K_i-Covers. II. K_i-Perfect Graphs

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ABSTRACT

A K_i is a complete subgraph of size i. A K_i-cover of a graph G(V, E) is a set **C** of K_{i-1}s of G such that every K_i in G contains at least one K_{i-1} in **C**. $c_i(G)$ is the cardinality of a smallest K_i-cover of G. A K_i-packing of G is a set of K_is such that no two K_is have $i - 1$ nodes in common. $p_i(G)$ is the cardinality of a largest K_i-packing of G. Let $\mathbf{F}_i(G)$ denote the set of K_is in G and define $c_i(F)$ and $p_i(F)$ analogously for $F \subseteq \mathbf{F}_i(G)$. G is K_i-perfect if $\forall F \subseteq \mathbf{F}_i(G)$, $c_i(F) = p_i(F)$. The K_2-perfect graphs are precisely the bipartite graphs. We present a characterization of K_i-perfect graphs that is similar to the Strong Perfect Graph Conjecture, and explore the relationships between K_i-perfect graphs and normal hypergraphs. Furthermore, if iA denotes the $0 - 1$ matrix of G where the rows are the elements of $\mathbf{F}_{i-1}(G)$ that belong to at least one K_i and the columns are the elements of $\mathbf{F}_i(G)$, then we show that iA is perfect iff G is a K_i-perfect graph. We also characterize the K_i-perfect graphs for which iA is balanced.

1. INTRODUCTION

Berge [1] has defined a graph G to be *perfect* if for every induced subgraph H of G the chromatic number of H equals the clique number of H. The concept of perfectness has two requirements. The first is a pair of parameters such that the

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value of one parameter is always greater than or equal to the value of the other parameter (the chromatic number of any graph is always greater than or equal to the clique number). The second requirement is a quantification over all subsets of a certain type (for example, induced subgraphs). In fact, Berge introduced a second notion of perfectness (later shown to be equivalent to the first by the perfect graph theorem [11]) using the parameters of clique covering number and stability number (definitions given below), and quantification over all induced subgraphs. Hell and Roberts [8] extended the notions of chromatic number and clique number to n-chromatic number and n-clique number, respectively. Using these parameters and quantification over all induced subgraphs they defined the concepts of n-perfectness and degree of perfectness (the smallest value of n such that the graph is n-perfect). Christen and Selkow [3] presented other types of perfectness based on pairs of parameters chosen from clique number, chromatic number, Grundy number, and achromatic number. Again quantification is over all induced subgraphs. For example, they showed that a graph is perfect with respect to the parameters of Grundy number and clique number iff it is perfect with respect to the parameters of Grundy number and chromatic number. These conditions are equivalent to the graph having no induced path on four vertices (i.e., a cograph [7]).

One of the outstanding open problems in graph theory is Berge's *Strong Perfect Graph Conjecture* (SPGC), which states that a graph G is perfect iff neither G nor $\overline{G}$ contains a hole (i.e., an odd chordless cycle) of size ≥ 5.

In this paper we continue our study of K_i-covers of graphs (see [5]). A K_i is a complete graph on i vertices. A *K_i-cover* of G is a set **C** of K_{i-1}s such that every K_i in G contains at least one K_{i-1} in **C**. Note that the definition of K_2-cover is equivalent to that of vertex cover. For a graph G, the *K_i-cover number* $c_i(G)$ is the cardinality of a smallest K_i-cover of G. In [5] we showed that the problem of determining whether $c_i(G) \leq k$ for a given graph G, and integers $i \geq 2$ and $k \geq 1$, is NP-complete. We also studied the complexity of the problem on the restricted family of chordal graphs. Having seen that the general problem is NP-complete, we examined some families of facets of the K_i-cover polytope and showed that various induced subgraphs of G define facets of this polytope.

A *K_i-packing* is a set of K_is such that no two K_is have $i - 1$ nodes in common. Note that the definition of a K_2-packing is equivalent to a matching. For a graph G, the *K_i-packing number* $p_i(G)$ is the cardinality of a largest K_i-packing of G. In [10] it was shown that the problem of determining whether $p_i(G) \geq k$ for a given graph G, and integers $i \geq 3$ and $k \geq 1$, is NP-complete. The complexity status of a generalized notion of K_i-packing was studied in [6].

Given a graph $G(V,E)$ we let $\mathbf{F}_i(G)$ denote $\{K_i | K_i \subseteq G\}$ [i.e., $\mathbf{F}_i(G)$ is the set of complete graphs on i vertices in G]. For any $F \subseteq \mathbf{F}_i(G)$ we may define $c_i(F)$ and $p_i(F)$ in a similar manner to their definitions for graphs. Thus $c_i(F)$ is the cardinality of a smallest set of K_{i-1}s that covers all K_is in F and $p_i(F)$ is the cardinality of a largest set of K_is in F such that no two of these K_is have $i - 1$ nodes in common. In the next section we will show that, for any such F,

$c_i(F) \geq p_i(F)$. We now define a graph to be *K_i-perfect* ($i \geq 2$) if for all $F \subseteq \mathbf{F}_i(G)$, $c_i(F) = p_i(F)$. Note that our definition of K_i-perfect uses the parameters of K_i-covering number and K_i-packing number as well as quantification over all subsets of the set of K_is in a graph G. It is clear from the definitions that, for $i = 2$, the K_2-perfect graphs are precisely the bipartite graphs. In section 2 we will prove a characterization of K_i-perfect graphs that is very similar to the SPGC. This characterization uses the Parthasarathy–Ravindra [13] proof that $(K_4 - e)$-free graphs (also called *diamond-free graphs*) are valid for the SPGC (i.e., the SPGC is true for this family of graphs). In that section we use Berge's second notion of perfectness using $\alpha(G)$, the *stability number of* G (the size of the largest maximal induced set of nonadjacent vertices) and $k(G)$, the *clique cover number of* G (the minimum number of maximal complete subgraphs of G whose union is G). Equivalently, $k(G)$ may be defined to be the minimum number of vertex disjoint complete subgraphs of G whose union is G. Throughout the paper n will denote $|V|$ and $K_{i,j}$ will refer to the complete bipartite graph with cell sizes i and j. C_i refers to a cycle on i vertices. An *antihole* is the complement of a hole, i.e., a chordless cycle of size ≥ 5.

In [11] Lovász studied normal hypergraphs and proved that the two previously mentioned definitions of perfect graphs are equivalent. In section 3 we study the relationship between K_i-perfect graphs and the normality of a hypergraph derived from the clique structure of the given graph. Relevant definitions are included in section 3.

Given a graph G and integer $i \geq 2$, we let iA denote the $0 - 1$ incidence matrix where the rows represent K_{i-1}s in G that belong to at least one K_i and the columns represent the K_is. Clearly each column has i ones and iA contains no zero rows. In the following $\mathbf{1}$ will denote a vector of all ones and a *cycle matrix* is a square $0 - 1$ matrix where each row and column contain exactly 2 ones. A *triangle matrix* is a cycle matrix of size three. A $0 - 1$ matrix A is *perfect* if the associated set packing polytope $\{x \mid Ax \leq \mathbf{1}, x \geq 0\}$ has all integral extreme points [12]. A $0 - 1$ matrix A is *balanced* if A does not contain any square submatrix of odd order with 2 ones per row and per column (see [2] and [9]). A matrix is *totally unimodular* iff every square submatrix has determinant 0, 1, or -1. Any balanced matrix is perfect, and any $0 - 1$ totally unimodular matrix is balanced.

In section 4 we will show that the matrix iA is perfect iff G is K_i-perfect. Furthermore, for $i \geq 3$ we will characterize those K_i-perfect graphs for which iA is balanced.

2. CHARACTERIZATION OF K_i-PERFECT GRAPHS

We now show that the K_i-perfect graphs have a characterization that is very similar to Berge's Strong Perfect Graph Conjecture for perfect graphs. As mentioned in section 1, the diamond-free [or $(K_4 - e)$-free] graphs are valid for this conjecture [13]. This result may be stated as follows:

Lemma 2.1 **[13].** A diamond-free graph is perfect iff it does not contain an odd hole.

In order to use this lemma to characterize the K_i-perfect graphs we need to establish a relationship between the parameters of K_i-perfectness, namely, $c_i(S)$ and $p_i(S)$, and the parameters of perfectness, namely, $\alpha(G)$ and $k(G)$. First we note the following relationship between $c_i(S)$ and $p_i(S)$.

Lemma 2.2. For any $i \geq 2$ and $S \subseteq \mathbf{F}_i(G)$ we have $p_i(S) \leq c_i(S) \leq i \cdot p_i(S)$.

Proof. By definition, no two K_is in P, any largest packing of S, have a K_{i-1} in common; therefore, each K_i in P must be covered by a distinct K_{i-1} and thus $c_i(S) \geq p_i(S)$.

To show that $c_i(S) \leq i \cdot p_i(S)$ we first note that all of the K_is in $S \backslash P$ have at least $i - 1$ nodes in common with a $K_i \in P$, otherwise P is not a maximal packing. Therefore, the family of K_{i-1}s contained in K_is in P is a K_i-cover of S. Since there are i K_{i-1}s in a K_i, the number of K_{i-1}s in P equals $i \cdot p_i(S)$. ∎

Note that the upper bound on $c_i(S)$ may be strengthened to $c_i(S) \leq i \cdot p_i^*(S)$, where $p_i^*(S)$ is the cardinality of the smallest maximal packing. The proof is exactly the same.

To establish the previously mentioned relationship between the parameters of K_i-perfectness and the parameters of perfectness, we define the *K_i-intersection graph of G*, denoted $I_i(G)$, as follows: The nodes of $I_i(G)$ are the K_is in $\mathbf{F}_i(G)$. Two nodes of $I_i(G)$ are adjacent iff the corresponding K_is in G have $i - 1$ nodes in common. If $S \subseteq \mathbf{F}_i(G)$ we let $I_i^S(G)$ denote the corresponding induced subgraph of $I_i(G)$. For $i = 2$ it is clear that $I_2(G)$ is the line graph $L(G)$. An important result by Whitney [14] states that if $n > 4$ then $G_1 \cong G_2 \Leftrightarrow L(G_1) \cong L(G_2)$ ($\cong$ means isomorphic). It is clear that if $i \geq 3$ we have $G_1 \cong G_2 \Rightarrow I_i(G_1) \cong I_i(G_2)$; however, the converse does not hold. The next two lemmas follow immediately from the definitions.

Lemma 2.3. $I_i(G)$ does not contain an induced $K_{1,i+1}$.

Lemma 2.4. If $S \subseteq \mathbf{F}_i(G)$ then $\alpha(I_i^S(G)) = p_i(S)$.

Lemma 2.4 leads us to hope that there is a relationship between $c_i(S)$ and $k(I_i^S(G))$. Although a K_i-cover of S does correspond to a clique cover of $I_i^S(G)$ of the same size, the converse does not always hold, as illustrated by the graph in Figure 1, where $i = 2$, $\{(a, b, c)\}$ is a clique cover of $I_2(G)$, yet $c_2(G) = 2$.

Although $c_i(S) \neq k(I_i^S(G))$ in general, we are able to characterize the $I_i(G)$s where equality does hold. $I_i(G)$ is *conformal* if, for any clique of size h, $h \geq 2$ in $I_i(G)$, the corresponding family of K_is in G have exactly $i - 1$ nodes in common. In this case we have equality between $k(I_i^S(G))$ and $c_i(S)$ for any $S \subseteq \mathbf{F}_i(G)$, namely,

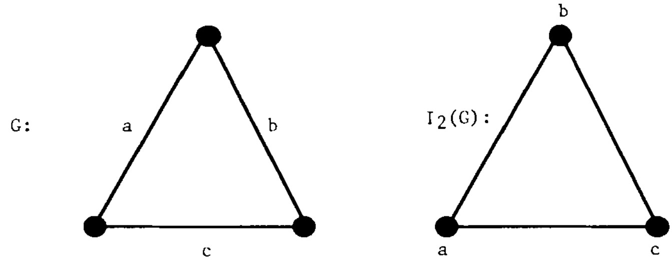

FIGURE 1. $c_2(G) \neq k(I_2(G))$.

Lemma 2.5. $I_i(G)$ is conformal iff, for all $S \subseteq \mathbf{F}_i(G)$, we have $k(I_i^S(G)) = c_i(S)$.

We now characterize conformal K_i-intersection graphs.

Theorem 2.6. $I_i(G)$ is conformal iff G does not contain a K_{i+1}.

Proof **($\Rightarrow$).** Suppose there exists a K_{i+1} in G; let S denote the set of $i + 1$ K_is in this K_{i+1}. The graph $I_i^S(G)$ is complete; however, it is clear that in G the intersection of the K_is in S is the null set since every vertex in the K_{i+1} is avoided by exactly one K_i in S. Thus $I_i(G)$ is not conformal.

($\Leftarrow$). Suppose G does not contain a K_{i+1} and that $I_i(G)$ is not conformal. Thus $I_i(G)$ contains a complete subgraph $I_i^S(G)$ but the corresponding K_is in G do not have a common set of $i - 1$ nodes. Clearly this can only happen if $|S| \geq 3$. Thus there are three K_is in S, A, B, C, such that $|A \cap B \cap C| < i - 1$. Assume that $A \cap B$ and $B \cap C$ represent different K_{i-1}s. In G let node $\{a\} = A \backslash B$ and $\{b\} = B \backslash A$. Similarly, let node $\{c\} = C \backslash B$. If $c \neq a$ then $|A \cap C| < i - 1$, contradicting the existence of edge (A, C) in $I_i(G)$. If $c = a$, then we have a K_{i+1} in G against assumption. ∎

We now establish the relationship between conformal intersection graphs and diamond-free graphs.

Lemma 2.7. If $I_i(G)$ is conformal then it is diamond-free.

Proof. Suppose $I_i(G)$ has a diamond on the vertices A, B, C, D where the edge (A, D) is missing. Since $I_i(G)$ is conformal, the K_is A, B, and C share a K_{i-1} in G and the K_is B, C, and D also share a different K_{i-1} in G. Therefore B and C have two distinct K_{i-1}s in common, which is impossible. ∎

We are now ready to characterize K_i-perfect graphs:

Theorem 2.8. G is K_i-perfect iff $I_i(G)$ is conformal and does not contain an odd hole of size ≥ 5.

***Proof* (⇒).** Suppose $I_i(G)$ is not conformal and let S be a complete subgraph in $I_i(G)$ that violates conformality. Since $\alpha(I_i^S(G)) = 1$ we know that $p_i(S) = 1$ by Lemma 2.4. However, since the K_is in S do not have a K_{i-1} in common, $c_i(S) \geq 2$, thereby contradicting the K_i-perfectness of G. We now assume that $I_i(G)$ is conformal and contains $I_i^S(G)$, an odd hole of size $2h + 1$, $(h \geq 2)$. For this odd hole $k(I_i^S(G)) = h + 1$, whereas $\alpha(I_i^S(G)) = h$. Since $I_i(G)$ is conformal, this implies that $c_i(S) > p_i(S)$ (by Lemmas 2.4 and 2.5), again contradicting the K_i-perfectness of G.

(⇐). Since $I_i(G)$ is conformal, it is diamond-free by Lemma 2.7. The nonexistence of an odd hole allows Lemma 2.1 to be applied and we conclude that $I_i(G)$ is perfect, and thus that $\alpha(I_i^S(G)) = k(I_i^S(G))$ for all induced subgraphs $I_i^S(G)$ of $I_i(G)$. Since $I_i(G)$ is conformal, we may use Lemmas 2.4 and 2.5 to conclude that, for any set $S \subseteq \mathbf{F}_i(G)$, $c_i(S) = p_i(S)$, thus implying that G is K_i-perfect. ∎

An equivalent form of Theorem 2.8 is the following:

Theorem 2.9. *G is K_i-perfect iff G contains neither a K_{i+1} nor a subgraph whose K_i-intersection graph contains an odd hole of size ≥ 5.*

As an immediate corollary to Theorem 2.8 we have

Corollary 2.10. *G is K_i perfect iff $I_i(G)$ is conformal and perfect.*

Theorems 2.8 and 2.9, when restricted to $i = 2$, state that a graph is K_2-perfect (i.e., bipartite) iff G does not contain a triangle and $L(G)$ does not contain an odd hole of size ≥ 5. Since $L(G)$ has such an odd hole iff G has an odd cycle of that size, our result is equivalent to the standard characterization of bipartite graphs.

We now relate K_i-perfectness to normality on hypergraphs.

3. RELATIONSHIP OF K_i-PERFECT GRAPHS WITH NORMAL HYPERGRAPHS

We first introduce notation and results presented in [11]. A *hypergraph* $H(V,E)$ consists of a nonempty set V of vertices and a nonempty set E of hyperedges where each hyperedge is a nonempty subset of V. The *degree* of a vertex is the number of hyperedges containing it. $\Delta(H)$ denotes the maximum degree of a vertex in the hypergraph H. $H'(V,E')$ is a partial hypergraph of $H(V,E)$ iff $E' \subseteq E$. $\chi'(H)$, the *chromatic index* of hypergraph H is the least number of colors needed to color the hyperedges of H such that hyperedges with the same color are disjoint. Obviously $\chi'(H) \geq \Delta(H)$ for any hypergraph H. A hypergraph H is *normal* if, for every partial hypergraph H' of H, we have $\chi'(H') = \Delta(H')$. Note that normality satisfies the criteria of perfectness discussed in section 1, where the parameters are chromatic index and maximum degree, and quantification is over all partial hypergraphs. Given $H(V, E)$, $T \subseteq V$ is a *vertex cover* or *transversal* if every hyperedge in E contains at least one vertex in T. $\tau(H)$ is the cardinality of the smallest transversal of H. We now define $\nu(H)$ to be the maximum number of pairwise disjoint hyperedges of H. Obviously $\tau(H) \geq \nu(H)$ for any hypergraph H. A hypergraph is *τ-normal* if, for every partial hypergraph H' of $H, \tau(H') = \nu(H')$. Again τ-normality satisfies the criteria of perfectness discussed in section 1. A hypergraph satisfies the *Helly property* if any collection of hyperedges whose intersection is empty contains two disjoint hyperedges. As pointed out by Lovász, normal and τ-normal hypergraphs satisfy the Helly property. Finally, given a hypergraph H the *edge graph* $L(H)$ is the intersection graph of the hyperedges of H. Thus the vertices of $L(H)$ represent the hyperedges of H; two vertices are adjacent iff the corresponding hyperedges intersect.

Lovász [11] proved the following:

Lemma 3.1.

(i) If hypergraph H has the Helly property, then H is τ-normal if $\overline{L(H)}$ is perfect.
(ii) A hypergraph is τ-normal iff it is normal.

We now relate the K_i-perfectness of a graph G to the normality of a hypergraph H_i^G defined on the clique structure of G as follows: The vertices of H_i^G are the K_{i-1}s in G; the hyperedges correspond to the K_is in G. From the definitions, Lemma 3.1 and the perfect graph theorem we have:

Lemma 3.2.

(i) $c_i(G) = \tau(H_i^G)$
(ii) $p_i(G) = \nu(H_i^G)$
(iii) G is K_i-perfect iff H_i^G is τ-normal.
(iv) $I_i(G) \cong L(H_i^G)$
(v) $I_i(G)$ is conformal iff H_i^G satisfies the Helly property.
(vi) H_i^G is normal iff H_i^G has the Helly property and $L(H_i^G)$ is perfect.

From Lemmas 3.1 and 3.2 we have another proof of Corollary 2.10, namely, that G is K_i-perfect iff $I_i(G)$ is conformal and perfect. Note that from Lemma 3.2 we cannot conclude the stronger statements of Theorems 2.8 or 2.9.

We now examine the matrix iA and show that iA is perfect iff G is K_i-perfect.

4. *iA* AND K_i-PERFECT GRAPHS

We first introduce some definitions. Given a $0 - 1$ matrix A, the *intersection graph* $I(A)$ is constructed by associating one node to each column of A and joining two nodes by an edge if the corresponding columns have at least one 1 in the same position (i.e., the columns are not orthogonal). A is a *clique matrix* if the undominated rows of A form the list of the representative vectors of all the cliques of $I(A)$. In [4] Chvátal has shown the following:

Lemma 4.1. A clique matrix A is perfect iff $I(A)$ is a perfect graph.

We now study the matrix iA for a graph G and see immediately that $I(iA) \equiv I_i(G)$. Furthermore, we have:

Lemma 4.2. $I_i(G)$ is conformal iff iA is a clique matrix.

Proof. We have the following identities: iA is a clique matrix $\Longleftrightarrow$ all cliques of $I_i(G)$ are represented by rows of iA $\Longleftrightarrow$ and $S \subseteq \mathbf{F}_i(G)$ such that any two K_is in S have $i - 1$ points in common has the property that $|\bigcap_{K_i^j \in S} K_i^j| = i - 1$ $\Longleftrightarrow$ $I_i(G)$ is conformal. (This last identity uses Theorem 2.6.) ∎

Theorem 4.3. iA is a clique matrix iff iA does not have a triangle submatrix.

Proof. From Lemma 4.2 and Theorem 2.6 we know that iA is a clique matrix iff $G \not\supset K_{i+1}$.

($\Rightarrow$). Assume that there is a triangle submatrix in iA. It is straightforward to see that the three K_is whose columns are in this triangle submatrix must form a K_{i+1} in G.

($\Leftarrow$). If a $K_{i+1} \subset G$, then for any $i \geq 2$ there must exist three distinct K_is in the K_{i+1} such that any two of them have $i-1$ points in common; however, the intersection of all three contains fewer than $i - 1$ points. Therefore, iA contains a triangle submatrix. ∎

We may now characterize K_i-perfect graphs in terms of perfect matrices.

Theorem 4.4. G is K_i-perfect iff iA is a perfect matrix.

Proof. Follows immediately from Corollary 2.10, Lemma 4.1 and Lemma 4.2. ∎

As a result of Theorem 4.4 we conclude that a graph G is K_i-perfect iff the polytope

$$P(G)\begin{cases} iAx \leq 1 \\ \quad x \geq 0 \end{cases}$$

has all its vertices in 0,1.

As a summary, we have the following theorem:

Theorem 4.5. Given a graph G, the following seven conditions are equivalent:

(i) G is K_i-perfect.
(ii) $I_i(G)$ is conformal and does not contain an odd hole of size ≥ 5.
(iii) $I_i(G)$ is conformal and perfect.
(iv) H_i^G is normal.
(v) iA is a perfect matrix.
(vi) iA is a clique matrix and $I_i(G)$ is perfect.
(vii) $P(G)$ has all its vertices in 0,1.

It is well known that for K_2-perfect graphs (bipartite graphs) $2A$ (the vertex-edge incidence matrix) is totally unimodular. One might hope that the structure of iA might allow a stronger statement than that made in Theorem 4.4, in particular that G is K_i-perfect iff iA is a balanced matrix (i.e., that, for matrices of the form iA, the perfection and balance properties are equivalent). However, already for $i = 3$ one may construct K_3-perfect graphs for which $3A$ is not balanced. The smallest such graph is shown in Fig. 2. $I_3(G)$ is presented in Fig. 3 and $3A$ is in Table 1. The first 13 rows and the 13 columns of $3A$ form a cycle submatrix.

We have seen that G is K_i-perfect iff iA is a perfect matrix and that for $i \geq 3$ there exist K_i-perfect graphs for which iA is not balanced. We now characterize the K_i-perfect graphs for which iA is balanced. Clearly, any K_i-perfect graph with unbalanced iA must have a partial odd hole C in $I_i(G)$ where the edges of

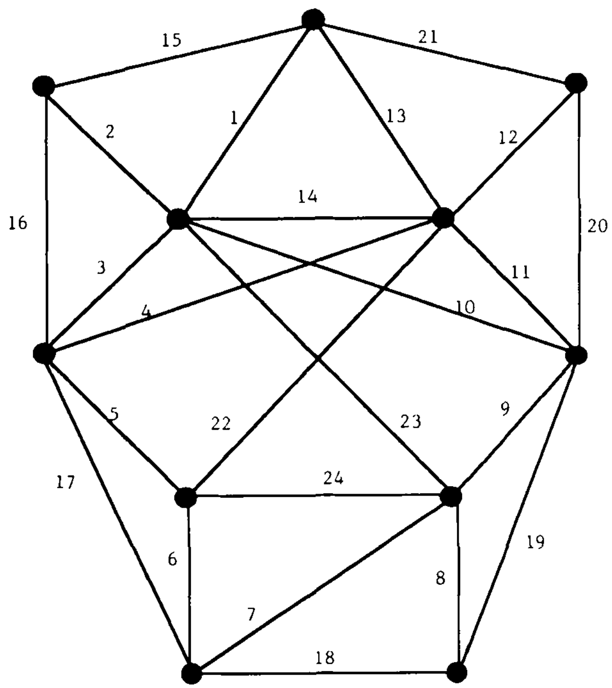

FIGURE 2. G, K_3-perfect, $3A$ not balanced.

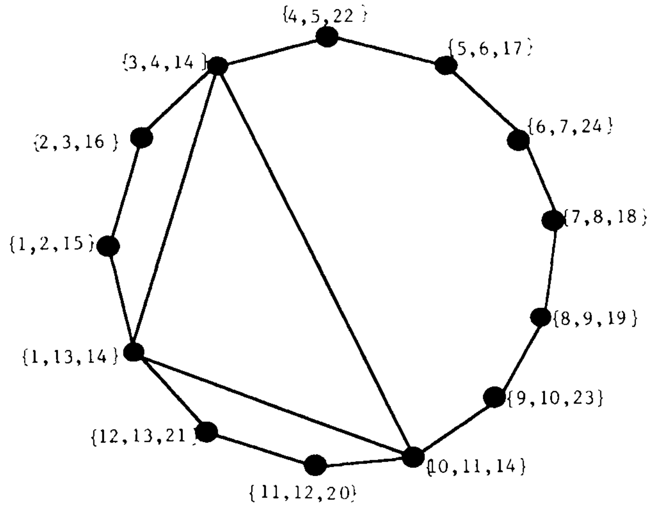

FIGURE 3. $I_3(G)$.

Table 1. 3*A*.

	1 13 14	1 2 15	2 3 16	3 4 14	4 5 22	5 6 17	6 7 24	7 8 18	8 9 19	9 10 23	10 11 14	11 12 20	12 13 21
1	1	1											
2		1	1										
3			1	1									
4				1	1								
5					1	1							
6						1	1						
7							1	1					
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13	1												1
14	1			1							1		
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24							1						

this hole correspond to distinct K_{i-1}s in G. Since $I_i(G)$ is perfect the induced subgraph on the nodes in C must contain chords. Furthermore, $I_i(G)$ is conformal and thus the edges of $I_i(G)$ may be partitioned into edge-disjoint cliques such that each vertex of $I_i(G)$ belongs to at most i such cliques. In the following we will characterize the number, size, and structure of the complete subgraphs formed by the chords of a smallest such partial odd hole.

Throughout this discussion, given C a partial odd hole of size $2k + 1 (k \geq 2)$ in $I_i(G)$, we let $\{K_i^0, \ldots, K_i^{2k}\}$ denote the K_is of G corresponding to the vertices of C and let $\{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$ denote the K_{i-1}s of G where K_{i-1}^j represents the edge (K_i^j, K_i^{j+1}) in $I_i(G)$ (addition modulo $2k$). The K_{i-1}s are all distinct, otherwise the submatrix of iA representing C would not be a cycle matrix. A subgraph of $I_i(G)$ that has the above properties together with at least one "chording" complete subgraph that corresponds to $K_{i-1}^* \notin \{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$ is called a *clique-chorded hole*. We now have

Lemma 4.6. iA is balanced iff G is K_i-perfect and $I_i(G)$ does not contain an odd clique-chorded hole.

Proof. This proof follows immediately from the definitions, Lemma 4.2, and Theorems 4.3 and 4.4. ∎

The definition of clique-chorded hole allows the chording complete subgraphs to have an unrestricted interaction. We will show in fact that this interaction must be greatly constrained. To this end we let X_C denote the set of chords of C in $I_i(G)$ and partition X_C into l edge disjoint complete subgraphs, X_j, $1 \leq j \leq l$. The vertices in each X_j may be considered ordered using the ordering of the K_is in C. Each X_j represents a K_{i-1} in G that does not belong to $\{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$. In the following lemmas, C will refer to a *minimum* C, namely, any partial odd hole in $I_i(G)$, G K_i-perfect, whose corresponding submatrix in iA is a cycle matrix and there does not exist any other such C' where $|C'| < |C|$. From Theorem 4.3 we know that $|C| > 3$.

Lemma 4.7. For each $X_j \in X_C$ the number of vertices of C between consecutive vertices in X_j is even.

Proof. Since K_{i-1}^*, the $K_{i-1} \in \mathbf{F}_{i-1}(G)$, which represents X_j, does not belong to $\{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$, consecutive vertices in C cannot be vertices in X_j. Now assume that there exist vertices K_i^α, K_i^β, $(\alpha < \beta)$ consecutive vertices in X_j such that $\beta - \alpha - 1$, the number of C vertices trapped between K_i^α and K_i^β is odd. We immediately see that the submatrix of iA defined on $\{K_i^\alpha, K_i^{\alpha+1}, \ldots, K_i^\beta\}$ is a cycle matrix, which contradicts the minimality of C. ∎

Corollary 4.8. Each X_j of X_C is of odd order.

Proof. Follows immediately from Lemma 4.7 since $|C|$ is odd. ∎

We now examine the situation where X_C contains at least two complete subgraphs. An *exterior edge* of such a complete subgraph is the chord of C between two consecutive vertices of the complete subgraph. Given two complete subgraphs X_1 and X_2 where (K_i^α, K_i^β) is a chord of X_1, we say that this chord *traps* vertex K_i^γ of X_2 if $\alpha \leq \gamma \leq \beta$. Note that X_1 and X_2 may have at most one vertex in common. We now state a lemma that determines the interaction between two complete subgraphs $X_1, X_2 \in X_C$. The proof is straightforward and will be left to the reader.

Lemma 4.9. There cannot exist two complete subgraphs $X_1, X_2 \in X_C$ where an exterior edge e of X_1 traps an odd number of vertices of X_2 unless the single vertex trapped by e belongs to e.

From this lemma we have the following corollary:

Corollary 4.10. There cannot exist two disjoint X_js in X_C.

Proof. If X_1 and X_2 are two disjoint elements of X_C, then from Lemma 4.9 every exterior edge of X_1 must trap an even number of vertices of X_2. This implies that X_2 has even cardinality, contradicting Corollary 4.8. ∎

Thus each pair of X_js in X_C must have a common vertex. We now show that all such X_js must have the same common vertex.

Lemma 4.11. If $|X_C| > 1$, then there exists a single vertex that belongs to all $X_j \in X_C$.

Proof. Assume there exist three complete subgraphs in X_C, X_1, X_2, X_3 such that $X_1 \cap X_2 = x$, $X_2 \cap X_3 = y$, and $X_1 \cap X_3 = z$ where x, y, and z are all different. In $I_i(G)$ the K_is x, y, and z are the vertices of a triangle. Conformality of $I_i(G)$ requires that these K_is all have a K_{i-1} in common, which contradicts the assumption that X_1, X_2, and X_3 are different. ∎

We can now make a statement about the maximum number of elements in X_C. All such elements have node $\bar{x}$ in common.

Lemma 4.12. The maximum number of complete graphs in X_C is $i - 2$.

Proof. From Lemma 2.3, there is no $K_{1,i+1}$ in $I_i(G)$. Since $\bar{x}$ is on C it already has two independent cycle edges emanating from it and thus since each $X_i \in X_C$ adds another independent edge, the maximum is $i - 2$. ∎

For $i = 3$, we thus see that there is exactly one element in X_C. We now examine the case where $i > 3$ and show that a certain nesting property must hold among the complete graphs in X_C. Given two $X_1, X_2 \in X_C$ we say that X_1

is nested in X_2, if the two exterior edges of X_2 from x trap no vertices of X_1. A set of complete graphs is called *nested* if, for any pair, one is nested in the other. We now show that the set X_C is nested in this sense.

Lemma 4.13. If $X_1, X_2 \in X_C$ then either X_1 is nested in X_2 or X_2 is nested in X_1.

Proof. If not, then from Lemma 4.9 we see that both X_1 and X_2 have an even number of vertices, which contradicts Corollary 4.8. ∎

We are now ready to characterize the K_i-perfect graphs for which iA is balanced. A graph $G(V, E)$ is called an *i-nested clique hole* if E may be partitioned into a Hamiltonian cycle C and a set of at most $i - 2$ nested odd cliques such that the intersection of any two cliques is $\bar{x} \in V$. Furthermore, for any clique each exterior edge traps an even (>2) number of vertices of C and an even number (possibly zero) of vertices of any other clique.

Theorem 4.14. The following conditions are equivalent:

(i) iA is balanced.
(ii) G is K_i-perfect and $I_i(G)$ does not contain an odd induced i-nested clique hole.
(iii) G is K_i-perfect and $I_i(G)$ does not contain an odd clique-chorded hole.

Proof. (i) $\Longleftrightarrow$ (iii) is Lemma 4.6.

(i) $\Rightarrow$ (ii). This proof is the same as the ($\Rightarrow$) proof in Lemma 4.6, except we must show that the K_{i-1}s corresponding to the edges of the cycle are distinct and different from the K_{i-1}s corresponding to the interior cliques. This follows immediately from the definition of i-nested clique hole since each exterior edge of an interior clique must trap at least four vertices of C. Thus if G is K_i-perfect and contains an odd induced i-nested clique hole, iA is not balanced.

(i) $\Leftarrow$ (ii). Assume iA is perfect but not balanced. Let C be a minimum size odd cycle matrix of iA. That the subgraph of $I_i(G)$ induced on the K_is in C is an odd i-nested clique hole follows from the definition of i-nested clique hole and Lemmas 4.7, 4.9, 4.11, 4.12, 4.13, and Corollary 4.8. ∎

5. CONCLUDING REMARKS

As stated in section 1, our definition of K_i-perfect uses quantification over all subsets F of $\mathbf{F}_i(G)$. We could of course have chosen the quantification to be over all induced subgraphs of G or all partial subgraphs of G.

We now examine each of these two alternative definitions and show by counterexample that they are not equivalent to the definition of K_i-perfect used in this paper.

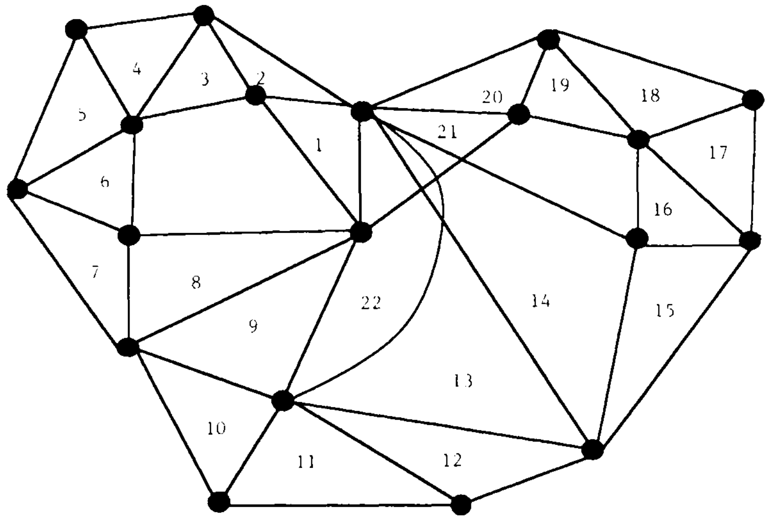

FIGURE 4. G_1.

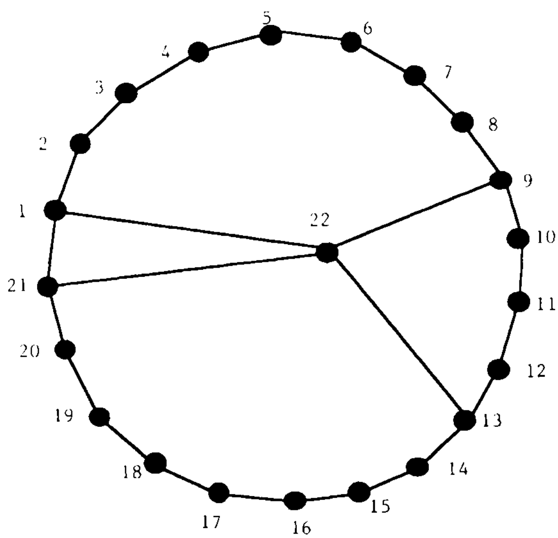

FIGURE 5. $I_3(G_1)$.

We define a graph to be *K_i-partial perfect* if $\forall H < G$, $c_i(H) = p_i(H)$, and a graph to be *K_i-induced perfect* if $\forall H \lhd G$, $c_i(H) = p_i(H)$. Figure 4 shows G_1, a graph that is both K_3-partial perfect and K_3-induced perfect, but that is not K_3-perfect. Its K_3 intersection graph is presented in Fig. 5. For this graph $c_3(G_1) = p_3(G_1) = 11$. It is easy to verify that all $H < G_1$ and all $H \lhd G_1$ satisfy $c_3(H) = p_3(H)$. The odd hole in $I_3(G_1)$ shows that G_1 is not K_3-perfect.

G_2 presented in Fig. 6 is a K_3-induced perfect graph that is neither K_3-perfect nor K_3-partial perfect. $I_3(G_2)$ is drawn in Fig. 7. It is seen that $p_3(G_2) = c_3(G_2) = 8$ and that $p_3(H) = c_3(H) \forall H \lhd G_2$. Again the odd hole in $I_3(G_2)$ shows that G_2 is not K_3-perfect. The removal of the bold edge yields a partial subgraph that is not K_3-partial perfect.

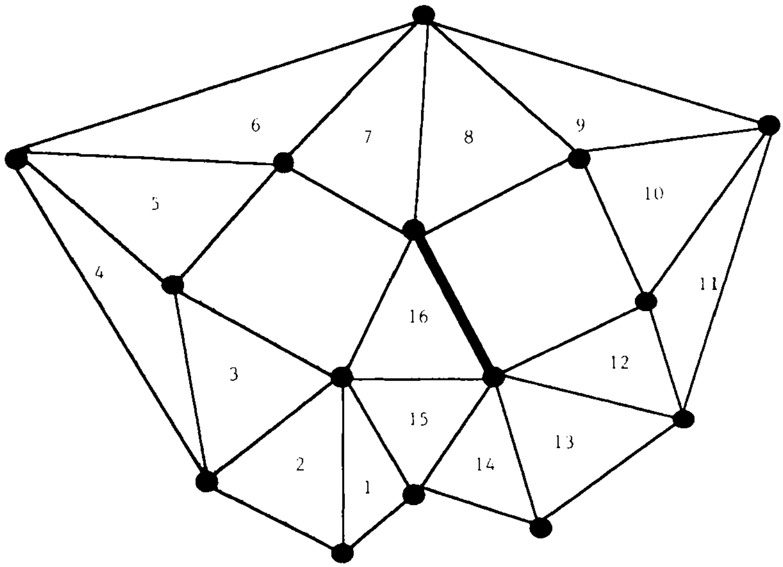

FIGURE 6. G_2.

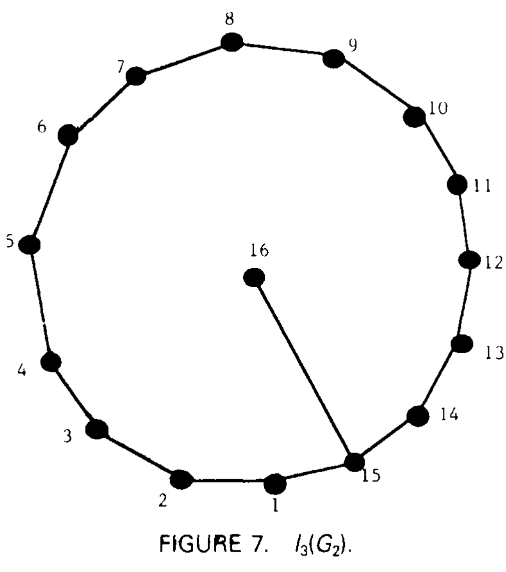

FIGURE 7. $I_3(G_2)$.

The characterizations of the K_i-partial perfect and the K_i-induced perfect graphs are left as open problems.

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