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K_i-COVERS I: COMPLEXITY AND POLYTOPES

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A K_i in a graph is a complete subgraph of size i. A K_i-cover of a graph $G(V, E)$ is a set $\mathscr{C}$ of K_{i-1}'s of G such that every K_i in G contains at least one K_{i-1} in $\mathscr{C}$. Thus a K_2-cover is a vertex cover. The problem of determining whether a graph has a K_i-cover ($i \geq 2$) of cardinality $\leq k$ is shown to be NP-complete for graphs in general. For chordal graphs with fixed maximum clique size, the problem is polynomial; however, it is NP-complete for arbitrary chordal graphs when $i \geq 3$. The NP-completeness results motivate the examination of some facets of the corresponding polytope. In particular we show that various induced subgraphs of G define facets of the K_i-cover polytope. Further results of this type are also produced for the K_3-cover polytope. We conclude by describing polynomial algorithms for solving the separation problem for some classes of facets of the K_i-cover polytope.

1. Introduction

In this paper we begin our study of various aspects of K_i-covers in graphs, a generalization of the notion of vertex cover. The concept of covering is well established in combinatorics and includes set covering and edge covering as well as vertex covering, clique covering and coloring. Other types of covering and applications are discussed in [1]. The paradigm for combinatorial covering may be stated as follows:

Given a set X and a family $\mathscr{S}$ of subsets of X, a set $\mathscr{C}$ of elements from 2^X having a prescribed property is a *cover* of $\mathscr{S}$ if, for each $\mathscr{S}_i \in \mathscr{S}$, there is a $\mathscr{C}_j \in \mathscr{C}$ such that $\mathscr{C}_j$ is contained in or 'covers' $\mathscr{S}_i$.

In many applications one is interested in finding a cover of minimum cardinality or more generally to find a cover with minimum weight where the elements in $\mathscr{C}$ have each been assigned a weight. To illustrate the general definition of cover, we examine each of the five particular coverings mentioned above. For the set covering problem, $\mathscr{C} \subseteq X$ and thus we require a subset of X such that each element in $\mathscr{S}$ contains at least one of these elements. For the graph theoretic

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coverings $X = V$ where the given graph has vertex set V and edge set E. In a vertex cover, $\mathcal{S} = E$ and $\mathcal{C} \subseteq V$; in an edge cover $\mathcal{S} = V$ and $\mathcal{C} \subseteq E$ (note here that edge e 'covers' vertex v if e is incident with v). In the clique covering, $X = V$, $\mathcal{S} = V$ and $\mathcal{C}$ is the family of completely connected nodesets; in the coloring X and $\mathcal{S}$ do not change, but $\mathcal{C}$ is the family of stable subsets of nodes of G. Incidentally, the only well solved covering problem is the edge cover, by matching theory.

It should be noted that some confusion arises in the naming of various covering problems. For vertex and edge covers the adjectives 'vertex' and 'edge' indicate the nature of the covering set $\mathcal{C}$, whereas for set covers, the adjective 'set' describes the contents of $\mathcal{S}$, the set to be covered. It is this latter convention that we follow in our definition of *K_i-cover.*

Given a graph $G(V, E)$ we let $\mathcal{F}_i(G)$ denote $\{K_i \mid K_i \subseteq G\}$ (i.e., $\mathcal{F}_i(G)$ is the set of complete subgraphs on i vertices of G). For the K_i-cover problem, $X = V$, $\mathcal{S} = \mathcal{F}_i(G)$ and $\mathcal{C} \subseteq \mathcal{F}_{i-1}(G)$, $i \geqslant 2$. In other words, a K_i-cover of G is a set $\mathcal{C}$ of K_{i-1}'s such that every K_i in G contains at least one K_{i-1} in $\mathcal{C}$.

Note that this definition of K_2-cover is equivalent to that of vertex cover and a K_3-cover is a set of edges meeting all the triangles of G. For a graph G, the *K_i-cover number* $c_i(G)$ is the cardinality of a smallest K_i-cover of G. If we associate a weight to each K_{i-1} of G, then $c_i(G)$ for the weighted version of the problem is defined as the minimum total weight of any K_i-cover, where the weight of a K_i-cover $\mathcal{C}_j$ is the sum of the weights of the K_{i-1}'s in $\mathcal{C}_j$.

In this paper we address various computational aspects of K_i-covers. First we examine the complexity of the K_i-cover problem on graphs in general and then on the restricted class of chordal graphs. The decision version of the K_i-cover problem is "given a graph G and integers $i \geqslant 2$ and $k \geqslant 1$, is $c_i(G) \leqslant k$?" Having seen that the general problem is NP-complete, we study some families of facets of the K_i-cover polytope and show that various induced subgraphs of G define facets of this polytope.

In [6] we continue the study of K_i-covers by examining the relationship between $c_i(G)$ and $p_i(G)$, the *K_i-packing number* defined to be the largest cardinality of any set of K_i's in G not having $i-1$ nodes in common. For any $F \subseteq \mathcal{F}_i(G)$ we may define $c_i(F)$ and $p_i(F)$ in a manner similar to their definitions for graphs. Clearly for all such F, $c_i(F) \geqslant p_i(F)$. We define a graph to be *K_i-perfect* if $\forall F \subseteq \mathcal{F}_i(G)$, $c_i(F) = p_i(F)$. In [6] we provide a characterization of K_i-perfect graphs in terms of a class of graphs which satisfies Berge's Strong Perfect Graph Conjecture [10]. Furthermore, we study the associated $\mathcal{F}_{i-1}(G) \times \mathcal{F}_i(G)$ intersection matrices.

Before presenting our material on the computational aspects of K_i-covers, we introduce the definitions and terminology used in this paper. Given a graph $G(V, E)$ with $X \subseteq V$, $G[X]$ denotes the induced subgraph of G restricted to X. For $v \in V$, $\Gamma(v) = \{u \mid u \in V, (u, v) \in E\}$. A vertex v is *universal* to $X \subseteq V \setminus \{v\}$ if $X \subseteq \Gamma(v)$. Similarly, a set of vertices may be universal to X. A *clique* in $G(V, E)$

is a maximal complete subgraph and $\omega(G)$ is the *clique number of* G, namely, the size of the largest clique in G.

A graph is *chordal* (or *triangulated*) [8, 10] if every cycle of length greater than three has a chord. A *simplicial vertex* v is one for which $\Gamma(v)$ is complete. A graph has a *perfect elimination scheme* if there exists an order of eliminating the vertices such that each vertex is simplicial at the time of its elimination. A graph is chordal iff it has a perfect elimination scheme (see [10]).

The relationship between the K_i's and the K_{i-1}'s in G may be represented by the *K_i-intersection graph*, $I_i(G)$. The vertices of $I_i(G)$ are the K_i's in $\mathcal{F}_i(G)$; two such vertices are adjacent iff they have a K_{i-1} in common. Note that $I_2(G)$ is the line graph $L(G)$. Throughout the paper $n = |V|$, and $k_i(G) = |\mathcal{F}_i(G)|$, the number of K_i's in G.

2. Complexity results

In this section we examine the complexity of the K_i-cover problem for different values of i and for restricted inputs. This examination consists of showing the problem to be NP-complete for graphs in general and also for chordal graphs. It is then shown that the problem is polynomial for chordal graphs with fixed clique size.

2.1. NP-*completeness*

As mentioned in the introduction, the K_2-cover problem is the well-known vertex cover problem, one of the first problems shown to be NP-complete [15]. The K_3-cover problem was shown to be NP-complete by Yannakakis [19] using a reduction from the vertex cover problem. We now use Yannakakis' proof technique to show that the K_i-cover problem is NP-complete for all $i \geq 2$.

Theorem 2.1. *For any $i \geq 2$, the K_i-cover problem is* NP-*complete.*

Proof. The reduction is from the vertex cover problem. Let $G(V, E)$ be the input graph to the vertex cover problem. As in [19] we may assume that G has no triangles by replacing each edge of G with a path on four vertices. It is clear that for this new graph G_1, $c_2(G_1) = c_2(G) + |E|$. We now form the graph G' by adding a universal K_{i-2}, C to G_1. We claim that $c_i(G') = c_2(G_1)$.

$c_i(G') \geq c_2(G_1)$

Let $\mathcal{C}$ be a K_i-cover of G' where $|\mathcal{C}| = \beta$. Examine all K_{i-1}'s in $\mathcal{C}$; if any K_{i-1}, say X, contains an edge (u, v) in G_1, then replace X with a new complete subgraph $X' = C \cup \{u\}$ or $X' = C \cup \{v\}$. Since G_1 is Δ-free, X covers only the K_i, $C \cup \{u, v\}$, whereas X' covers this K_i and possibly others. Thus the set $\mathcal{C}'$ resulting from the replacement of all such K_{i-1}'s in $\mathcal{C}$ is also a K_i-cover of G' with cardinality β. Now let A be the set of all vertices in G_1 such that the vertex belongs to a K_{i-1} in $\mathscr{C}'$. It is clear that A is a vertex-cover of G_1.

$c_i(G') \leqslant c_2(G_1)$

Assume A is an optimum vertex cover of G_1. Set $\mathscr{C} = \{C \cup \{x\} \mid x \in A\}$. $\mathscr{C}$ is a K_i-cover of G' since any K_i not covered by $\mathscr{C}$ would imply the existence of an edge in G_1 which is not covered by A. □

Using the above construction we immediately have

Corollary 2.2. *For any $i \geqslant 2$, the K_i-cover problem is* NP*-complete when restricted to graphs with $\omega(G) = i$.*

Although the vertex-cover problem is polynomial for chordal graphs [8], we now show that for any fixed $i \geqslant 3$, the K_i-cover problem is NP-complete for chordal graphs.

Corollary 2.3. *For any fixed $i \geqslant 3$, the K_i-cover problem on chordal graphs is* NP*-complete.*

Proof. The reduction will be from the general K_i-cover problem. Given a graph $G(V, E)$ we construct a chordal graph $G'(V', E')$ as follows:

(i) Set $C = K_n$, $(n = |V|)$ where v_j' in C represents $v_j \in V$.

(ii) Examine all $\binom{n}{i-1}$ subsets of $i-1$ vertices in V. If such a subset S does not form a K_{i-1}, then add a new vertex v_s to V', where v_s is adjacent to all vertices in C corresponding to the vertices in S.

Clearly, G' is chordal and since i is fixed, G' may be constructed in polynomial time. We claim that $c_i(G') = c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$. (Recall that $k_{i-1}(G) = |\mathscr{F}_{i-1}(G)|$.)

$c_i(G') \geqslant c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$

Let $\mathscr{C}$ be a K_i-cover of G' where $|\mathscr{C}| = c_i(G')$. Examine each K_{i-1} in $\mathscr{C}$; if any such K_{i-1}, say X, contains a v_s, then replace X with $X' = S$. If S is already in $\mathscr{C}$, then choose any other K_{i-1} in G. Thus X' covers the K_i, $\{v_s\} \cup S$ and possibly others. Thus the set $\mathscr{C}'$, resulting from the replacement of all such K_{i-1}'s in $\mathscr{C}$ is also a K_i-cover of G' with cardinality $c_i(G')$. Form $\mathscr{C}^*$ from $\mathscr{C}'$ by removing all $\binom{n}{i-1} - k_{i-1}(G)$ elements of $\mathscr{C}'$ which correspond to subsets of G which do not form a K_{i-1}. None of the K_i's in G is covered by any K_{i-1} in $\mathscr{C}' \backslash \mathscr{C}^*$ and thus these K_i's are covered by the K_{i-1}'s in $\mathscr{C}^*$. Therefore $\mathscr{C}^*$ is a K_i-cover of G with cardinality $\leqslant c_i(G') - \binom{n}{i-1} + k_{i-1}(G)$.

$c_i(G') \leqslant c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$

Assume $\mathscr{C}^*$ is an optimal K_i-cover of G. Set $\mathscr{C} = \mathscr{C}^* \bigcup_{v_s \in V'} \Gamma(v_s)$. Clearly, $|\mathscr{C}| = c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$. $\mathscr{C}$ is a K_i-cover of G' since any uncovered K_i must be in G and furthermore must be uncovered in G, which contradicts the assumption that $\mathscr{C}^*$ is a K_i-cover of G. □

2.2 Polynomial algorithm

In the light of Corollaries 2.2 and 2.3, it is somewhat surprising to note that for any fixed j, the K_i-cover problem is polynomial on chordal graphs with maximum clique size j. Before presenting this algorithm we note some facts about chordal graphs. As mentioned in Section 1, a graph is chordal iff it has a perfect elimination scheme thereby indicating that the cliques of the chordal graph G interlock in a very tree-like way. Given a chordal graph G and an associated perfect elimination scheme, we may construct a rooted clique tree T [8] where the nodes of T are the cliques of G. Furthermore, this tree may be constructed in linear time [17]. Given G with a rooted clique tree T and a clique C of G (note $|C| \leq j$) we let $G(C)$ denote the subgraph of G induced by the cliques in the subtree of T rooted at C. Thus if R is the root of T then $G(R) \equiv G$. We let $C_1, C_2, \ldots, C_l$ denote the children of C in T ($l = 0$ iff C is a leaf of T). If C_k is a child of C, then we let x_k denote the vertex $C_k \backslash C$. See Fig. 1, where the vertex number is its order in a perfect elimination scheme. Clique $C_k = \{3, 5, 8\}$ is a child of $\{5, 6, 7, 8\}$ and the node x_k is 3. $G(\{3, 5, 8\})$ is the subgraph induced by the nodeset $\{1, 3, 5, 8\}$.

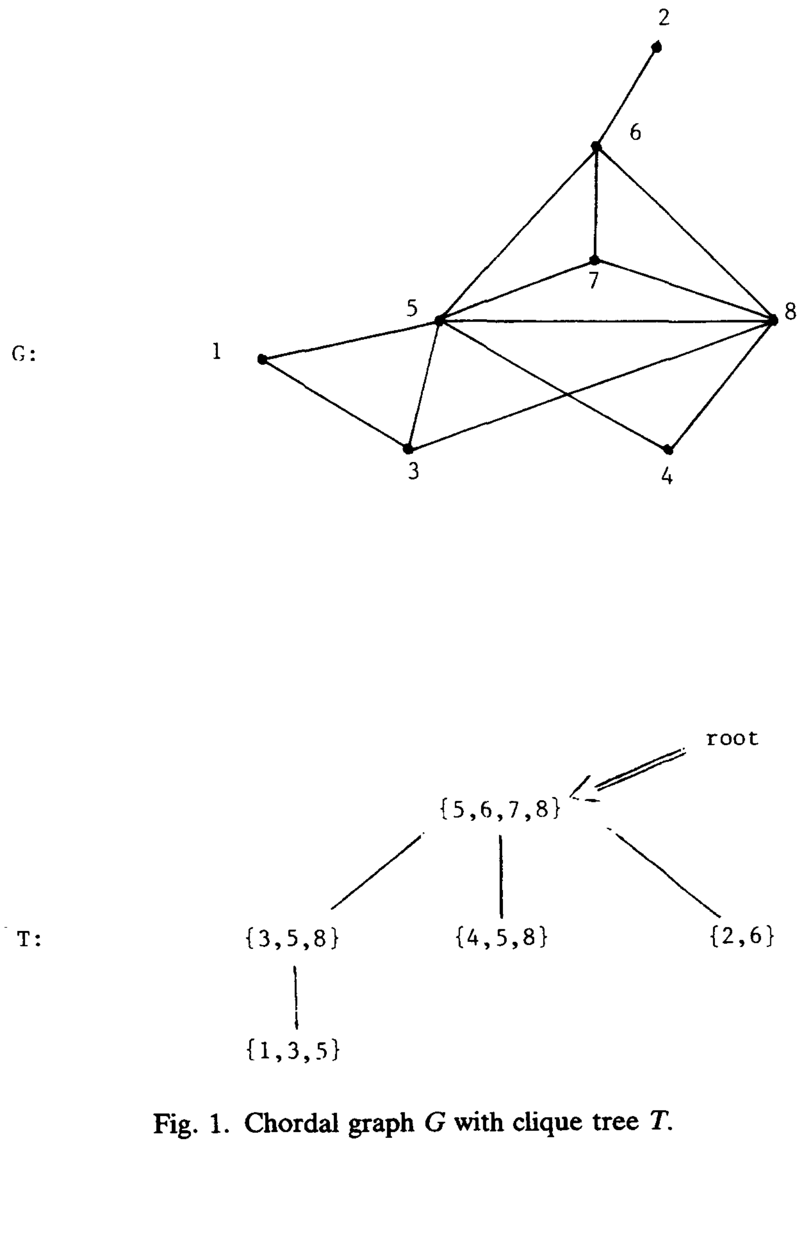

Fig. 1. Chordal graph G with clique tree T.

Given a clique C of G and $\mathscr{C}$ any set of K_{i-1}'s which cover all K_i's in C, we define COVi$(G(C), \mathscr{C})$ to be a minimum cardinality set of K_{i-1}'s in $G(C)$ which is a K_i-cover of $G(C)$ and COVi$(G(C), \mathscr{C}) \cap F_{i-1}(C) = \mathscr{C}$ (i.e., $\mathscr{C} \subset$ COVi$(G(C), C)$; however, no other K_{i-1}'s in $\mathscr{F}_{i-1}(C) \setminus \mathscr{C}$ belong to COVi$(G(C), \mathscr{C})$).

The algorithm will find COVi$(G(C), \mathscr{C})$ for all cliques C in G and for all covering sets $\mathscr{C}$ of C. The general form of this dynamic programming is described in [7].

Algorithm 2.1. K_i-cover of chordal graphs with fixed clique size j.

Input: Chordal graph G with $\omega(G) \leq j$, a constant; integer i.

Output: A set $\mathscr{C}^*$ of K_{i-1}'s such that $\mathscr{C}^*$ covers $\mathscr{F}_i(G)$ and $|\mathscr{C}^*| = c_i(G)$.

1. If $i > j$, then set $\mathscr{C}^* = \emptyset$ and stop.
2. Construct a clique tree T rooted at R.
3. Do a bottom-up scan of T calculating the COVi$(G(C), \mathscr{C})$'s for each node C in T in the following way: Assume node C has children $C_1, C_2, \ldots, C_l$ ($l = 0$ iff C is a leaf of T).
   (i) If C is a leaf then for all $\mathscr{C} \subset \mathscr{F}_{i-1}(C)$ such that $\mathscr{C}$ covers C set
   $$\text{COVi}(G(C), \mathscr{C}) = \mathscr{C}.$$
   (ii) If C is not a leaf, then for all $\mathscr{C}$ covering C do the following: For each clique C_k ($1 \leq k \leq l$) examine all sets $\mathscr{D}_k$ of K_{i-1}'s such that for each $\mathscr{D}_k$, $\mathscr{D}_k$ covers all K_i's in C_k and $\mathscr{D}_k$ does not include any new K_{i-1}'s in C that are not in $\mathscr{C}$. Let B_k be a set with minimum cardinality in $\{\mathscr{C} \cup \text{COVi}(G(C_k), \mathscr{D}_k)\}$. Set
   $$\text{COVi}(G(C), \mathscr{C}) = \bigcup_{k=1}^{l} B_k.$$
4. If C is the root R, then set $\mathscr{C}^*$ to be any set of minimum cardinality in $\{\text{COVi}(G(R), \mathscr{C})\}$ where $\mathscr{C}$ is a set of K_{i-1}'s which cover all K_i's in R.

As an example of this algorithm consider the graph G in Fig. 1, where $i = 3$. For cliques $\{1, 3, 5\}$ and $\{4, 5, 8\}$ the COV3 sets are calculated immediately by step 3(i). Now examine clique $\{3, 5, 8\}$. Its different $\mathscr{C}$ sets and the corresponding COV3 sets are listed in Table 1. For the root $R = \{5, 6, 7, 8\}$, all of its $\mathscr{C}$ sets of

Table 1. COV3 sets for $C = \{3, 5, 8\}$.

$\mathscr{C}$	COV3$(G(C), \mathscr{C})$
{(3, 5)}	{(3, 5)}
{(3, 8)}	{(3, 8), (1, 3)}
{(5, 8)}	{(5, 8), (1, 5)}
{(3, 5)(3, 8)}	{(3, 5), (3, 8)}
{(3, 5)(5, 8)}	{(3, 5), (5, 8)}
{(3, 8)(5, 8)}	{(3, 8), (5, 8), (1, 3)}
{(3, 5)(3, 8)(5, 8)}	{(3, 5), (3, 8), (5, 8)}

cardinality 2 or 3 and the corresponding COV3 sets are shown in Table 2. From this table we see that $\mathscr{C}^* = \{(5, 8)(6, 7)(1, 5)\}$ is a K_3-cover of G with minimum cardinality.

Table 2. Some COV3 sets for $C = \{5, 6, 7, 8\}$

$\mathscr{C}$	COV3(G, $\mathscr{C}$)
{(5, 6)(7, 8)}	{(5, 6)(7, 8)(3, 5)(4, 8)}
{(5, 8)(6, 7)}	{(5, 8)(6, 7)(1, 5)} *
{(6, 8)(5, 7)}	{(6, 8)(5, 7)(3, 5)(4, 5)}
{(5, 6)(7, 8)(5, 7)}	{(5, 6)(7, 8)(5, 7)(3, 5)(4, 5)}
{(5, 6)(7, 8)(5, 8)}	{(5, 6)(7, 8)(5, 8)(1, 5)}
{(5, 6)(7, 8)(6, 7)}	{(5, 6)(7, 8)(6, 7)(3, 5)(4, 5)}
{(5, 6)(7, 8)(6, 8)}	{(5, 6)(7, 8)(6, 8)(3, 5)(4, 5)}
{(5, 8)(6, 7)(5, 6)}	{(5, 8)(6, 7)(5, 6)(3, 5)}
{(5, 8)(6, 7)(5, 7)}	{(5, 8)(6, 7)(5, 7)(3, 5)}
{(5, 8)(6, 7)(6, 8)}	{(5, 8)(6, 7)(6, 8)(3, 5)}
{(5, 8)(6, 7)(7, 8)}	{(5, 8)(6, 7)(7, 8)(3, 5)}
{(6, 8)(5, 7)(5, 6)}	{(6, 8)(5, 7)(5, 6)(3, 5)(4, 5)}
{(6, 8)(5, 7)(5, 8)}	{(6, 8)(5, 7)(5, 8)(3, 5)}
{(6, 8)(5, 7)(6, 7)}	{(6, 8)(5, 7)(6, 7)(3, 5)(4, 5)}
{(6, 8)(5, 7)(7, 8)}	{(6, 8)(5, 7)(7, 8)(3, 5)(4, 5)}
{(5, 6)(5, 7)(6, 7)}	{(5, 6)(5, 7)(6, 7)(3, 5)(4, 5)}
{(5, 7)(5, 8)(7, 8)}	{(5, 7)(5, 8)(7, 8)(3, 5)}
{(5, 6)(5, 8)(6, 8)}	{(5, 6)(5, 8)(6, 8)(3, 5)}
{(6, 7)(6, 8)(7, 8)}	{(6, 7)(6, 8)(7, 8)(3, 5)(4, 5)}

We now establish the correctness of Algorithm 2.1 and then discuss its efficiency. First we state some straightforward lemmas.

Lemma 2.4. *Let $\mathscr{C}$ be a K_i-cover of $G(V, E)$. For any $X \subseteq V$, the restriction of $\mathscr{C}$ to X covers all K_i's in $G[X]$.*

Lemma 2.5. *Let $G(V, E)$ be a chordal graph with clique tree T and C a node of T with children $C_1, \ldots, C_l$. Then for any j, k, $1 \leq j < k \leq l$, $G(C_j) \cap G(C_k) = C_j \cap C_k \subset C$.*

Corollary 2.6. *Under the conditions of Lemma 2.5, $x \in G(C_j)\backslash C$,*

$$y \in G(C_k)\backslash C \Rightarrow (x, y) \notin E.$$

Corollary 2.7. *Under the conditions of Lemma 2.5, let $\mathscr{C}_j$ be a K_i-cover of $G(C_j)$ and let $\mathscr{C}_k$ be a K_i-cover of $G(C_k)$. Then $\mathscr{C}_j \cap \mathscr{C}_k \subseteq \mathscr{F}_{i-1}(C)$.*

Theorem 2.8. *Algorithm 2.1 determines a minimum cardinality K_i-cover for a chordal graph with $\omega(G) = j$.*

Proof. We only need to show that in step 3 the algorithm correctly determines a COVi($G(C)$, $\mathscr{C}$) for each possible $\mathscr{C}$ of C. Under this assumption, step 4 will obviously find a minimum cardinality K_i-cover of G.

The proof of the correct calculation of the COVi sets proceeds by induction on the height of the clique tree T. If C is a leaf, then obviously the COVi sets are calculated correctly. Assume now that C is not a leaf and for all children $C_1, C_2, \ldots, C_l$ of C, the COVi sets are determined accurately. We now show that $\mathscr{C}' = \text{COVi}(G(C), \mathscr{C})$ is calculated correctly for $\mathscr{C} \subseteq \mathscr{F}_{i-1}(C)$ such that $\mathscr{C}$ covers all K_i's in C.

Clearly, $\mathscr{C} \subseteq \mathscr{C}'$. First we show that $\mathscr{C}'$ covers $G(C)$ and then that it is of minimum cardinality. From the inductive assumption $G(C_k)$ is covered $\forall k$, $1 \leq k \leq l$. From Corollary 2.6 we see that all K_i's in $G(C)$ are of one of the following two types:

(i) entirely within $G(C_k)$ for some k, in which case it is covered, or

(ii) entirely within C, in which case it is covered by $\mathscr{C}$.

Furthermore, it is clear that $\mathscr{C}' \cap \mathscr{F}_{i-1}(C) = \mathscr{C}$ as required.

To establish the minimum cardinality of $\mathscr{C}'$ we assume to the contrary that there exists X which satisfies the various conditions for $\mathscr{C}'$ and $|X| < |\mathscr{C}'|$. By Lemma 2.4 the restriction of $\mathscr{C}'$ to $G(C_k)$ covers $G(C_k)\,\forall k$, $1 \leq k \leq l$. Let X_k denote the restriction of X to $G(C_k)$. By Corollary 2.7, $X_j \cap X_k \subseteq \mathscr{F}_{i-1}(C)$, $1 \leq j < k \leq l$. In fact, since $X \cap \mathscr{F}_{i-1}(C) = \mathscr{C}$, $X_j \cap X_k \subseteq \mathscr{C}$. Thus $|X| = \sum_{j=1}^{l} |X_j \setminus (\mathscr{C} \cap \mathscr{F}_{i-1}(C_j))| + |\mathscr{C}|$. Similarly, $|\mathscr{C}'| = \sum_{j=1}^{l} |\mathscr{C}'_j \setminus (\mathscr{C} \cap \mathscr{F}_{i-1}(C_j))| + |\mathscr{C}|$. Since $|X| < |\mathscr{C}'|$ there exists k such that $|X_k \setminus (\mathscr{C} \cap \mathscr{F}_{i-1}(C_k))| < |\mathscr{C}'_k \setminus (\mathscr{C} \cap \mathscr{F}_{i-1}(C_k))|$. Since both X and C' must contain all K_i's in $\mathscr{C} \cap \mathscr{F}_{i-1}(C_k)$ this means that $|X_k| < |\mathscr{C}'_k|$. But $X \cap \mathscr{F}_{i-1}(C_k)$ covers C_k and $X \cap \mathscr{F}_{i-1}(C) = \mathscr{C}$, and thus $X \cap \mathscr{F}_{i-1}(C_k)$ is one of the $\mathscr{D}_k$'s considered by the algorithm. However, by the inductive assumption $\text{COVi}(G(C_k), \mathscr{D}_k)$ is of minimum cardinality thereby contradicting the assumption that $|X| < |\mathscr{C}'|$. □

Since for any clique in G, the number of subsets of K_{i-1}'s to be examined is bounded by $2^{\binom{j}{i-1}}$ which is a constant (albeit quite large!) we see that Algorithm 2.1's running time and storage requirements are bounded by polynomials in the size of the input graph. In the light of Corollary 2.3, this exponential growth with j is hardly surprising. For particular values of i and j it is expected that algorithms which are more efficient than Algorithm 2.1 can be developed; as an example for $i = 2, j = 3$ a straightforward greedy algorithm suffices. We also note that Algorithm 2.1 shows that the K_i-cover problem is polynomial for chordal graphs whose largest degree is bounded by a constant.

We now turn our attention to examining the facets of the K_i-cover polytope.

3. Facets of the K_i-cover polytope

As before, let $\mathscr{F}_i(G)$ and $\mathscr{F}_{i-1}(G)$ be the families of K_i's and K_{i-1}'s in G. Throughout this section the following terminology and notation will be used. The matrix ${}_iA$ will denote the $K_{i-1}(G)$ versus $K_i(G)$ incidence matrix. The (j, k)th entry of ${}_iA$ will equal 1 iff the jth K_{i-1} in $\mathcal{F}_{i-1}(G)$ is contained in the kth K_i in $\mathcal{F}_i(G)$, and equal 0 otherwise. $|\mathcal{F}_{i-1}(G)|$ will be denoted by h. To any $\mathscr{C} \subseteq \mathcal{F}_{i-1}(G)$ we may associate the *incidence vector* $x^{\mathscr{C}} \in (0, 1)^h$, where

$$x_j^{\mathscr{C}} = \begin{cases} 1 & \text{if } K_{i-1}^j \in \mathscr{C} \\ 0 & \text{otherwise.} \end{cases}$$

The 'all ones' vector will be denoted by $\mathbf{1}$, and ${}_iA_j$ refers to the jth row of ${}_iA$. Very often we will wish to refer to a specific complete graph in a given graph. In the notation ${}^{\alpha}K_i^j$, i gives the size of the complete graph, j is an index and α is a list of nodes included in or excluded from the complete graph. For example, $\alpha = x, \bar{y}, \bar{z}$ indicates that node x is included, whereas nodes y and z are excluded.

We now examine the minimum weight K_i-cover problem by presenting a partial characterization of the corresponding polytope. The problem of finding a minimum weight K_i-cover of G is equivalent to solving the following integer linear programming problem:

$$(P) = \begin{cases} {}_iA^{\mathrm{T}}x \geq \mathbf{1} \\ x_j \in \{0, 1\}, \quad j = 1, \ldots, h \\ \min Wx, \end{cases}$$

where W is the system of weights associated with $\mathcal{F}_{i-1}(G)$. By relaxing the integrality constraint on the x_j's in (P) we get the following linear program:

$$(P') = \begin{cases} {}_iA^{\mathrm{T}}x \geq \mathbf{1} & (1) \\ 0 \leq x_j \leq 1, \quad j = 1, \ldots, h & (2) \\ \min Wx. \end{cases}$$

If the polyhedron defined by (1) and (2) has integer valued vertices, then the problems (P) and (P') are equivalent. If this is not the case, it becomes necessary to determine the hyperplanes (facets) which in addition to constraints (1) and (2) defined the convex hull of the integer solutions of (P'). This convex hull will be called the *K_i-cover polytope of G* and will be denoted by $P_{K_i}(G)$. The minimum weight K_i-cover problem may be stated as the following linear program:

min Wx, $x \in P_{K_i}(G)$. If the constraint matrix in (1) is a general 0–1 matrix, then (P) is known as the set covering problem. Hence our K_i-cover polytope is a special case of the set covering polytope.

An *independent set* is a set of nonadjacent vertices. If S is an independent set, then $V - S$ is a K_2-cover. Thus, for $i = 2$, the K_2-cover polytope of G is equivalent to the independence polytope (each vertex of the polytope is the incidence vector of an independent set of G). A great deal of work has been done on this polytope [16, 5, 18, 4]; in particular Padberg [16] has studied it for arbitrary graphs and has described some classes of its hyperplanes.

Since the K_i-cover problem is NP-complete, in the light of the implications brought by the ellipsoid method [12], there is very little hope of completely characterizing $P_{K_i}(G)$ for an arbitrary graph G. Nevertheless, it is interesting to produce a partial characterization of the polytope corresponding to such an NP-complete problem. In particular, one often focuses such attention on facets for which the separation problem is polynomial. The separation problem is to decide whether a point x belongs to the polytope and, if not, to find a hyperplane which separates x from the polytope. For examples of this approach see [13, 11] on the travelling salesman problem and [3, 14] on the max cut problem.

3.1. *Facets of* $P_{K_i}(G)$

We now present a partial non-redundant system of inequalities defining some hyperplanes of $P_{K_i}(G)$. These inequalities are essential inequalities or facets of $P_{K_i}(G)$. For two of these families we present polynomial algorithms for solving the separation problem (see Section 3.3). First we prove the following

Lemma 3.1. *The polytope* $P_{K_i}(G)$ *is of full dimension (i.e.,* $\dim(P_{K_i}(G)) = h$).

Proof. The sets $\mathscr{F}_{i-1}(G)\backslash K_{i-1}^j$, $j = 1, \ldots, h$ and the set $\mathscr{F}_{i-1}(G)$ form a family of $h+1$ K_i-covers of G whose incidence vectors are affinely independent. □

Thus a valid inequality $a^{\mathrm{T}}x \geqslant a_0$ (satisfied by all points of $P_{K_i}(G)$) defines a facet of $P_{K_i}(G)$ iff $\emptyset \neq \{P_{K_i}(G) \cap \{x \mid a^{\mathrm{T}}x = a_0\}\} \neq P_{K_i}(G)$, and there exists h affinely independent points in $P_{K_i}(G) \cap \{x \mid a^{\mathrm{T}}x = a_0\}$. It is easy to see that the trivial constraints $x_j \leqslant 1$, $\forall j$ define facets for $P_{K_i}(G)$ for all $i \geqslant 2$, and the constraints $x_j \geqslant 0$, $\forall j$ define facets for $P_{K_i}(G)$ only if $i \geqslant 3$. We now present three different families of facets. The first is defined on the K_i's and K_{i+1}'s.

3.1.1. *Complete subgraphs* K_i *and* K_{i+1} *and facets*

Theorem 3.2. *For* $i \geqslant 3$, *the constraints* ${}_iA^{\mathrm{T}}x \geqslant 1$ *define facets of* $P_{K_i}(G)$.

Proof. It is clear that these constraints are valid for $P_{K_i}(G)$. Given $K_i^j \in \mathscr{F}_i(G)$, let $\{K_{i-1}^1, K_{i-1}^2, \ldots, K_{i-1}^i\}$ be the K_{i-1}'s in K_i^j. We now examine the following K_i-covers of G:

$$\mathscr{C}_l = \{K_{i-1}^l, K_{i-1}^{i+1}, \ldots, K_{i-1}^h\}, \quad l = 1, 2, \ldots, i$$
$$\mathscr{C}_r = \{K_{i-1}^1\} \cup \{K_{i-1}^j, i+1 \leqslant j \leqslant h, j \neq r\}, \quad r = i+1, \ldots, h.$$

The vectors $x^{\mathscr{C}_1}, \ldots, x^{\mathscr{C}_h}$ all verify $\sum_{k=1}^i x_k = {}_iA_j^{\mathrm{T}}x = 1$ and they are linearly independent. Thus ${}_iA_j^{\mathrm{T}}x \geqslant 1$ is a facet of $P_{K_i}(G)$. □

Lemma 3.3. *Every facet defining inequality of* $P_{K_i}(G)$ *except those given by* $x_j \leqslant 1$, $\forall j$, *is of the form* $\sum_{j=1,\ldots,h} a_j x_j \geqslant a_0$, *with* $a_j \geqslant 0$, $\forall j = 0, 1, \ldots, h$.

Proof. Suppose that $a_{j_0}<0$ for $j_0 \in \{1, \ldots, h\}$. Since $\sum_{j=1,\ldots,h} a_j x_j \geqslant a_0$ is different from $x_{j_0} \leqslant 1$, there exists a K_i-cover $\mathscr{C}_2$ with incidence vector $x^{\mathscr{C}_1}$ such that $K_{i-1}^{j_0} \notin \mathscr{C}_1$ and $\sum_{j=1,\ldots,h} a_j x_j^{\mathscr{C}_1} = a_0$. Let $\mathscr{C}_2 = \mathscr{C}_1 \cup \{K_{i-1}^{j_0}\}$. It is obvious that $\mathscr{C}_2$ is a K_i-cover of G, but $\sum_{j=1,\ldots,h} a_j x_j^{\mathscr{C}_2} < a_0$, where $x^{\mathscr{C}_2}$ is the incidence vector of $\mathscr{C}_2$. This is a contradiction. □

Theorem 3.4. *Let K_{i+1}^0 be a complete subgraph of size $i+1$ in G, then*

$$\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geqslant \left\lceil \frac{i+1}{2} \right\rceil \tag{3}$$

is a valid inequality for $P_{K_i}(G)$. Furthermore, (3) defines a facet of $P_{K_i}(G)$ iff i is even.

Proof. First we show that (3) is valid for $P_{K_i}(G)$. For any $K_i^j \subset K_{i+1}^0$, $\sum_{K_{i-1}^k \subset K_i^j} x_k \geqslant 1$ is valid for $P_{K_i}(G)$. By summing all of these inequalities for $j = 1, 2, \ldots, i+1$ we get $2\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geqslant i+1$. Thus $\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geqslant (i+1)/2$. Since the sum of the x_j's is an integer, (3) is valid for $P_{K_i}(G)$. This also implies that (3) does not define a facet of $P_{K_i}(G)$ when i is odd since (3) could then be written as the linear combination of constraints defined on the K_i's contained in K_{i+1}^0.

We now assume that $i = 2r$ $(r \geqslant 1)$ and denote the corresponding constraint $\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geqslant r+1$ by $a^{\mathrm{T}}x \geqslant a_0$. Furthermore, assume that there exists an inequality $b^{\mathrm{T}}x \geqslant b_0$ which defines a facet of $P_{K_i}(G)$ and which also obeys the following:

if $x \in P_{K_i}(G)$ verifies $a^{\mathrm{T}}x = a_0$, then x also verifies $b^{\mathrm{T}}x = b_0$.

If we are able to show the existence of $\rho > 0$ such that $b = \rho a$, then we may conclude that $a^{\mathrm{T}}x \geqslant a_0$ is a facet of $P_{K_i}(G)$. To do this, we first show that for two complete graphs K_{i-1}^j and K_{i-1}^k in K_{i+1}^0, $b_j = b_k$.

Let $V_0 = \{v_0, v_1, \ldots, v_{2r}\}$ denote the set of vertices of K_{i+1}^0. For $v_\alpha, v_\beta \in V_0$, ${}^{\bar{\alpha},\bar{\beta}}K_{i-1}$ denotes the K_{i-1} defined on $V_0 \backslash \{v_\alpha, v_\beta\}$. Let $\mathscr{C}_0 = \{K_{i-1}^k \mid K_{i-1}^k \in \mathscr{F}_{i-1}(G), K_{i-1}^k \not\subset K_{i+1}^0\}$, and let $j \in \{0, 1, \ldots, 2r\}$. We now examine the following sets of K_{i-1}'s where the indices are modulo $2r+1$:

$$\bar{\mathscr{C}}_j = \{{}^{\bar{\alpha},\overline{\alpha+1}}K_{i-1} \mid \alpha = j, j+2, \ldots, j+2r\} \cup \mathscr{C}_0$$

$$\bar{\mathscr{C}}_j^k = (\bar{\mathscr{C}}_j \backslash \{{}^{\overline{j-1},\bar{j}}K_{i-1}\}) \cup \{{}^{\overline{j-1},\bar{k}}K_{i-1}\}, \quad k \in \{0, 1, \ldots, 2r\} \backslash \{j-1, j\}.$$

It is clear that for any j, the sets $\bar{\mathscr{C}}_j$ and $\bar{\mathscr{C}}_j^k$ are each K_i-covers of G and that their vectors $x^{\bar{\mathscr{C}}_j}$ and $x^{\bar{\mathscr{C}}_j^k}$ satisfy $a^{\mathrm{T}}x^{\bar{\mathscr{C}}_j} = a^{\mathrm{T}}x^{\bar{\mathscr{C}}_j^k} = a_0$. Thus $b^{\mathrm{T}}x^{\bar{\mathscr{C}}_j} = b^{\mathrm{T}}x^{\bar{\mathscr{C}}_j^k} = b_0$, which implies that $b_{j,j-1} = b_{j-1,k}$ for $k \neq j, j-1$ and $0 \leqslant k \leqslant 2r$, where $b_{\underline{j,j-1}}$ and $b_{j-1,k}$ are respectively the coefficients of b associated with ${}^{\bar{j},\overline{j-1}}K_{i-1}$ and ${}^{\overline{j-1},\bar{k}}K_{i-1}$.

Since j is chosen arbitrarily, we conclude that there exists $\rho \in \mathbb{R}$ such that $b_k = \rho$ for all $K_{i-1}^k \subset K_{i+1}^0$. For any $K_{i-1}^k \in \mathscr{C}_0$ note that the set $\mathscr{C}_k = \bar{\mathscr{C}}_j \backslash \{K_{i-1}^k\}$ is a K_i-cover of G and that $x^{\mathscr{C}_k}$ verifies $a^{\mathrm{T}}x = a_0$. Thus $0 = b^{\mathrm{T}}x^{\bar{\mathscr{C}}_j} - b^{\mathrm{T}}x^{\mathscr{C}_k} = b_k$ and $b_k = 0\ \forall K_{i-1}^k \in \mathscr{C}_0$. Furthermore, it is easy to see that $\forall K_{i-1}^j \in \mathscr{F}_{i-1}(G)$, there exists a K_i-cover $\mathscr{C}$ of G such that $K_{i-1}^j \notin \mathscr{C}$ and $a^{\mathrm{T}}x^{\mathscr{C}} = a_0$. This implies that the facet $a^{\mathrm{T}}x \geqslant a_0$ is not contained in a trivial facet $\{x \in P_{K_i}(G) \mid x_j = 1\}$ for a $K_{i-1}^j \in \mathscr{F}_{i-1}(G)$. Therefore $b^{\mathrm{T}}x \geqslant b_0$ defines a non-trivial facet of $P_{K_i}(G)$. By Lemma 3.3 it follows that $\rho > 0$ and thus $b = \rho a$. □

3.1.2. *Chordless cycles in $I_i(G)$ and facets*

Our second family of facets of $P_{K_i}(G)$ is defined by graphs called K_i-p-holes, defined as follows. Let H be a graph where $|\mathscr{F}_i(H)| = p$ and each K_{i-1} in H is contained in at least one of these p K_i's. H is a *K_i-p-hole* (recall that a hole is a chordless cycle) if the K_i-intersection graph $I_i(H)$ is a hole of size p. Three nonisomorphic K_3-9-holes are presented in Fig. 2.

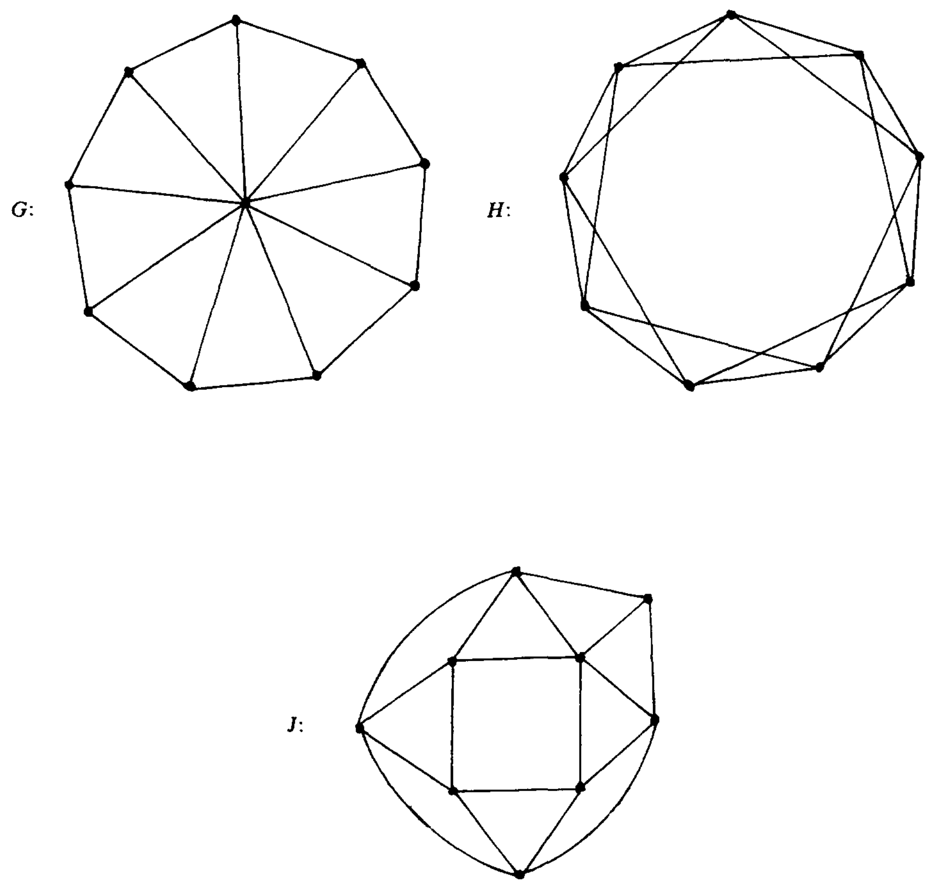

Fig. 2. K_3-9-holes.

Remark 3.1. If H is a K_i-p-hole with $\mathscr{F}_i(H) = \{K_i^0, K_i^1, \ldots, K_i^{p-1}\}$, then $\mathscr{F}_{i-1}(H)$ may be partitioned into $p+1$ pairwise disjoint sets $\mathscr{C}, \mathscr{C}_0, \ldots, \mathscr{C}_{p-1}$ such that $\mathscr{C} = \{\bar{K}_{i-1}^0, \ldots, \bar{K}_{i-1}^{p-1}\}$ is formed by a bijection with the edges of $I_i(H)$ where $\bar{K}_{i-1}^k = K_i^k \cap K_i^{k+1}$ (superscripts modulo p) and $\mathscr{C}_j = \{K_{i-1}^l \mid K_{i-1}^l \subset K_i^j$ and $K_{i-1}^l \notin \mathscr{C}\}$ for $j = 0, \ldots, p-1$. This notation will be used throughout this subsection.

Theorem 3.5. *Let G be a graph with an induced subgraph H, which is a K_i-p-hole, $i \geq 3$. Then the inequality*

$$\sum_{K_{i-1}^j \in \mathscr{F}_{i-1}(h)} x_j \geq \left\lceil \frac{p}{2} \right\rceil \tag{4}$$

is valid for $P_{K_i}(G)$, and is a facet of $P_{K_i}(G)$ iff p is odd.

Proof. From Remark 3.1 we note that $\sum_{K_{i-1}^j \subset K_i^l} x_j \geq 1$, for all $K_i^l \in \mathscr{F}_i(H)$ yields (by summing these inequalities)

$$2 \sum_{\bar{K}_{i-1}^l \in \mathscr{C}} x_j + \sum_{K_{i-1}^k \in \mathscr{C}_0 \cup \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_{p-1}} x_k \geq p,$$

which implies $\sum_{K_{i-1}^j \in \mathscr{F}_{i-1}(H)} x_j \geq \frac{1}{2}p$. Since the sum is an integer, inequality (4) is valid for $P_{K_i}(G)$. Furthermore, when p is even (4) may be written as a linear combination of the constraints defined on the K_i's of H, and then (4) does not define a facet of $P_{K_i}(G)$.

Now assume that $p = 2r + 1$ and denote the constraint $\sum_{K_{i-1}^j \in \mathscr{F}_{i-1}(H)} x_j \geq \lceil \frac{1}{2}p \rceil$ by $a^{\mathrm{T}}x \geq a_0$. As in the proof of Theorem 3.4 we assume that $b^{\mathrm{T}}x \geq b_0$ is a facet of $P_{K_i}(G)$ such that $\{x \in P_{K_i}(G) \mid a^{\mathrm{T}}x = a_0\} \subseteq \{x \in P_{K_i}(G) \mid b^{\mathrm{T}}x = b_0\}$. It suffices to show that there exists $\rho > 0$ such that $b = \rho a$. To do this we will show that for any $K_i^j \in \mathscr{F}_i(H)$ with $K_{i-1}^{j_1}, \ldots, K_{i-1}^{j_i} \subseteq K_i^j$ we have $b_{j_k} = b_{j_l}$, $1 \leq k < l \leq i$. Let $\mathscr{C} = \mathscr{F}_{i-1}(G) \backslash \mathscr{F}_{i-1}(H)$ and let $j \in \{0, 1, \ldots, 2r\}$. We now examine the following sets, where indices are modulo $2r + 1$:

$$\mathscr{C}_{jl} = \{\bar{K}_{i-1}^{j+1}, \bar{K}_{i-1}^{j+3}, \ldots, \bar{K}_{i-1}^{j+2r-1}\} \cup \{K_{i-1}^{j_l}\} \cup \mathscr{C}, \quad l = 1, \ldots, i.$$

All $\mathscr{C}_{jl}$'s are K_i-covers of G and the incidence vectors satisfy

$$a^{\mathrm{T}}x^{\mathscr{C}_{j1}} = a^{\mathrm{T}}x^{\mathscr{C}_{j2}} = \cdots = a^{\mathrm{T}}x^{\mathscr{C}_{ji}} = a_0.$$

Thus

$$b^{\mathrm{T}}x^{\mathscr{C}_{j1}} = b^{\mathrm{T}}x^{\mathscr{C}_{j2}} = \cdots = b^{\mathrm{T}}x^{\mathscr{C}_{ji}} = b_0,$$

which implies that $b_{j_l} = b_{j_k}$ for $1 \leq l < k \leq i$.

Since each K_i^k in H intersects K_i^{k+1} in $\bar{K}_{i-1}^k$ we may conclude that there exists $\rho \in \mathbb{R}$ such that $b_j = \rho$ for $K_{i-1}^j \in \mathscr{F}_i(H)$. It is easy to see that $b_j = 0$ for $K_{i-1}^j \notin \mathscr{F}_i(H)$. As in the proof of Theorem 3.4, $\rho > 0$ and thus $b = \rho a$. □

If H is one of the K_3-9-holes presented in Fig. 2, the inequality $\sum_{K_2^k} x_k \geq 5$ is a facet of $P_{K_3}(G)$. Furthermore, note that if $i = 2$, (4) does not always define a facet of $P_{K_2}(G)$. In this case Padberg [16] has presented a general procedure to generate the facets.

Remark 3.2. The facets associated with the K_i-p-holes as given in Theorem 3.5 may be generalized to facets of the set covering polytope, as defined in Section 3. In fact, consider the intersection graph Γ associated with A, the constraint matrix of our set covering problem, denoted by (P), in which the nodes correspond to the rows of A and two nodes i, j are adjacent if and only if the corresponding rows a_i, a_j verify $a_i^{\mathrm{T}} a_j > 0$. Let H be a hole of Γ of size p (where p is odd). Let $r_0, \ldots, r_{p-1}$ be the rows of A that correspond to nodes $i_0, \ldots, i_{p-1}$ of H. Suppose that $i_0 i_1, i_1 i_2, \ldots, i_{p-2} i_{p-1}, i_0 i_{p-1}$ are the edges of H and let $c_0, \ldots, c_{p-1}$ be the columns of A such that each c_k has a 1 in rows r_k, r_{k+1} for $k = 0, \ldots, p-1$ (where the indices are modulo p). (Note that the submatrix of A where the rows are $r_0, \ldots, r_{p-1}$ and columns are $c_0, \ldots, c_{p-1}$ is an odd cycle submatrix of A.) Let $r_p, \ldots, r_{m-1}$; $c_p, \ldots, c_{n-1}$ be the other rows and columns of A. Suppose that each column c_j, $j \geq p$, has a 1 in at most one of the rows $r_0, \ldots, r_{p-1}$ and each row r_i, $i \geq p$, has a 1 in at least two of the columns $c_p, \ldots, c_{n-1}$. Furthermore, let $T = \{j \mid \text{column } c_j \text{ has a 1 in one of the rows } r_0, \ldots, r_{p-1}\}$. Then $\sum_{j \in T} x_j \geq \lceil \frac{1}{2} p \rceil$ defines a facet of the polyhedron associated with (P).

The proof is similar to the proof of Theorem 3.5.

3.1.3. *p-Wheels and facets*

Our third and final family of facets of $P_{K_i}(G)$ involves a subfamily of K_i-p-holes, the p-wheels of order i defined as follows: Graph H is a *p-wheel of order i* if H consists of a hole C of length $p \geq 3$ and a K_i^* universal to C. See Fig. 3 for the 5-wheel of order 3. Note that a p-wheel of order $i-2$ is a K_i-p-hole; for example, graph G in Fig. 2 is a 9-wheel of order 1.

We now show that if p and i are odd, then the p-wheels of order i define facets of $P_{K_i}(G)$.

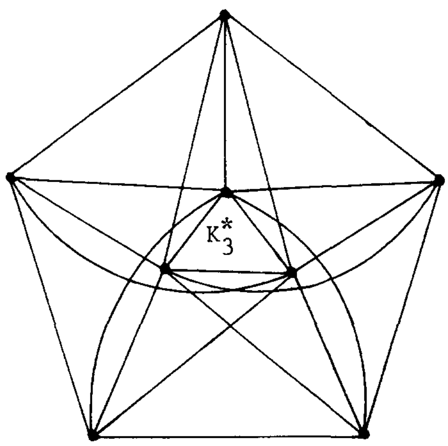

Fig. 3. 5-Wheel of order 3.

Theorem 3.6. *Let $G(V, E)$ be a graph with an induced subgraph H, which is a $(2k+1)$-wheel of order $i-1$, where $k \geq 2$ and $i \geq 3$. Let $\mathscr{C} = F_{i-1}(H) \backslash \{K_{i-1}^*\}$ (i.e., the set of K_{i-1}'s in H excluding the universal K_{i-1}); then the inequality*

$$\sum_{K_{i-1}^j \in \mathscr{C}} x_j \geq k(i-1) + \left\lceil \frac{i-1}{2} \right\rceil \tag{5}$$

is valid for $P_{K_i}(G)$. Furthermore (5) defines a facet of $P_{K_i}(G)$ iff i is even.

Proof. First we show that (5) is valid. In H we let $V^* = \{v_0, v_1, \ldots, v_{i-2}\}$ denote the vertices of K^*_{i-1} and $U = \{u_0, u_1, \ldots, u_{2k}\}$ denote the vertices of the hole C where u_j is adjacent to u_{j-1} and u_{j+1} (subscripts modulo $2k+1$). Each edge (u_j, u_{j+1}) is contained in $i-1$ K_i's of the form ${}^{\bar{l}}K^j_i = \{u_j, u_{j+1}, V^*\backslash\{v_l\}\}$, $l = 0, 1, \ldots, i-2$. Summing all the constraints (1) defined by these K_i's for all edges of C we get

$$2 \sum_{K^j_{i-1} \in \mathscr{C}} x_j \geq (2k+1)(i-1), \quad \text{thus} \quad \sum_{K^j_{i-1} \in \mathscr{C}} x_j \geq k(i-1) + \frac{(i-1)}{2},$$

which implies that (5) is valid for $P_{K_i}(G)$. Furthermore, if i is odd, (5) does not define a facet.

Now we consider the case where $i = 2r+2$ and denote the inequality $\sum_{K^j_{i-1} \in \mathscr{C}} x_j \geq r(2k+1) + k + 1$ by $a^{\mathrm{T}}x \geq a_0$. We assume that $b^{\mathrm{T}}x \geq b_0$ is a facet of $P_{K_i}(G)$ such that $\{x \in P_{K_i}(G) \mid a^{\mathrm{T}}x = a_0\} \subseteq \{x \in P_{k_i}(G) \mid b^{\mathrm{T}}x = b_0\}$. To prove this it suffices to show that $b = \rho a$ with $\rho > 0$. Any vertex u_j in C belongs to two different types of K_{i-1}'s:

$$\begin{aligned} {}^{\bar{f},\bar{l}}K^j_{i-1} &= \{u_j, u_{j+1}, V^*\backslash\{v_f, v_l\}\}, && 0 \leq f \neq l \leq 2r, \\ {}^{\bar{f}}K^j_{i-1} &= \{u_j, V^*\backslash\{v_f\}\}, && 0 \leq f \leq 2r. \end{aligned}$$

Note that $\mathscr{C} = \{{}^{\bar{f},\bar{l}}K^j_{i-1}, {}^{\bar{f}}K^j_{i-1}, j = 0, \ldots, 2k, 0 \leq f \neq l \leq 2r\}$ and the subscripts are modulo $2k+1$ and $2r+1$ respectively.

Let u_q and v_s be two arbitrary vertices in U and V^* respectively. We now define the following three sets of K_{i-1}'s:

$$\begin{aligned} \mathscr{C}_0 &= \mathscr{F}_{i-1}(G)\backslash\mathscr{F}_{i-1}(H), \\ \mathscr{C}^s &= \{{}^{\bar{f},\overline{f+1}}K^j_{i-1} \mid j = 0, 1, \ldots, 2k; f = s+1, s+3, \ldots, s+2r-1\}, \\ \mathscr{C}^q_s &= \{{}^{\bar{s}}K^j_{i-1} \mid j = q+2, q+4, \ldots, q+2k\}. \end{aligned}$$

Given t and t' (different from s), where $0 \leq t < t' \leq 2r$ define

$$\begin{aligned} \mathscr{C}_1 &= \mathscr{C}_0 \cup \mathscr{C}^s \cup \mathscr{C}^q_s \cup \{K^*_{i-1}, {}^{\bar{s},\bar{t}}K^q_{i-1}\}, \\ \mathscr{C}_2 &= (\mathscr{C}_1\backslash\{{}^{\bar{s},\bar{t}}K^q_{i-1}\}) \cup \{{}^{\bar{s},\bar{t}'}K^q_{i-1}\}. \end{aligned}$$

It is easily seen that $\mathscr{C}_1$ and $\mathscr{C}_2$ are both K_i-covers of G and that their incidence vectors $x^{\mathscr{C}_1}$ and $x^{\mathscr{C}_2}$ verify $a^{\mathrm{T}}x = a_0$. Thus $x^{\mathscr{C}_1}$ and $x^{\mathscr{C}_2}$ verify $b^{\mathrm{T}}x = b_0$ and

$$b_{q,\bar{s},\bar{t}} = b_{q,\bar{s},\bar{t}'}, \tag{6}$$

where $b_{q,\bar{s},\bar{t}}$ and $b_{q,\bar{s},\bar{t}'}$ are the coefficients of b associated with ${}^{\bar{s},\bar{t}}K^q_{i-1}$ and ${}^{\bar{s},\bar{t}'}K^q_{i-1}$ respectively. Now look at the K_i-covers

$$\begin{aligned} \mathscr{C}_3 &= (\mathscr{C}_1\backslash\{{}^{\bar{s},\bar{t}}K^q_{i-1}\}) \cup \{{}^{\bar{s}}K^q_{i-1}\}, \\ \mathscr{C}_4 &= (\mathscr{C}_1\backslash\{{}^{\bar{s},\bar{t}}K^q_{i-1}\}) \cup \{{}^{\bar{s}}K^{q+1}_{i-1}\}. \end{aligned}$$

Again we see that $a^{\mathrm{T}}x^{\mathscr{C}_3} = a^{\mathrm{T}}x^{\mathscr{C}_4} = a_0$ and thus $b^{\mathrm{T}}x^{\mathscr{C}_3} = b^{\mathrm{T}}x^{\mathscr{C}_4} = b_0$. Thus

$$b_{q,\bar{s},\bar{t}} = b_{q,\bar{s}} = b_{q+1,\bar{s}}. \tag{7}$$

Since s, q, t and t' are chosen arbitrarily, we may conclude from (6) and (7) that $\exists\rho \in \mathbb{R}$ such that $b_j = \rho\ \forall K^j_{i-1} \in \mathscr{C}$. For $K^j_{i-1} \in \mathscr{C}_0$, the set $\mathscr{C}'_j = \mathscr{C}_1 \setminus \{K^j_{i-1}\}$ is a K_i-cover of G and $x^{\mathscr{C}'_j}$ verifies $a^{\mathrm{T}}x = a_0$ and thus $b_j = 0$ for $K^j_{i-1} \in \mathscr{C}_0$.

Now we must show that $b_* = 0$ (b_* corresponds to K^*_{i-1}). Let

$$\bar{\mathscr{C}} = \begin{cases} \{^{\bar{l},\overline{l+1}}K^j_{i-1} \mid j = 0, 1, \ldots, 2k;\ l = 4, 6, \ldots, 2r\} & \text{if } r \geq 2, \\ \emptyset & \text{if } r < 2. \end{cases}$$

Using $\bar{\mathscr{C}}$ we have the following two K_i-covers of G:

$$\bar{\mathscr{C}}_0 = \bar{\mathscr{C}} \cup \mathscr{C}^0_1 \cup \mathscr{C}^1_2 \cup \mathscr{C}^1_3 \cup \mathscr{C}_0 \cup \{^{\bar{2},\bar{3}}K^1_{i-1}, {}^{\bar{1}}K^1_{i-1}\},$$

$$\bar{\mathscr{C}}'_0 = \bar{\mathscr{C}}_0 \cup \{K^*_{i-1}\},$$

where $a^{\mathrm{T}}x^{\bar{\mathscr{C}}_0} = a^{\mathrm{T}}x^{\bar{\mathscr{C}}'_0} = a_0$. Thus $0 = b^{\mathrm{T}}x^{\bar{\mathscr{C}}'} - b^{\mathrm{T}}x^{\bar{\mathscr{C}}_0} = b_*$. We have thus shown that

$$b_j = \begin{cases} \rho & \text{for } K^j_{i-1} \in \mathscr{C}, \\ 0 & \text{for } K^j_{i-1} \notin \mathscr{C}. \end{cases}$$

To complete the proof we note that, as before, $b^{\mathrm{T}}x \geq b_0$ defines a nontrivial facet of $P_{k_i}(G)$. Then, by Lemma 3.3, $\rho > 0$ and thus $b = \rho a$. □

Note that if $k = 1$, (5) is still valid but does not always define a facet. If, however, we set $\mathscr{C}'$ to be $\mathscr{C}$ without the K_{i-1}'s containing the three vertices of the exterior cycle, then a proof similar to that used for Theorem 3.6 shows that

$$\sum_{K^j_{i-1} \in \mathscr{C}'} x_j \geq k(i-1) + \left\lceil \frac{i-1}{2} \right\rceil$$

defines a facet for $P_{K_i}(G)$ iff i is even. As an example of Theorem 3.6 consider the 5-wheel of order 3 (Fig. 3). For this graph, the inequality $\sum_{K^3_j \neq K^*_3} x_j \geq 8$ is a facet of $P_{K_4}(G)$.

Remark 3.3. For G a $(2k+1)$-wheel of order $i - 1 = 2r$ ($r > 1$), it is easy to see that $c_i(G) = r(2k+1)$; however, for $r = 1$, then $i = 3$ and $c_3(G) = 2k + 2$ (see Section 3.2.1). Then from the previous theorem, the constraint $\sum_{K^j_{i-1} \in \mathscr{F}_{i-1}(G)} x_j \geq c_i(G)$ does not define a facet of $P_{K_i}(G)$ for $i = 2r + 1 > 3$; however, if $i = 3$, the corresponding constraint does define a facet of $P_{K_3}(G)$. In the next section we show this result and examine other facets of $P_{K_3}(G)$ related to the polytope of the bipartite subgraphs.

3.2. *Facets of $P_{K_3}(G)$*

3.2.1. *The polytope of the bipartite subgraphs and facets of $P_{K_3}(G)$*

$P_B(G)$, the polytope of the bipartite subgraphs of a graph $G(V, E)$, is the convex hull of the incidence vectors of the bipartite subgraphs of G. In [2]

Barahona, Grötschel and Mahjoub have presented a large family of facets of this polytope. It is clear that if (V, F) is a bipartite subgraph of G, then $E\backslash F$ is a K_3-cover of G. Thus if a constraint $a^T\bar{x} \leqslant \bar{a}_0$ is a facet of $P_B(G)$ and $a^Tx \geqslant a_0$ (the inequality resulting by changing $\bar{x}$ to $1-x$ in $a^T\bar{x} \leqslant \bar{a}_0$ is valid for $P_{K_3}(G)$, then $a^Tx \geqslant a_0$ is a facet of $P_{K_3}(G)$.

In the following we describe three families of this type of facet of $P_{K_3}(G)$. The first family arises from the $(2k+1)$-wheels of order 2. The notation developed in Section 3.1.3 will be used here.

Theorem 3.7. *Let $H(W, F)$, a $(2k+1)$-wheel $(k \geqslant 1)$ of order 2 be an induced subgraph of G. The inequality*

$$\sum_{e \in F} x_e \geqslant 2(k+1) \tag{8}$$

defines a facet of $P_{K_3}(G)$.

Proof. In [2] it was shown that

$$\sum_{e \in F} \bar{x}_e \leqslant 2(2k+1) \tag{9}$$

is a facet of $P_B(G)$. We now show that (8) is valid for $P_{K_3}(G)$. Set $K_2^* = \{v_1, v_2\}$ and let $H_1(W\backslash\{v_2\}, F_1)$ be the $(2k+1)$-wheel of order 1 defined by $H\backslash\{v_2\}$. Similarly, define $H_2(W\backslash\{v_1\}, F_2)$. Given the triangles defined by $K_3^j = \{u_j, v_1, v_2\}$, $j = 0, 1, \ldots, 2k$ the following constraints are valid for $P_{K_3}(G)$:

$$\sum_{e \in F_i} x_e \geqslant k+1, \quad i = 1, 2,$$

$$\sum_{e \in K_3^j} x_e \geqslant 1, \quad j = 0, 1, \ldots, 2k.$$

Summing these constraints yields

$$2 \sum_{e \in F\backslash\{(v_1, v_2)\}} x_e + (2k+1)x_{(v_1, v_2)} \geqslant 2(k+1) + 2k + 1. \tag{10}$$

From Theorem 3.6 we know that

$$\sum_{e \in F\backslash\{(v_1, v_2)\}} x_e \geqslant 2k+1 \tag{11}$$

is valid for $P_{K_3}(G)$. Therefore, $(2k+1)\sum_{e \in F} x_e \geqslant (2k+1)(2k+1)+1$ which implies $\sum_{e \in F} x_e \geqslant (2k+1) + 1/(2k+1)$. Since the sum is an integer, we conclude that (8) is valid for $P_{K_3}(G)$ and thus is a facet. □

The second family of facets of $P_{K_3}(G)$ consists of complete subgraphs of odd order.

Theorem 3.8. *Let $H(W, F)$ be a complete subgraph of G, where $|W| = 2k + 1$. Then*

$$\sum_{e \in F} x_e \geq k^2 \tag{12}$$

is a facet of $P_{K_3}(G)$.

Proof. Since $\sum_{e \in F} \bar{x}_e \leq k(k+1)$ is a facet of $P_B(G)$ [2] we only need to show that (12) is valid for $P_{K_3}(G)$. This may be shown by induction on k and by examining the subgraphs of size $2k-1$ of H. □

We now present the third family of facets of $P_{K_3}(G)$.

Theorem 3.9. *Let $H(W, F)$ be a complete subgraph of $G(V, E)$, where $W = \{1, 2, \ldots, q\}$. Let positive integers t_i $(1 \leq i \leq q)$ satisfy $\sum_{i=1}^{q} t_i = 2k + 1$, $k \geq 3$ and $\sum_{t_i > 1} t_i \leq k - 1$. Set*

$$a_{ij} = \begin{cases} t_i t_j, & 1 \leq i < j \leq q, \\ 0, & (i, j) \in E \backslash F. \end{cases}$$

Then

$$a^{\mathrm{T}} x \geq k^2 - \sum_{i=1}^{q} \frac{t_i(t_i - 1)}{2} \tag{13}$$

is a facet of $P_{K_3}(G)$.

Proof. In [2] the inequality $a^{\mathrm{T}} \bar{x} \leq k(k+1)$ was shown to be a facet of $P_B(G)$. To prove that (13) is valid for $P_{K_3}(G)$ we let $\mathscr{C}_1 \subseteq E$ be a K_3-cover of G. Now form the graph $G'(V', E')$ from G by replacing node i of W with V^i a K_{t_i} (if $t_i \geq 1$). All vertices in V^i are completely connected to any vertex in G adjacent to i. Thus H has been replaced by a K_{2k+1}. We now construct $\mathscr{C}_1'$ a K_3-cover of G' as follows:

(i) If $(i, j) \in \mathscr{C}_1 \cap F$, then put all edges between V^i and V^j into $\mathscr{C}_1'$.

(ii) If $(v, j) \in \mathscr{C}_1$, $v \in V \backslash W$, $j \in W$, then add all edges between v and V^j into $\mathscr{C}_1'$.

(iii) For each $1 \leq i \leq q$, where $t_i > 1$, add to $\mathscr{C}_1'$ all edges (u, v), such that $u, v \in V^i$.

Since $\mathscr{C}_1$ is a K_3-cover of G, it is clear that $\mathscr{C}_1'$ is a K_3-cover of G'. Furthermore, $a^{\mathrm{T}} x^{\mathscr{C}_1'} = a^{\mathrm{T}} x^{C_1} + \sum_{i-1}^{q} t_i(t_i - 1)/2 \geq k^2$; therefore (13) is valid and thus is a facet of $P_{k_2}(G)$. □

As an example let $H(W, F)$ be a complete subgraph K_9 then for all edges $(u, v) \in F$, the inequality

$$4x_{u,v} + 2 \sum_{i \neq v} x_{u,i} + 2 \sum_{j \neq u} x_{v,j} + \sum_{(i,j) \neq (u,v)} x_{i,j} \geq 23 \tag{14}$$

is a facet of $P_{K_3}(G)$. (14) may be obtained from Theorem 3.9 by setting $t_u = t_v = 2$, $t_i = 1$ $u \neq i \neq v$.

3.2.2. *Construction of facets*

Let G' be obtained from G by the addition of an edge. We now show how to construct the facets of $P_{K_3}(G')$ from the facets of $P_{K_3}(G)$.

Theorem 3.10. *Let $a^{\mathrm{T}}x \geqslant a_0$ be a facet of $P_{K_3}(G)$ for a graph $G(V, E)$ and denote by G' the graph obtained from G by the addition of an edge e_0. Let a be a system of weights of E and let γ be the minimum weight of $\mathscr{C}_0 \subset E$, where $\mathscr{C}_0$ is a K_3-cover of both G and G' ($\gamma \geqslant a_0$). Set*

$$\bar{a}_{e_0} = \gamma - a_0,$$

$$\bar{a}_e = a_e \quad \text{for } e \neq e_0,$$

$$\bar{a}_0 = \gamma.$$

Then

$$\bar{a}^{\mathrm{T}}x \geqslant \bar{a}_0 \tag{15}$$

defines a facet of $P_{K_3}(G')$.

Proof. (15) is valid for $P_{K_3}(G')$ since if $\mathscr{C}'$ is a K_3-cover of G' which contains e_0, then $x^{\mathscr{C}}$ verifies (15). If $\mathscr{C}'$ does not contain e_0, then $a^{\mathrm{T}}x^{\mathscr{C}} \geqslant \gamma$ and again (15) is verified.

Since $a^{\mathrm{T}}x \geqslant a_0$ is a facet of $P_{K_3}(G)$, there exist $|E|$ K_3-covers of G, $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{|E|}$ such that $x^{\mathscr{C}_1}, \ldots, x^{\mathscr{C}_{|E|}}$ verify $a^{\mathrm{T}}x = a_0$ and are affinely independent. Let $\mathscr{C}_i' = \mathscr{C}_i \cup \{e_0\}$, $i = 1, 2, \ldots, |E|$ and $\mathscr{C}'_{|E|+1} = \mathscr{C}_0$. The sets $\mathscr{C}_i'$, $i = 1, \ldots, |E| + 1$ are all K_3-covers of G' and their incidence vectors $x^{\mathscr{C}_i'}$ verify $\bar{a}^{\mathrm{T}}x = \bar{a}_0$ and are affinely independent. Thus (15) is a facet of $P_{K_3}(G')$. $\square$

To illustrate this method consider the graph $H(W, F)$ in Figure 4. H is obtained from the K_3-9-hole J presented in Fig. 2 by the addition of edge e_0.

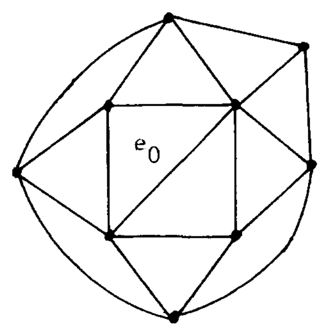

Fig. 4. $H(W, F)$.

From Theorem 3.5 we know that

$$\sum_{e \in F \setminus \{e_0\}} x_e \geqslant 5 \tag{16}$$

is a facet of $P_{K_3}(J)$. It is easy to see that $c_3(H) = 6$ and thus by Theorem 3.10 $\sum_{e \in F} x_e \geqslant 6$ is a facet of $P_{K_3}(H)$.

In this example we note that although $H\backslash\{e_0\}$ is a partial subgraph of H, constraint (16) does not define a facet of $P_{K_3}(H)$. From Theorem 3.10 we see that $a^{\mathrm{T}}x \geqslant a_0$ is also a facet of $P_{K_3}(G')$ iff $a_1 = a_0$. This gives the following corollary.

Corollary 3.11. *Let $H(W, F)$ be a partial subgraph of $G(V, E)$, where $\sum_{e\in F} x_e \geqslant a_0$ is a facet of $P_{K_3}(H)$. If for all edges $e_0 \in E\backslash F \cap (W \times W)$ there exists a K_3-cover of $H + \{e_0\}$ of cardinality a_0, then $\sum_{e\in F} x_e \geqslant a_0$ is a facet of $P_{K_3}(G)$.*

It is easy to see that the $(2k+1)$-wheels of order 1 satisfy the condition of Corollary 3.11. Therefore, if a graph contains a partial subgraph $H(W, F)$ of this type, then $\sum_{e\in F} x_e \geqslant k+1$ is a facet of $P_{K_3}(G)$.

3.3. *Polynomial algorithms to test the facets defined by the $(2k+1)$-wheels*

Using the ellipsoid method, Grötschel, Lovász and Schrijver [12] have shown that there exists a polynomial time algorithm for a linear optimization problem on a polyhedron iff for each constraint of the polyhedron, there exists a polynomial time algorithm for the separation problem. Knowledge of an efficient method solving the separation problem not only shows that the problem is polynomial but also allows the use of these facets as cutting planes in a 'dual' algorithm for the solution of our problem.

We now present polynomial algorithms to solve the separation problem associated with some facets of $P_{K_i}(G)$. First of all, it is easy to test the facets defined by the inequalities (1), (2) and (3). Gerards [9] has presented a polynomial algorithm to test the facets of $P_B(G)$ defined by the $(2k+1)$-wheels of order 2 (see inequality (9)). The same algorithm may be used to test (8). Gerards' algorithm reduces the problem to finding all cycles of minimum length in a graph with positive edge weights. Grötschel and Pulleyblank [14] have shown that this problem has a polynomial solution. We also use this idea to develop a polynomial algorithm to test the facets of the polytope $P_{K_i}(G)$ defined by the $(2k+1)$-wheels H of order $2r-2$ or $2r-1$ (where $i = 2r$), whose inequalities are:

$$\sum_{K_{i-1}^j \in \mathscr{F}_{i-1}(H)} x_j \geqslant k+1 \tag{17}$$

and

$$\sum_{K_{i-1}^j \in (F_{i-1}(H)\backslash K_{i-1}^*)} x_j \geqslant (r-1)(2k+1)+k+1. \tag{18}$$

(See (4) and (5) respectively.)

Algorithm 3.1. Testing facets of $P_{K_i}(G)$ defined by $(2k+1)$-wheel subgraphs.

Given an element $x \in \mathbb{R}^h$, without loss of generality we may assume that the constraints (1), (2) and (3) are verified by x. To test (17) (respectively (18)), do the following:

For each K_{i-2}^j (resp. K_{i-1}^j) in G set:

$$V_j = \{w \in V \mid (w, v) \in E, \forall v \in K_{i-2}^j \text{ (resp. } K_{i-1}^j)\},$$
$$E_j = \{(w, w') \mid w, w' \in V_j\}.$$

For $(w, w') \in E_j$, let ${}^{ww'}\mathscr{C}_0^j$ denote the set of K_{i-1}'s in G which contain nodes w and w' and $i-3$ nodes of K_{i-2}^j (resp. K_{i-1}^j) and let ${}^{w}\mathscr{C}_1^j$ denote the set of K_{i-1}'s in G which contain node w and $i-2$ nodes of K_{i-2}^j (resp. K_{i-1}^j). Set

$$\alpha_{ww'} = \sum_{K_{i-1}^l \in {}^{ww'}\mathscr{C}_0^j} x_l, \qquad \beta_w = \sum_{K_{i-1}^l \in {}^{w}\mathscr{C}_1^j} x_l.$$

For $(w, w') \in E_j$ we set $y_{ww'}^j = -\frac{1}{2} + (\alpha_{ww'} + \frac{1}{2}(\beta_w + \beta_{w'}))$ (resp. $y_{ww'}^j = -r + \frac{1}{2} + \alpha_{ww'} + \frac{1}{2}(\beta_w + \beta_{w'})$).

It is straightforward to show that there is a $(2k+1)$-wheel of order $i-2$ where $K_{i-2}^* = K_{i-2}^j$ (resp. of order $i-1$ where $K_{i-1}^* = K_{i-1}^j$) for which (17) (resp. (18)) is not verified by x iff the minimum weight of an odd cycle in (V_j, E_j) is less than $\frac{1}{2}$. Furthermore, by summing all the constraints (1) corresponding to K_i's which contain nodes w and w' and $i-2$ nodes of K_{i-2}^j (resp. K_{i-1}^j) we get

$$2\alpha_{ww'} + \beta_w + \beta_{w'} \geqslant 1 \qquad (\text{resp. } 2\alpha_{ww'} + \beta_w + \beta_{w'} \geqslant 2r - 1).$$

Since the constraints are assumed to be verified by x, $y_{ww'}^j \geqslant 0$. We now apply the Grötschel–Pulleyblank minimum weight odd cycles algorithm [14] on the graphs (V_j, E_j) weighted by y^j to test the constraints of type (17) and (18).

For constraints defined by the K_i-p-holes, other than those given by the p-wheels of order $i-2$, no polynomial testing algorithm is known. The existence of such an algorithm seems to be an interesting open problem.

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