemma 2.2. *Every facet defining inequality of $P_D(G)$ except those given by $x(u) \leqslant 1$, is of the form $\sum_{i=1,\dots,k} a_i x_i \geqslant a_0$ with $a_i \geqslant 0$ for $i = 1,\dots,k$.*

Proof. Suppose that $a_{i_0} < 0$ for some $i_0 \in \{1,\dots,k\}$. Since $\sum_{i=1,\dots,k} a_i x_i \geqslant a_0$ is different from $x_{i_0} \leqslant 1$, there must exist a dominating set S such that $i_0 \notin S$ and $\sum_{i=1,\dots,k} a_i x_i^S = a_0$. Let $S' = S \cup \{i_0\}$. Obviously, S' is a dominating set, but $\sum_{i=1,\dots,k} a_i x_i^{S'} < a_0$. This is a contradiction. $\square$

Theorem 2.3. (i) *Inequality* (2.1) *defines a facet of $P_D(G)$ if and only if $|N(v)| \geqslant 3$ for every $v \in N(u)$.*

(ii) *Inequality* (2.2) *defines a facet of $P_D(G)$.*

Proof. (i) Suppose $|N(v)| \leqslant 2$ for some node v of $N(u)$. Since G does not have isolated nodes, there must exist a node u', $u' \neq u$, such that $N(v) = \{u, u'\}$ where v is either equal to u or to u'. Thus, for every dominating set of G, the following holds:

$$u \notin S \Longrightarrow u' \in S.$$

But this implies that the face $\{x \in P_D(G) \mid x(u) = 0\}$ is contained in the face $\{x \in P_D(G) \mid x(u') = 1\}$. Hence, (2.1) cannot define a facet.

Conversely, suppose that $|N(v)| \geqslant 3$ for every $v \in N(u)$. Thus, the sets $V \backslash \{u\}$, $V \backslash \{u, v\}$, for all $v \in V \backslash \{u\}$, define a family of $|V|$ dominating sets whose incidence vectors satisfy inequality (2.1) with equality and are affinely independent. It then follows that (2.1) defines a facet.

(ii) Since G does not contain an isolated node, the sets V, $V\setminus\{v\}$, $v \neq u$, define a family of $|V|$ dominating sets of G whose incidence vectors satisfy inequality (2.2) with equality and are affinely independent. $\square$

Theorem 2.4. *Inequality* (2.3) *defines a facet of* $P_D(G)$ *if and only if the two following conditions are satisfied*:

(i) *There is no node* u' *such that* $N(u') \subset N(u)$.

(ii) *For every node* $w \in V\setminus N(u)$ *such that* $N^*(w) \subseteq N^*(u)$, *there is at least one node* $z \in N^*(w)$ *such that every node* $y \in N^*(w)\setminus\{z\}$ *is adjacent either to* z *or to a node of* $V\setminus(N(u) \cup \{w\})$.

Proof. If there is a node u' such that $N(u') \subset N(u)$, then the inequality (2.3) is implied by that associated with $N(u')$ together with inequalities (2.1). Thus, it does not define a facet. If (ii) does not hold for some node $w \in V\setminus N(u)$, then for every dominating set S of G, the following holds:

$$|S \cap N(u)| = 1 \Rightarrow w \in S. \tag{2.4}$$

Indeed, suppose that $\{S \cap N(u)\} = \{z\}$. If $S \cap N^*(w) = \emptyset$, then clearly, $w \in S$. If $S \cap N^*(w) = \{z\}$, then, as (ii) is not satisfied for w, there must exist a node $y \in N^*(w)\setminus\{z\}$ which is not adjacent neither to z nor to a node of $V\setminus(N(u) \cup \{w\})$. As a consequence, one should have $w \in S$ and thus (2.4) holds. Thus, every dominating set containing exactly one node of $N(u)$ must also contain w. But this implies that the face $\{x \in P_D(G) \mid x(N(u)) = 1\}$ is contained in the face $\{x \in P_D(G) \mid x(w) = 1\}$. Hence, (2.3) is not facet defining.

Now suppose that both (i) and (ii) hold. Let us denote inequality (2.3) by $a^{\mathrm{T}}x \geqslant a_0$ and assume that there is a facet defining inequality $b^{\mathrm{T}}x \geqslant b_0$ such that

$$\{x \in P_D(G) \mid a^{\mathrm{T}}x = a_0\} \subseteq \{x \in P_D(G) \mid b^{\mathrm{T}}x = b_0\}.$$

We will show that there is a scalar $\rho > 0$ such that $b = \rho a$, which implies that (2.3) defines a facet. First we show that $b(u) = b(v)$ for all $v \in N(u)\setminus\{u\}$. For this consider the sets

$$S_1 = V\setminus N^*(u),$$
$$S_v = (V\setminus N(u)) \cup \{v\}, \quad \text{for all } v \in N^*(u),$$

which define dominating sets of G. This is clear for S_1. If S_v is not a dominating set, then there is a node $u' \in N(u)\setminus\{v\}$ that is not adjacent to any node of $(V\setminus N(u)) \cup \{v\}$. But this implies that $N(u') \subset N(u)$, a contradiction.

Moreover, the incidence vectors of S_1, S_v satisfy constraint (2.3) with equality. Thus,

$$0 = bx^{S_1} - bx^{S_v} = b(u) - b(v).$$

Hence,

$$b(u) = b(v) = \rho, \quad \text{for all } v \in N^*(u) \text{ and some } \rho \in \mathbb{R}.$$

Next we show that $b(w) = 0$ for all $w \in V \backslash N(u)$. First consider the case where $N^*(w) \cap (V \backslash N(u)) \neq \emptyset$. Let $\tilde{S} = V \backslash (N^*(u) \cup \{w\})$. Clearly, $\tilde{S}$ is a dominating set of G. Since $a^{\mathrm{T}} x^{\tilde{S}} = a_0$, it follows that $b^{\mathrm{T}} x^{\tilde{S}} = b_0$ and thus

$$\begin{aligned} 0 &= bx^{S_1} - bx^{\tilde{S}}, \\ &= b(w), \end{aligned}$$

where S_1 is the dominating set introduced above. If $N^*(w) \cap (V \backslash N(u)) = \emptyset$, then by (ii) there is a node $z \in N^*(w)$ such that for every node $y \in N^*(w) \backslash \{z\}$, y is adjacent either to z or to a node of $V \backslash (N(u) \cup \{w\})$. Let $\bar{S} = S_z \backslash \{w\}$, where S_z is as defined above. It easy to see that $\bar{S}$ is a dominating set of G. Moreover, we have $a^{\mathrm{T}} x^{\bar{S}} = a_0$. This implies that $b^{\mathrm{T}} x^{\bar{S}} = b_0$ and, as above, it fallows that $b(w) = 0$.

Thus, altogether we have

$$b(v) = \begin{cases} \rho & \text{for all } v \in N(u), \\ 0 & \text{for all } v \in V \backslash N(u). \end{cases}$$

Since for every $v \in V$, there is a dominating set S such that $a^{\mathrm{T}} x^S = a_0$ and $v \notin S$, this implies that the facet defined by $b^{\mathrm{T}} x \geqslant b_0$ is not contained in a trivial facet $\{x \in P_D(G) \mid x(w) = 1\}$ for some $w \in V$. Therefore, $b^{\mathrm{T}} x \geqslant b_0$ defines a nontrivial facet. By Lemma 2.2 it follows that $\rho > 0$ and our proof is complete. □

Lemma 2.4. *Let C_n be a chordless cycle on n nodes. Then the inequality*

$$x(C_n) \geqslant \left\lceil \frac{|C_n|}{3} \right\rceil \tag{2.5}$$

is valid for $P_D(C_n)$. Moreover, it defines a facet of $P_D(C_n)$ if and only if $|C_n|$ is not a multiple of 3.

Proof. Easy. □

2.2. Structural properties

In what follows we shall study some structural properties of the facets of $P_D(G)$. These propreties will be used later for the composition of polyhedra.

Let $a^{\mathrm{T}} x \geqslant a_0$ be an inequality that defines a nontrivial facet of $P_D(G)$, i.e a constaint with at least two nonzero components. Hence by Lemma 2.2 we have $a \geqslant 0$ and $a_0 > 0$. We denote by V_a the set

$$V_a = \{v \in V \mid a(v) > 0\}.$$

The graph $G_a = (V_a, E(V_a))$ is called a *facet-inducing graph.* We denote by $D(G)$ the set of dominating sets of G and by D_a the set

$$D_a = \{S \subseteq V \mid S \in D(G) \text{ and } a^{\mathrm{T}} x = a_0\}.$$

We have the following lemmas.

Lemma 2.5. *The graph G_a is connected.*

Proof. Suppose that G_a is the union of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let a_1 and a_2 be the restrictions of a on V_1 and V_2, respectively.

Letting

$$a_0^i = \min\{a^{\mathrm{T}}x^{S\setminus V_{i'}},\ S \in D(G)\},$$

where $i' = \{1,2\}\setminus\{i\}$, we obtain that $a_0 = a_0^1 + a_0^2$ and $a_i^{\mathrm{T}}x \geqslant a_0^i$ is valid for $P_D(G_i)$, for $i = 1,2$. Thus, $a^{\mathrm{T}}x \geqslant a_0$ can be obtained as the sum of two valid inequalities. But this contradicts the fact that $a^{\mathrm{T}}x \geqslant a_0$ defines a facet. $\square$

Lemma 2.6. *Suppose that G is the 1-sum of a graph $G_1 = (V_1, E_1)$ and a 5-cycle C where $C = \{u, w_1, w_2, w_3, w_4\}$ with $u = C \cap V_1$ (see Fig. 1). Suppose that $a^{\mathrm{T}}x \geqslant a_0$ is different from a neighbourhood inequality and the inequality (2.5) associated with C. Then*

(1) $a(w_1) = a(w_4) \leqslant a(u)$,

(2) $a(w_2) = a(w_3) = 0$.

Proof. First we show that $a(w_1) = a(w_4)$. For this we will show that $a(w_4) \geqslant a(w_1)$. The statement then follows by symmetry. W.l.o.g., we may suppose that $a(w_1) > 0$. Since $a^{\mathrm{T}}x \geqslant a_0$ is different from the inequality $x(C) \geqslant 2$, there must exist a dominating set $S^1 \in D_a$ with $|S^1 \cap C| \geqslant 3$. We shall distinguish three cases.

Case 1: $w_1, w_4 \notin S^1$. Then $\{u, w_2, w_3\} \subseteq S^1$. Since the sets $S^1\setminus\{w_2\}$ and $S^1\setminus\{w_3\}$ are both dominating sets of G, one should have $a(w_2) = a(w_3) = 0$. Now, since $a^{\mathrm{T}}x \geqslant \alpha$ is different from a trivial inequality, there must exist a dominating set $S^2 \in D_a$ with $w_1 \in S^2$. Let $\tilde{S}^2 = (S^2\setminus\{w_1\}) \cup \{w_2, w_3, w_4\}$. It is clear that $\tilde{S}^2 \in D(G)$. Since $a(w_2) = a(w_3) = 0$, it follows that $a(w_4) \geqslant a(w_1)$.

Case 2: $w_1, w_4 \in S^1$. Suppose first that $\{w_2, w_3\} \cap S^1 \neq \emptyset$.

— If $w_2 \in S^1$, then the set $S^1\setminus\{w_1\} \in D(G)$. But this yields $a(w_1) = 0$, which is impossible.

— If $w_3 \in S^1$, then the set $S^1\setminus\{w_4\}$ and $S^1\setminus\{w_3\}$ are both dominating sets of G, and thus we get $a(w_3) = a(w_4) = 0$. Since $a^{\mathrm{T}}x \geqslant a_0$ is different from the neighbourhood inequality associated with w_1, there is a dominating set S^3 such that $|S^3 \cap \{u, w_1, w_2\}| \geqslant 2$. If $\{u, w_1\} \subseteq S^3$ (resp. $\{w_1, w_2\} \subseteq S^3$, $\{u, w_2\} \subseteq S^3$) then the

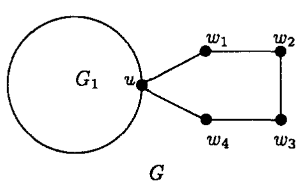

Fig. 1.

set $(S^3\backslash\{w_1\})\cup\{w_3\}$ (resp. $(S^3\backslash\{w_1\})\cup\{w_4\}$, $S^3\backslash\{w_1\}$) is a dominating set of G. Since $a(w_3)=a(w_4)=0$, in all cases we obtain that $a(w_1)=0$, a contradiction.

Consequently, $\{w_2,w_3\}\cap S^1=\emptyset$ and thus we have $\{u,w_1,w_4\}\subseteq S^1$. Then the sets $(S^1\backslash\{w_1\})\cup\{w_3\}$ and $(S^1\backslash\{w_4\})\cup\{w_3\}$ are dominating sets of G. This yields

$$a(w_3)\geqslant a(w_1), a(w_4). \tag{2.6}$$

Since $a^{\mathrm{T}}x\geqslant a_0$ is different from the neighbourhood inequality associated with w_2, there is a dominating set $S^4\in D_a$ with $|S^4\cap\{w_1,w_2,w_3\}|\geqslant 2$.

— If $\{w_1,w_2\}\subseteq S^4$, then the set $(S^4\backslash\{w_1\})\cup\{w_4\}$ is a dominating set of G and thus $a(w_4)\geqslant a(w_1)$.

— If either $\{w_1,w_3\}\subseteq S^4$ or $\{w_2,w_3\}\subseteq S^4$, then the set $(S^4\backslash\{w_3\})\cup\{w_4\}$ is a dominating set of G and thus $a(w_4)\geqslant a(w_3)$. By (2.6), it follows that $a(w_4)\geqslant a(w_1)$.

Case 3: $|\{w_1,w_4\}\cap S^1|=1$. Consider first the case where $w_1\in S^1$. If $u\in S^1$, since $S^1\cap\{w_2,w_3\}\neq\emptyset$, it follows that $S^1\backslash\{w_1\}$ is a dominating set of G. Hence $a(w_1)=0$, a contradiction. As a consequence, we have $\{w_1,w_2,w_3\}\subseteq S^1$. As $S^1\backslash\{w_2\}$ belongs to $D(G)$, we obtain that $a(w_2)=0$. Now consider the set S^2 introduced above. Note that $w_1\in S^2$. Since $(S^2\backslash\{w_1\})\cup\{w_2,w_4\}$ is a dominating set, it then follows that $a(w_4)\geqslant a(w_1)$. If $w_4\in S^1$ we can show in a similar way that $a(w_4)\geqslant a(w_1)$.

In all cases we obtain that $a(w_4)\geqslant a(w_1)$. Since by symmetry we also have $a(w_1)\geqslant a(w_4)$, it then follows that $a(w_1)=a(w_4)$.

Next we show that $a(w_2)=a(w_3)=0$. For this we shall consider two cases.

Case 1′: $a(w_1)=a(w_4)=0$. Since $a^{\mathrm{T}}x\geqslant a_0$ is a nontrivial inequality, there are two dominating sets S and T of D_a such that $w_2\in S$ and $w_3\in T$. Let $S'=(S\backslash\{w_2\})\cup\{w_1,w_4\}$ and $T'=(T\backslash\{w_3\})\cup\{w_1,w_4\}$. Obviously, S' and T' are dominating sets of G, which yield $a(w_2)\leqslant a(w_1)+a(w_4)$ and $a(w_3)\leqslant a(w_1)+a(w_4)$. Since $a(w_2)\geqslant 0$ and $a(w_3)\geqslant 0$, by our hypothesis, it follows that $a(w_2)=a(w_3)=0$.

Case 2′. $a(w_1)=a(w_4)>0$. Consider the dominating set S^1 of D_a introduced above. Note that $|S^1\cap C|\geqslant 3$. First suppose that $\{w_2,w_3\}\cap S^1=\emptyset$. Thus, $S^1\cap C=\{u,w_1,w_4\}$. Let $T^1=(S^1\backslash\{w_1\})\cup\{w_2\}$ and $T^2=(S^1\backslash\{w_4\})\cup\{w_3\}$. Clearly, both T^1 and T^2 are in $D(G)$. Hence,

$$\begin{aligned} &a(w_2)\geqslant a(w_1),\\ &a(w_3)\geqslant a(w_4). \end{aligned} \tag{2.7}$$

We claim that there is $i\in\{2,3\}$ such that $a(w_1)=a(w_4)=a(w_i)$. In fact, consider again the dominating set $S^4\in D_a$ introduced in Case 2, satisfying $|S^3\cap\{w_1,w_2,w_3\}|\geqslant 2$.

— If $\{w_1,w_2\}\subseteq S^4$, then $(S^4\backslash\{w_2\})\cup\{w_4\}\in D(G)$, and thus $a(w_4)\geqslant a(w_2)$. Since $a(w_1)=a(w_4)$, by (2.7) we obtain that $a(w_1)=a(w_4)=a(w_2)$.

— If $\{w_1,w_3\}\subseteq S^4$, then $(S^4\backslash\{w_3\})\cup\{w_4\}\in D(G)$ and we obtain similarly that $a(w_1)=a(w_4)=a(w_3)$.

— If $\{w_2,w_3\}\subseteq S^4$, then the sets $(S^4\backslash\{w_2\})\cup\{w_1\}$ and $(S^4\backslash\{w_3\})\cup\{w_4\}$ are both in $D(G)$. Thus, $a(w_1)\geqslant a(w_2)$ and $a(w_4)\geqslant a(w_3)$, which implies by (2.7) that $a(w_1)=a(w_4)=a(w_2)=a(w_3)$.

Hence, our claim is proved. Now suppose, for instance, that $a(w_1) = a(w_2) = a(w_4)$. Let $\bar{S}^1 = (S^1 \backslash \{w_1, w_4\}) \cup \{w_2\}$. It is clear that $\bar{S}^1 \in D(G)$. Thus, $a(w_2) \geqslant a(w_1) + a(w_4)$. As $a(w_2) = a(w_4)$, it follows that $a(w_1) = 0$. But this is a contradiction.

Now suppose that $|\{w_2, w_3\} \cap S^1| = 1$. W.l.o.g., we may suppose $S^1 \cap \{w_2, w_3\} = w_2$. If $w_1 \in S^1$ (resp. $w_4 \in S^1$), since $|S^1 \cap C| \geqslant 3$ and thus $S^1 \cap \{u, w_4\} \neq \emptyset$ (resp. $S^1 \cap \{u, w_1\} \neq \emptyset$), we have that $S^1 \backslash \{w_1\}$ (resp. $S^1 \backslash \{w_4\}$) is a dominating set of G. But this implies that $a(w_1) = 0$ (resp. $a(w_4) = 0$), a contradiction.

Consequently, $\{w_2, w_3\} \subseteq S^1$. If $u \in S^1$, then the sets $S^1 \backslash \{w_2\}$ and $S^1 \backslash \{w_3\}$ are both in $D(G)$, implying that $a(w_2) = a(w_3) = 0$. If not then we may, w.l.o.g., assume that $w_1 \in S^1$. Clearly, the sets $S^1 \backslash \{w_2\}$ and $(S^1 \backslash \{w_1, w_3\}) \cup \{w_4\}$ belong to $D(G)$. As $a(w_1) = a(w_4)$, it follows that $a(w_2) = a(w_3) = 0$.

Finally, we show that $a(w_1) (= a(w_4)) \leqslant a(u)$. For this consider again the dominating set $S^2 \in D_a$ introduced in Case 1, containing w_1. Since the set $(S^2 \backslash \{w_1\}) \cup \{u, w_2\}$ is also a dominating set and $a(w_2) = 0$, it follows that $a(w_1) \leqslant a(u)$, and our proof is complete. □

3. Composition of polyhedra

In this section we derive a system of inequalities that defines $P_D(G)$ provided that G is the 1-sum, of two graphs and such a system is known for two graphs related to the pieces.

Consider a graph $G = (V, E)$ that is a 1-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $\{u\} = V_1 \cap V_2$. Let $\bar{G}_1 = (\bar{V}_1, \bar{E}_1)$ ($\bar{G}_2 = (\bar{V}_2, \bar{E}_2)$) be the graphs obtained from G_1 (G_2) by adding the nodes w_1, w_2, w_3, w_4 and the edges $\{uw_1, w_1w_2, w_2w_3, w_3w_4, w_4u\}$ (see Fig. 2) Let $C = (u, w_1, w_2, w_3, w_4)$. In what follows, we shall show that a system of inequalities defining $P_D(G)$ can be obtained whenever such a system is known for $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$.

Lemmas 2.4 and 2.6 imply that the polytopes $P_D(\bar{G}_k)$, for $k = 1, 2$, can be assumed to be described by two minimal linear inequality systems of the following

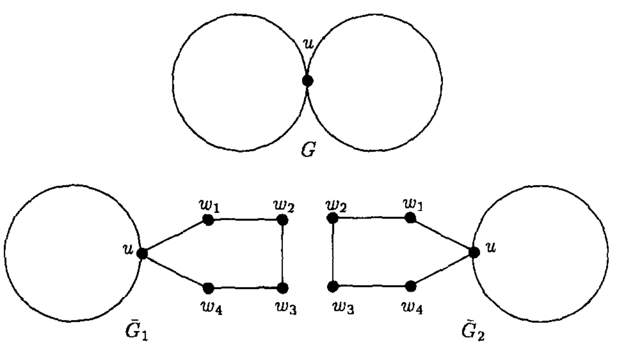

Fig. 2.

form:

$$P_D(\bar{G}_k)\begin{cases}\sum_{j\in V_k} a_i^k(j)x(j)\geqslant\alpha_i^k, \quad i\in I_1^k, & (3.1)\\ \sum_{j\in V_k} a_i^k(j)x(j)+x(w_1)+x(w_4)\geqslant\alpha_i^k, \quad i\in I_2^k, & (3.2)\\ x(u)+x(w_1)+x(w_2)\geqslant 1, & (3.3)\\ x(w_1)+x(w_2)+x(w_3)\geqslant 1, & (3.4)\\ x(w_2)+x(w_3)+x(w_4)\geqslant 1, & (3.5)\\ x(u)+x(w_4)+x(w_3)\geqslant 1, & (3.6)\\ x(u)+x(w_1)+x(w_2)+x(w_3)+x(w_4)\geqslant 2, & (3.7)\\ x(j)\leqslant 1, \quad j\in\bar{V}_k, & (3.8)\\ x(j)\geqslant 0, \quad j\in\bar{V}_k. & (3.9)\end{cases}$$

The set I_1^k consists of the nontrivial inequalities whose support does not intersect C. The inequalities of I_2^k have a support that contains both w_1 and w_4, and has empty intersection with $\{w_2, w_3\}$.

A constraint of type (3.2) corresponding to $i\in I_2^k$ will be denoted by $[i,k]$. Given two inequalities $[i,1]$ and $[l,2]$, we call *mixed inequality* of $[i,1]$ and $[l,2]$, denoted by $\langle i,l\rangle$, the inequality given by

$$\sum_{j\in V_1\setminus\{u\}} a_i^1(j)x(j)+\sum_{j\in V_2\setminus\{u\}} a_l^2(j)x(j)+(a_i^1(u)+a_l^2(u)-1)x(u)\geqslant\alpha_i^1+\alpha_l^2-1 \tag{3.10}$$

Note that any mixed inequalities have nonnegative coefficients. Moreover we have the following.

Lemma 3.1. *Mixed inequalities are valid for $P_D(G)$.*

Proof. Let S be a dominating set of G. We distinguish two cases.

Case 1: $u\in S$. Let $S_1=(S\cap V_1)\cup\{w_2\}$ and $S_2=(S\cap V_2)\cup\{w_2\}$. It is clear that S_1 and S_2 are dominating sets of $\bar{G}_1$ and $\bar{G}_2$, respectively. Thus, we have

$$\sum_{j\in V_1} a_i^1(j)x^{S_1}(j)\geqslant\alpha_i^1, \tag{3.11}$$

$$\sum_{j\in V_2} a_l^2(j)x^{S_2}(j)\geqslant\alpha_l^2. \tag{3.12}$$

From (3.11) and (3.12) together with $x(u)\leqslant 1$, it follows that (3.10) is satisfied by x^S.

Case 2: $u\notin S$. Thus, there exists a node $v\in N^*(u)$ that belongs to S. W.l.o.g., we may suppose that $v\in V_1$. Let

$$S_3=(S\cap V_1)\cup\{w_2,w_3\},$$

$$S_4=(S\cap V_2)\cup\{w_1,w_3\}.$$

Obviously, S_3 and S_4 are dominating sets of $\bar{G}_1$ and $\bar{G}_2$, respectively. Thus,

$$\sum_{j\in V_1\setminus\{u\}} a_i^1(j)x^{S_3}(j)\geqslant\alpha_i^1, \tag{3.13}$$

$$\sum_{j\in V_2\setminus\{u\}} a_l^2(j)x^{S_4}(j)\geqslant\alpha_l^2-1. \tag{3.14}$$

By adding (3.13) and (3.14) we obtain that (3.10) is satisfied by x^S. □

Definition 3.2. Let $G=(V,E)$ be a graph. Suppose that $P_D(G)$ is given by the system $\{Ax\geqslant b,\ x\geqslant 0\}$ where A is an (m,n) matrix and b is an m-column vector. If $a_1^{\mathrm{T}}x\geqslant\alpha_1$ and $a_2^{\mathrm{T}}x\geqslant\alpha_2$ are two valid constraints of $P_D(G)$, then we say that $a_2^{\mathrm{T}}x\geqslant\alpha_2$ *dominates* $a_1^{\mathrm{T}}x\geqslant\alpha_1$ if (i) there exists an m-row vector $y\geqslant 0$ such that $a_2=yA$, $\alpha_2=yb$ (i.e. $a_2^{\mathrm{T}}x\geqslant\alpha_2$ is a linear combination of the constraints of the system $Ax\geqslant b$), and
(ii) $a_2\leqslant a_1$, $\alpha_2\geqslant\alpha_1$.

We then have the following.

Lemma 3.3. *Let $a^{\mathrm{T}}x\geqslant\alpha$ be a valid inequality of $P_D(\bar{G}_k)$ $k=1,2$ with $a(w_2)=a(w_3)=0$ and $a(w_1)=a(w_4)$. Then there exists a valid inequality $\bar{a}^{\mathrm{T}}x\geqslant\bar{\alpha}$ of $P_D(\bar{G}_k)$, $k=1,2$, that dominates $a^{\mathrm{T}}x\geqslant\alpha$ with $\bar{a}(w_2)=\bar{a}(w_3)=0$ and $\bar{a}(w_1)=\bar{a}(w_4)$. Moreover, if $a(w_1)\neq 0\neq a(w_4)$ (resp. $a(w_1)=a(w_4)=0$) then $\bar{a}^{\mathrm{T}}x\geqslant\bar{\alpha}$ can be chosen as a linear combination of the inequalities* (3.1),(3.2) *(resp.* (3.1)*) and the constraints $x(i)\leqslant 1$ for $i\in V_k$.*

Proof. First notice that if $a^{\mathrm{T}}x\geqslant\alpha$ is a linear combination of the constraints of $P_D(\bar{G}_k)$, then one can take $\bar{a}=a$ and $\bar{\alpha}=\alpha$. Now assume that this is not the case and, for instance, that $a^{\mathrm{T}}x\geqslant\alpha$ is valid for $P_D(\bar{G}_1)$. Let $A_1x\geqslant b_1$ denote the system given by inequalities (3.1) and (3.2). Then the linear program

$$\min\{a^{\mathrm{T}}x;\ x\in P_D(\bar{G}_1)\} \tag{3.15}$$

has an optimal solution x_0 such that $a^{\mathrm{T}}x_0\geqslant\alpha$. (Note that x_0 can be considered as the incidence vector of some dominating set of $\bar{G}_1$.) By LP-duality there exists an optimal dual solution $(y,z,t)\geqslant 0$, where $y=(y(i),\ i\in I_1^1\cup I_2^1)$ is associated with the constraints (3.1) and (3.2), $z=(z(j),\ j\in\bar{V}_1)$ is associated with the constraints $x(j)\geqslant 1$, $j\in\bar{V}_1$ and $t=(t(w_1),\dots,t(w_4),t(C))$ is associated with the constraints (3.3)–(3.7), such that

$$yA_1^{(i)}-z(i)\leqslant a(i)\quad\text{for } i\in\bar{V}_1\setminus C,$$

$$yA_1^{(u)}-z(u)+t(w_1)+t(w_4)+t(C)\leqslant a(u),$$

$$yA_1^{(w_1)}+t(w_1)+t(w_2)+t(C)-z(w_1)\leqslant a(w_1), \tag{3.16}$$

$$yA_1^{(w_4)}+t(w_3)+t(w_4)+t(C)-z(w_4)\leqslant a(w_4), \tag{3.17}$$

$$t(w_1)+t(w_2)+t(w_3)+t(C)-z(w_2)\leqslant 0, \tag{3.18}$$

$$t(w_2)+t(w_3)+t(w_4)+t(C)-z(w_3)\leqslant 0, \tag{3.19}$$

$$yb_1-\sum_{i\in\bar{V}_1} z(i)+t(w_1)+t(w_2)+t(w_3)+t(w_4)+2t(C)=a^{\mathrm{T}}x_0\geqslant\alpha,$$

where $A^{(i)}$ is the column of A associated with node i.

Since the dual program is to be maximized, by (3.18) and (3.19), it follows that

$$z(w_2)=z(w_3)=0,$$

$$t(w_1)=t(w_2)=t(w_3)=t(w_4)=t(C)=0.$$

Moreover, we should have $z(w_1)=z(w_4)=0$. For this, first note that $x_0(w_i)=0$ for at least one node of $\{w_1,w_4\}$. In fact, if $x_0(w_1)=x_0(w_4)=1$, then let $\bar{x}_0\in\mathbb{R}^{|\bar{V}_1|}$ such that

$$\bar{x}_0(i)=\begin{cases} x_0(i) & \text{if } i\in V_1,\\ 1 & \text{if } i\in\{w_1,w_3\},\\ 0 & \text{if } i\in\{w_2,w_4\}.\end{cases}$$

Clearly, $\bar{x}_0$ induces a dominating set of G. Thus, $\bar{x}_0$ is a solution of (3.15). Since $a^{\mathrm{T}}\bar{x}_0<a^{\mathrm{T}}x_0$, this contradicts the fact that x_0 is an optimal solution of (3.15).

Consequently, we may suppose, for instance, that $x_0(w_1)=0$. By complementary slackness, it then follows that $z(w_1)=0$. Now if $x_0(w_4)=0$, then by the same argument we have $z(w_4)=0$. If not, then consider again the solution $\bar{x}_0$ defined above. Since $a(w_1)=a(w_4)$, we have that $a^{\mathrm{T}}\bar{x}_0=a^{\mathrm{T}}x_0$. Thus, $\bar{x}_0$ is an optimal solution for (3.15), which yields $z(w_4)=0$.

Let

$$\begin{aligned} &\bar{a}(i)=yA_1^{(i)}z(i) \quad \text{for } i\in V_1,\\ &\bar{a}(i)=yA_1^{(i)} \quad \text{for } i\in\{w_1,w_4\},\\ &\bar{a}(w_2)=\bar{a}(w_3)=0,\\ &\bar{\alpha}=yb_1\sum_{i\in V_1} z(i). \end{aligned} \tag{3.20}$$

Then $\bar{a}^{\mathrm{T}}x\geqslant\bar{\alpha}$ is the required constraint.

Now if $a(w_1)\neq 0\neq a(w_4)$, by (3.20) it is clear that $\bar{a}^{\mathrm{T}}x\geqslant\bar{\alpha}$ is a linear combination of inequalities (3.1), (3.2) and the constraints $x(i)\leqslant 1$, for $i\in V_1$. If $a(w_1)=a(w_4)=0$, since $z(w_1)=z(w_4)=0$, by (3.16) and (3.17) it follows that

$$yA_1^{(w_1)}=yA_1^{(w_4)}=\sum_{i\in I_2^1} y(i)\leqslant 0.$$

As $y(i)\geqslant 0$ for $i\in I_2^1$, one obtains that $y(i)=0$ for $i\in I_2^1$. By (3.20), it then follows that $\bar{a}^{\mathrm{T}}x\geqslant\bar{\alpha}$ is a linear combination of inequalities (3.1) and the constraints $x(i)\leqslant 1$, for $i\in V_1$. $\square$

Lemma 3.4. *Let*

$$\sum_{i\in V_1} a^1(i)x(i) + a^1(w_1)x(w_1) + a^1(w_4)x(w_4) \geqslant \alpha_1, \tag{3.21}$$

$$\sum_{i\in V_2} a^2(i)x(i) + a^2(w_1)x(w_1) + a^2(w_4)x(w_4) \geqslant \alpha_2 \tag{3.22}$$

be two valid inequalities of $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$, respectively, such that $a^1(w_1) = a^1(w_4) = a^2(w_1) = a^2(w_4) = \sigma \neq 0$. Suppose that inequality (3.21) *(resp.* (3.22)*) is a linear combination of inequalities* (3.1), (3.2) *and the constraints $x(i) \leqslant 1$, for $i \in V_1$ (resp. $i \in V_2$). Then the inequality*

$$\sum_{i\in V_1\setminus\{u\}} a^1(i)x(i) + \sum_{i\in V_2\setminus\{u\}} a^2(i)x(i) + (a^1(u) + a^2(u) - \sigma)x(u) \geqslant \alpha_1 + \alpha_2 - \sigma \tag{3.23}$$

is redundant with respect to the system defining $P_D(G)$.

Proof. Inequality (3.21) (resp. (3.22)) can be written as the sum of three inequalities $c^1x \geqslant \alpha^1$, $b^1x + b^1(w_1)x(w_1) + b^1(w_4)x(w_4) \geqslant \beta^1$ and $d^1x \geqslant \delta^1$ (resp. $c^2x \geqslant \alpha^2$, $b^2x + b^2(w_1)x(w_1) + b^2(w_4)x(w_4) \geqslant \beta^2$ and $d^2x \geqslant \delta^2$) that are linear combinations of the inequalities (3.1), (3.2) and constraints $x(i) \leqslant 1$, $i \in V_1$ (resp. $i \in V_2$), respectively. Let $y^1 = (y_i^1,\ i \in I_2^1)$ (resp. $y^2 = (y_i^2,\ i \in I_2^2)$) be the vector yielding the inequality involving w_1 and w_4. We have

$$\sum_{i\in I_2^1} y_i^1 = \sum_{j\in I_2^2} y_j^2 = \sigma.$$

Thus, there are $|I_2^1| \times |I_2^2|$ values $x_{ij} \geqslant 0$ that are a solution to the transportation constraints

$$\sum_{i\in I_2^1} x_{ij} = y_j^2 \quad \text{for } j \in I_2^2,$$

$$\sum_{j\in I_2^2} x_{ij} = y_i^1 \quad \text{for } i \in I_2^1.$$

Now for every $(i,j) \in I_2^1 \times I_2^2$ such that $x_{ij} \neq 0$, mix the constrains $[i,1]$ and $[j,2]$ and multiply the resulting constraint by x_{ij}. Let $\tilde{a}^{\mathrm{T}}x \geqslant \tilde{\alpha}$ be the sum of all the mixed constraints thus obtained. We have

$$\tilde{a}(v) = \sum_{i\in I_2^1} y_i^1 a_i^1(v) \quad \text{if } v \in V_1\setminus\{u\},$$

$$\tilde{a}(v) = \sum_{j\in I_2^2} y_j^2 a_j^2(v) \quad \text{if } v \in V_2\setminus\{u\},$$

$$\tilde{a}(u) = \sum_{i \in I_2^1} y_i^1 a_i^1(u) + \sum_{j \in I_2^2} y_j^2 a_j^2(u) - \sigma,$$

$$\tilde{\alpha} = \sum_{i \in I_2^1} y_i^1 \alpha_i^1 + \sum_{j \in I_2^2} y_j^2 \alpha_j^2 - \sigma.$$

Now by summing the inequalities $\tilde{a}^{\mathrm{T}} x \geqslant \tilde{\alpha}$ and $c^k x \geqslant \alpha^k$, and $d^k x \geqslant \delta^k$, for $k = 1,2$, we obtain inequality (3.23). □

Note that inequalities (3.1), for $k = 1,2$, are valid for $P_D(G)$.

Let $P_0(G)$ be the polytope in $\mathbb{R}^{|V|}$ given by the trivial inequalities together with the inequalities (3.1) and (3.10). What we are going to show in the following is that $P_0(G)$ is precisely the polytope $P_D(G)$. To this end we need some lemmas.

Lemma 3.5. *Neighbourhood inequalities are redundant with respect to the system defining $P_0(G)$.*

Proof. Let $v \in V$. Suppose, w.l.o.g. that $v \in V_1$. We shall distinguish two cases.

Case 1: $v \neq u$. Then the inequality

$$\sum_{i \in N(v)} x(i) \geqslant 1 \tag{3.24}$$

is valid for $P_D(\bar{G}_1)$. By Lemma 3.3 it follows that (3.24) is dominated by a valid inequality of $P_D(\bar{G}_1)$ that is a linear combination of inequalities (3.1) and $x(i) \leqslant 1$, for $i \in V_1$. Thus, (3.24) is redundant with respect to $P_0(G)$.

Case 2: $v = u$. Then the inequalities

$$\sum_{i \in V_1 \cap N(u)} x(i) + x(w_1) + x(w_4) \geqslant 1, \tag{3.25}$$

$$\sum_{i \in V_2 \cap N(u)} x(i) + x(w_1) + x(w_4) \geqslant 1, \tag{3.26}$$

associated with the neighbourhood sets of u in $\bar{G}_1$ and $\bar{G}_2$, respectively, are valid for $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$. Thus, by Lemma (3.3) there is a valid inequality $a^{1^{\mathrm{T}}} x \geqslant \alpha^1$ (resp. $a^{2^{\mathrm{T}}} x \geqslant \alpha^2$) of $P_D(\bar{G}_1)$ (resp. $P_D(\bar{G}_2)$), that dominates (3.25) (resp. (3.26)). Moreover, this inequality is a linear combination of the inequalities (3.1), (3.2) and $x(i) \leqslant 1$, $i \in V_1$ (resp. $i \in V_2$). Hence, we have

$$a^1(w_1) = a^1(w_4) \leqslant 1,$$

$$a^2(w_1) = a^2(w_4) \leqslant 1,$$

$$a^k(i) \leqslant 1 \quad \text{for all } i \in \bar{V}_k \cap N(u),\ k = 1,2,$$

$$a^k(i) = 0 \quad \text{for all } i \in \bar{V}_k \backslash N(u),\ k = 1,2,$$

$$\alpha^1 \geqslant 1, \qquad \alpha^2 \geqslant 1.$$

Let $\rho^1 = 1 - a^1(w_1)$, $\rho^2 = 1 - a^2(w_1)$. Note that $\rho^1 \geqslant 0$ and $\rho^2 \geqslant 0$. Let $b^{k^T}x \geqslant \beta^k$, for $k = 1,2$, be the inequality obtained by summing $a^{k^T}x \geqslant \alpha^k$ and the inequalities $\rho^k(w_1) \geqslant 0$ and $\rho^k(w_4) \geqslant 0$. Hence $b^k(w_1) = b^k(w_4) = 1$, for $k = 1,2$. By Lemma 3.4 the mixed inequality

$$\sum_{i\in V_1\setminus\{u\}} b^1(i)x(i) + \sum_{j\in V_2\setminus\{u\}} b^2(j)x(j) + (b^1(u) + b^2(u) - 1)x(u) \geqslant \beta^1 + \beta^2 - 1$$

is redundant with respect to the system defining $P_0(G)$. Now it is not hard to see that this inequality dominates (3.24). $\square$

Lemma 3.6. *Let* $\sum_{i\in V_k} a(i)x(i) + x(w_1) + x(w_4) \geqslant \alpha$ *be a valid constraint for* $P_D(\bar{G}_k)$. *Then the constraint*

$$\sum_{i\in V_k\setminus\{u\}} a(i)x(i) + (a(u) - 1)x(u) \geqslant \alpha - 1 \tag{3.27}$$

is valid for $P_D(G)$.

Proof. Suppose $k = 1$. Let S be a dominating set of G. If $u \in S$, then let $S' = (S \cap V_1) \cup \{w_3\}$. Since $S' \in D(\bar{G}_1)$, we then have $\sum_{i\in \bar{V}_1} a(i)x^{S'}(i) \geqslant \alpha$. This implies that (3.27) is satisfied by x^S. If $u \notin S$, then there exists a node $w \neq u$ such that $w \in N(u) \cap S$. If $w \in V_1\setminus\{u\}$, then let $S^1 = (S \cap V_1) \cup \{w_2, w_3\}$. Thus, $S^1 \in D(\bar{G}_1)$ and, consequently, $\sum_{i\in V_1} a(i)x^{S^1}(i) \geqslant \alpha$. Which implies that (3.27) is satisfied by x^S. If $w \in V_2\setminus\{u\}$, then let $S^2 = (S \cap V_1) \cup \{w_1, w_3\}$. Clearly, $S^2 \in D(\bar{G}_1)$, and thus $\sum_{i\in V_1\setminus\{u\}} a(i)x^{S^2}(i) + x(w_1) \geqslant \alpha$. It then follows that $\sum_{i\in V_1\setminus\{u\}} a(i)x^S(i) = \sum_{i\in V_1\setminus\{u\}} a(i)x^{S^2}(i) \geqslant \alpha - 1$, and hence (3.27) is satisfied by x^S, which ends the proof of our lemma. $\square$

Now we are ready to state the main result of this section.

Theorem 3.7. $P_0(G) = P_D(G)$.

Proof. By Lemma 3.5 every integral feasable solution of $P_0(G)$ is the incidence vector of a dominating set of G. Since $P_D(G) \subseteq P_0(G)$, to prove the result, it suffices to show that every extreme point of $P_0(G)$ is integral.

Suppose not, then let x be a fractional extreme point of $P_0(G)$. We shall examine two cases.

Case 1: $x(u) = 1$. Let $\bar{x}_k \in \mathbb{R}^{|\bar{V}_k|}$, for $k = 1,2$, be the solution given by

$$\bar{x}_k(i) = \begin{cases} x(i) & \text{if } i \in V_k, \\ 1 & \text{if } i = w_3, \\ 0 & \text{if } i \in \{w_1, w_2, w_4\}. \end{cases}$$

We claim that $\bar{x}_k \in P_D(\bar{G}_k)$ for $k = 1,2$. We shall show the claim for $\bar{x}_1$ (the proof is similar for $\bar{x}_2$). In fact, it is clear that $\bar{x}_1$ satisfies all the constraints of $P_D(\bar{G}_1)$ whose support does not intersect $\{w_1, w_4\}$.

Now consider a constraint

$$\sum_{j\in V_1} a_i^1(j)x(j)+x(w_1)+x(w_4)\geqslant\alpha_i^1 \tag{3.28}$$

of type (3.2). By Lemma 3.6, the constraint

$$\sum_{j\in V_1\setminus\{u\}} a_i^1(j)x(j)+(a_i^1(u)-1)x(u)\geqslant\alpha_i^1-1$$

is valid for $P_D(G)$. Since $x(u)=1$, it then follows that constraint (3.28) is satisfied by $\bar{x}_1$.

Thus, $\bar{x}_1\in P_D(\bar{G}_1)$ and $\bar{x}_2\in P_D(\bar{G}_2)$. As a consequence, $\bar{x}_1$ and $\bar{x}_2$ can be written as

$$\bar{x}_1=\sum_{i=1,\dots,s}\lambda_i y_i, \tag{3.29}$$

$$\bar{x}_2=\sum_{j=1,\dots,t}\mu_j z_j, \tag{3.30}$$

where $\sum_{i=1,\dots,s}\lambda_i=1$, $\sum_{j=1,\dots,t}\mu_j=1$, $\lambda_i\geqslant 0$, $\mu_j\geqslant 0$ and y_i and z_j are integer extreme points of $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$, respectively, for $i=1,\dots,s$, and $j=1,\dots,t$. At this point we should remark that any constraint of $P_D(\bar{G}_1)$ (resp. $P_D(\bar{G}_2)$) which is tight for $\bar{x}_1$ (resp. $\bar{x}_2$) is at the same time tight for y_i, $i=1,\dots,s$ (resp. z_j, $j=1,\dots,t$). In particular, since $\bar{x}_1(u)=\bar{x}_2(u)=1$, we should have $y_i(u)=1$ and $z_j(u)=1$, for $i=1,\dots,s$, and $j=1,\dots,t$.

Let $x^*\in\mathbb{R}^{|V|}$ be the solution such that

$$x^*(i)=\begin{cases} y_1(i) & \text{if } i\in V_1,\\ z_1(i) & \text{if } i\in V_2\setminus\{u\}.\end{cases}$$

We claim that every constraint of $P_0(G)$ that is tight for x is also tight for x^*. (A constraint is tight for a solution x if it is satisfied by x as equation). In fact, by the remark above, any constraint among (3.1), (3.8), (3.9) that is tight for x and thus for $\bar{x}_k$ is also tight for x^*. Now consider a mixed constraint $\langle i,j\rangle$ that is tight for x. Since $x(u)=1$, we have

$$\sum_{v\in V_1\setminus\{u\}} a_i^1(v)x(v)+\sum_{w\in V_2\setminus\{u\}} a_j^2(w)x(w)=\alpha_i^1+\alpha_j^2-a_i^1(u)-a_j^2(u).$$

By Lemma 3.6, we also have

$$\sum_{v\in V_1\setminus\{u\}} a_i^1(v)x(v)\geqslant\alpha_i^1-a_i^1(u),$$

$$\sum_{w\in V_2\setminus\{u\}} a_j^2(w)x(w)\geqslant\alpha_j^2-a_j^2(u).$$

It then follows that the above inequalities are tight for x and thus for $\bar{x}_1$ and $\bar{x}_2$, respectively, which implies that $\langle i,j\rangle$ is tight for x^*. Consequently, any constraint of $P_0(G)$ that is tight for x is also tight for x^*. Since $x\neq x^*$, this contradicts the extremality of x.

Case 2: $x(u) < 1$.

Case 2.1: None of the mixed inequalities is tight for x. Then any nontrivial inequality of $P_D(G)$ that is tight for x is among inequalities (3.1). Let $\bar{x}_1 \in \mathbb{R}^{\bar{V}_1}$ and $\bar{x}_2 \in \mathbb{R}^{\bar{V}_2}$ be the solutions given by

$$\bar{x}_1(j) = \begin{cases} x(j) & \text{if } j \in V_1, \\ 1 & \text{if } j \in \{w_1, w_4\}, \\ 0 & \text{if } j \in \{w_2, w_3\}, \end{cases}$$

and

$$\bar{x}_2(j) = \begin{cases} x(j) & \text{if } j \in V_2, \\ 1 & \text{if } j \in \{w_1, w_4\}, \\ 0 & \text{if } j \in \{w_2, w_3\}. \end{cases}$$

It is easy to see that $\bar{x}_1$ and $\bar{x}_2$ belong to $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$, respectively. Thus, $\bar{x}_1$ and $\bar{x}_2$ can be written as in (3.29) and (3.30). Since $x(u) < 1$ and thus $\bar{x}_1(u) < 1$ and $\bar{x}_2(u) < 1$, there must exist $i_0 \in \{1, \ldots, s\}$ and $j_0 \in \{1, \ldots, t\}$ such that $y_{i_0}(u) = z_{j_0}(u) = 0$.

Let $\bar{x} \in \mathbb{R}^{|V|}$ be such that

$$\bar{x}(j) = \begin{cases} y_{i_0}(j) & \text{if } j \in V_1, \\ z_{j_0}(j) & \text{if } j \in V_2 \setminus \{u\}. \end{cases} \tag{3.31}$$

We have that every constraint that is tight for x is also tight for $\bar{x}$. Since $x \neq \bar{x}$, this is a contradiction.

Case 2.2: One of the mixed constraints is tight for x. Let $r \in I_2^1$ and $s \in I_2^2$ be such that the mixed inequality $\langle r, s \rangle$ is tight for x, that is

$$\sum_{j \in V^1 \setminus \{u\}} a_r^1(j)x(j) + \sum_{k \in V^2 \setminus \{u\}} a_s^2(k)x(k) + (a_r^1(u) + a_s^2(u) - 1)x(u) = \alpha_r^1 + \alpha_s^2 - 1. \tag{3.32}$$

Let $\varepsilon = x(u)$. We claim that there is $0 \leqslant \lambda \leqslant 1$ such that

$$\sum_{j \in V^1} a_r^1(j)x(j) = \alpha_r^1 - 1 + \lambda, \tag{3.33}$$

$$\sum_{k \in V^2} a_s^2(k)x(k) = \alpha_s^2 - \lambda + \varepsilon. \tag{3.34}$$

In fact, first note that by Lemma 3.36 together with (3.32) it follows that (3.33) and (3.34) hold for some $\lambda \geqslant \varepsilon \geqslant 0$. Now if $\lambda > 1$, then from (3.32) it follows that

$$\sum_{k \in V^2 \setminus \{u\}} a_s^2(k)x(k) + a_s^2(u)x(u) < \alpha_s^2 - 1 + \varepsilon.$$

Since $x(u) = \varepsilon$, this contradicts Lemma 3.36.

Now let $\bar{x}_1 \in \mathbb{R}^{|V_1|}$ and $\bar{x}_2 \in \mathbb{R}^{|V_2|}$ be the solutions given by

$$\bar{x}_1(j) = \begin{cases} x(j) & \text{if } j \in V_1, \\ 1-\lambda & \text{if } j = w_1, \\ 1 & \text{if } j = w_3, \\ \lambda - \varepsilon & \text{if } j = w_2, \\ 0 & \text{if } j = w_4 \end{cases}$$

and

$$\bar{x}_2(j) = \begin{cases} x(j) & \text{if } j \in V_2, \\ \lambda - \varepsilon & \text{if } j = w_1, \\ 1 & \text{if } j = w_3, \\ 1-\lambda & \text{if } j = w_2, \\ 0 & \text{if } j = w_4. \end{cases}$$

We claim that $\bar{x}_1$ and $\bar{x}_2$ belong to $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$, respectively. We will show that $\bar{x}_1 \in P_D(\bar{G}_1)$, the proof for $\bar{x}_2 \in P_D(\bar{G}_2)$ is similar. It is clear that $\bar{x}_1$ ($\bar{x}_2$) verifies the constraints (3.1) and the trivial inequalities. Thus, consider an inequality of the form

$$\sum_{j \in V^1} a_i^1(j)x(j) + x(w_1) + x(w_4) \geqslant \alpha_i^1. \tag{3.35}$$

We claim that, there exists $\lambda' \geqslant \lambda$, such that

$$\sum_{j \in V^1} a_i^1(j)\bar{x}_1(j) = \sum_{j \in V^1} a_i^1(j)x(j) = \alpha_i^1 - 1 + \lambda', \tag{3.36}$$

Indeed, from Lemma 3.36, there is $\lambda' \geqslant 0$ for which (3.36) holds. Now if $\lambda' < \lambda$, then it is easy to see that by mixing inequality (3.35) and inequality $[s,2]$, one gets an inequality that is not satisfied by x, a contradiction. Thus, $\lambda' \geqslant \lambda$, which implies that inequality (3.35) is satisfied by $\bar{x}_1$. Consequently, $\bar{x}_1 \in P_D(\bar{G}_1)$ and $\bar{x}_2 \in P_D(\bar{G}_2)$ and hence $\bar{x}_1$ and $\bar{x}_2$ can be written as in (3.29) and (3.30), respectively. Since $0 \leqslant \lambda \leqslant 1$ and $\varepsilon < 1$, we have $\sigma = \max(1-\lambda, \lambda - \varepsilon) > 0$. W.l.o.g, we may suppose that $\sigma = 1 - \lambda > 0$. Hence, there must exist $i_0 \in \{1,\ldots,s\}$ such that $y_{i_0}(w_1) = 1$. Also, by the same argument, there must exist $j_0 \in \{1,\ldots,t\}$ such that $z_{j_0}(w_2) = 1$. As y_{i_0} (z_{i_0}) satisfies as equation the constraints of $P_D(\bar{G}_1)$ ($P_D(\bar{G}_2)$) that are tight for $\bar{x}_1$ ($\bar{x}_2$), it follows that

$$\begin{aligned} y_{i_0}(u) &= z_{j_0}(u) = 0, \\ y_{i_0}(w_3) &= z_{j_0}(w_3) = 1, \\ y_{i_0}(C) &= z_{j_0}(C) = 2. \end{aligned}$$

Now let $\bar{x} \in \mathbb{R}^{|V|}$ be the solution defined as in (3.31). We claim that every constraint that is tight for x is at the same time tight for $\bar{x}$. In fact, if an inequality of type (3.1) is tight for x and thus for $\bar{x}_1$ ($\bar{x}_2$), then it is also tight for y_{i_0} (z_{i_0}) and thus for $\bar{x}$.

Now consider a mixed inequality $\langle i, l\rangle$ which is tight for x, that is

$$\sum_{j\in V^1\setminus\{u\}} a_i^1(j)x(j) + \sum_{k\in V^2\setminus\{u\}} a_l^2(k)x(k) + (a_i^1(u)+a_l^2(u)-1)x(u) = \alpha_i^1 + \alpha_l^2 - 1. \tag{3.37}$$

We claim that the following hold:

$$\sum_{j\in V^1} a_i^1(j)x(j) = \alpha_i^1 - 1 + \lambda, \tag{3.38}$$

$$\sum_{k\in V^2} a_l^2(k)x(k) = \alpha_l^2 - \lambda + \varepsilon. \tag{3.39}$$

In fact, by the claim above, there is $\lambda' \geqslant \lambda$ such that (3.36) holds. Now if $\lambda' > \lambda$, by (3.32), it follows that

$$\sum_{k\in V^2} a_l^2(k)x(k) < \alpha_l^2 - \lambda + \varepsilon. \tag{3.40}$$

By summing (3.33) and (3.40) we obtain

$$\begin{aligned}&\sum_{j\in V^1\setminus\{u\}} a_r^1(j)x(j) + \sum_{k\in V^2\setminus\{u\}} a_l^2(k)x(k) + (a_r^1(u)+a_l^2(u)-1)x(u)\\ &= \alpha_r^1 - 1 + \lambda + \alpha_l^2 - \lambda' + \varepsilon\\ &< \alpha_r^1 + \alpha_l^2 - 1,\end{aligned}$$

a contradiction. Thus, $\lambda' = \lambda$ and hence, (3.38) and (3.39) hold. In consequence, we have that $\bar{x}_1$ and $\bar{x}_2$ satisfy constraints $[i,1]$ and $[l,2]$ as equation. Hence, inequalities $[i,1]$ and $[l,2]$ are tight for y_{i_0} and z_{i_0}, respectively. This implies that constraint $\langle i,l\rangle$ is tight for $\bar{x}$, which finishes the proof of our theorem. □

Theorem 3.37 permits us to give a system that characterizes the dominating set polytope $P_D(G)$ of a graph G that is a 1-sum of two graphs G_1 and G_2, provided that this characterization is known for $P_D(\bar{G}_1)$ and $P_D(\bar{G}_2)$. The next theorem shows that this system is indeed minimal. For this let us first give a lemma.

Lemma 3.8. *If A is a dominating set whose incidence vector satisfies a constraint $a_i^k x \geqslant \alpha_i$ of type* (3.2) *with equality, then* $|A \cap \{u, w_1, w_4\}| \leqslant 1$.

Proof. If $|A \cap \{u, w_1, w_4\}| \geqslant 2$, then there is a node $z \in \{u, w_1, w_4\}$ such that $A' = (A\setminus\{z\}) \cup \{w_2, w_3\}$ is a dominating set of G. But this implies that $a_i^k x^{A'} < a_i^k x^A = \alpha_i$, a contradiction. □

Theorem 3.9. *Inequalities* (3.10) *define facets of* $P_D(G)$.

Proof. We show that there exists a vector $\bar{x} \in P_D(G)$ that satisfies (3.10) with equality and all others with strict inequalities. For the inequality $[i,1]$, $i \in I_2^1$ (resp. $[l,2]$, $l \in I_2^2$), let $S = \{S_1, \dots, S_n\}$ (resp. $T = \{T_1, \dots, T_m\}$) be the set of dominating sets whose incidence vectors satisfy it with equality. By Lemma 3.8 we have $|S_i \cap \{u, w_1, w_4\}| \leqslant 1$, for $i = 1, \dots, n$ (resp. $|T_j \cap \{u, w_1, w_4\}| \leqslant 1$, for $j = 1, \dots, m$). Since (3.2) is different from a trivial inequality, there must exist $S_{i_1}, S_{i_2}, S_{i_3} \in S$ such that $u \in S_{i_1}$, $w_1 \in S_{i_2}$ and $w_4 \in S_{i_3}$. We claim that there is a set $S_{i_4} \in S$ with $S_{i_4} \cap \{u, w_1, w_4\} = \emptyset$. In fact, if this is not the case, then for every set S_i we would have $|S_i \cap \{u, w_1, w_4\}| = 1$. But this implies that (3.2) is a positive multiple of the equation $x(u)+x(w_1)+x(w_4) = 1$ which is impossible. We can show along the same line that there are $T_{j_1}, T_{j_2}, T_{j_3}, T_{j_4} \in T$ such that $u \in T_{i_1}$, $w_1 \in T_{i_2}$, $w_4 \in T_{i_3}$ and $T_{i_4} \cap \{u, w_1, w_4\} = \emptyset$. As a consequence, for each set S_i, $i = 1, \dots, n$, one can find a set T_{j_i} such that $Z_i = (S_i \cup T_{j_i}) \backslash \{w_1, \dots, w_4\}$ is a dominating set of G whose incidence vector satisfies (3.10) with equality. Similarly, for each set T_j, $j = 1, \dots, m$, one may find a set S_{i_j} such that $Z_{n+i} = (S_{i_j} \cup T_j) \backslash \{w_1, \dots, w_4\}$ is a dominating set of G whose incidence vector satisfies (3.10) with equality. Let $\{Z_1, \dots, Z_r\}$ be the set of all dominating sets thus obtained ($r = n + m$). Then

$$\bar{x} = \frac{1}{r}(x^{Z_1} + \cdots + x^{Z_r})$$

is the required vector. □

4. Applications to the cactus

The technique discussed in the previous section is useful for classes of graphs that are decomposable by one-node cutsets. This is the case, for instance, of the cactus.

A *Cactus* is a graph that can be decomposed by one-node cutsets into cycles and edges (see Fig. 3).

To describe the polytope $P_D(G)$ when G is a cactus we should recursively decompose the graph. For each one-node cutset we have to add a 5-cycle to each piece. We might have to add more than one 5-cycle for one-node cutset. Let Γ (Γ') be the class of graphs G that may be obtained by means of 1-sums from a chordless cycle (an edge) and a family of 5-cycles. To give a complete description of $P_D(G)$ when G is a cactus, one then has to know such a description for the classes Γ and Γ'.

If G is a graph of Γ', then it is not hard to see that $P_D(G)$ is given by inequalities (2.1)–(2.3). Thus by Theorem 3.37, it follows that when G is a tree, the polytope $P_D(G)$ is completely described by inequalities (2.1)–(2.3) together with the inequalities of type (2.5) associated with the 5-cycles. This can also be obtained from [16] as a special case of a more general result related to strongly chordal graphs.

A strongly elimination ordering of a graph $G = (V, E)$ is an ordering $v_1, v_2, \dots, v_n$ of V with the property that for each i, j, k and l, if $i < j$, $k < l$, $v_k, v_l \in N(v_i)$, and $v_k \in N(v_j)$, then $v_l \in N(v_j)$.

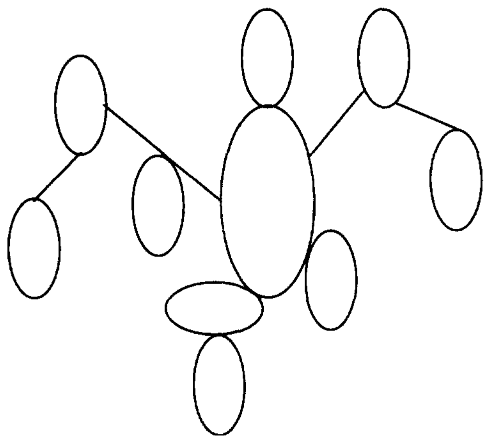

Fig. 3. A cactus.

A graph is said to be *strongly chordal* if it admits a strong elimination ordering. The class of strongly chordal graphs contains, for instance, the trees.

A 0–1 matrix is called *balanced* [7] if it does not contain as a submatrix an incidence matrix of an odd cycle. (A cycle matrix is a square matrix such that the sum of the coefficients in each row and each column equals two.) And it is called *totally balanced* [23] if it does not contain as a submatrix an incidence matrix of any cycle of lenght at least three.

In [19] Fulkerson et al. showed the following.

Theorem 4.1 (Fulkerson et al. [19]). *If A is a balanced matrix, then the polytope*

$$Ax \geqslant 1, \quad 0 \leqslant x \leqslant 1$$

has 0–1 *extreme points.*

The *neighbourhood matrix* of a graph $G = (V,E)$ is the $(|V| \times |V|)$-matrix given by the neighbourhood inequalities. In [16] Farber discussed the relationship between strongly chordal graphs and totally balanced matrices. He showed that a graph is strongly chordal if and only if its neighbourhood matrix is totally balanced. This together with Theorem 4.1 yield the following.

Theorem 4.2. *If $G = (V,E)$ is strongly chordal then $P_D(G)$ is given by inequalities* (2.1)–(2.3).

If G is a cactus that contains cycles, it may be that the inequalities (2.1)–(2.3) do not suffice to give a complete description of $P_D(G)$. In fact, consider for instance the cactus $G = (V,E)$ shown in Fig. 4.

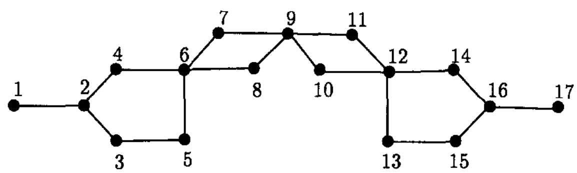

Fig. 4.

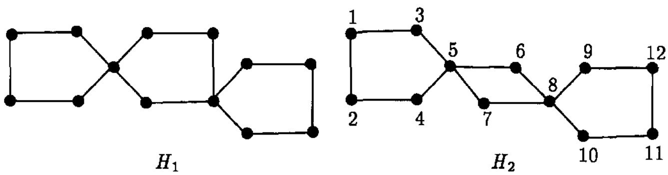

Fig. 5.

To give a linear description for $P_D(G)$, we need such a description for the polytopes $P_D(H_1)$ and $P_D(H_2)$ where H_1 and H_2 are the graphs shown in Fig. 5.

It can be shown that the minimal system describing $P_D(H_1)$ is given by the inequalities (2.1)–(2.3) together with the inequalities of type (2.5) associated with the 5-cycles of the graph. And a minimal system describing $P_D(H_2)$ is given by these inequalities together with the inequality

$$\sum_{u=3,\dots,10} x(u) \geqslant 2.$$

From Theorems 3.37 and 3.9 it follows that a minimal description of $P_D(G)$ is given by the following system of inequalities:

$$0 \leqslant x(u) \leqslant 1 \quad \text{for all } u \in V,$$

$$x(N(u)) \geqslant 1 \quad \text{for } u \in V \setminus \{2, 16\},$$

$$\sum_{u=2,\dots,6} x(u) \geqslant 2,$$

$$\sum_{u=12,\dots,16} x(u) \geqslant 2,$$

$$\sum_{u=4,\dots,14} x(u) \geqslant 3.$$

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