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Minimal arc-sets spanning dicycles

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We give a structural characterization of the inclusionwise minimal arc subsets whose vertices induce a subgraph containing at least one directed cycle. These arc sets arise in a linear formulation of a binary quadratic problem.

1. Introduction

Let $G = (V, E)$ be a digraph. A subset $D \subseteq E$ of arcs is called a *dicyclic set of* G if the subgraph $G[V(D)]$, induced by the set $V(D) \subseteq V$ composed of all the vertices incident with at least one arc from D, contains a dicycle (that is, a directed cycle). A dicyclic set D of G is (inclusionwise) *minimal* if the subgraph of G induced by $V(D')$ is acyclic for any proper subset D' of D. In other words, a minimal dicyclic set $D \subseteq E$ fulfills the following properties:

(α) $G[V(D)]$ contains a dicycle,
(β) for any dicycle C of $G[V(D)]$ and any arc e of D, there exists a vertex of C whose only incident arc in D is e.

Indeed property (β) prevents from having a dicycle C of $G[V(D)]$ and an arc e of D such that $V(D \setminus \{e\})$ contains all the vertices of C or equivalently, such that C still is a dicycle of $G[V(D \setminus \{e\})]$. The dicyclic sets of G clearly correspond to the dependent set of the independence system $(E, \mathcal{I} = \{D \subseteq E : G[V(D)] \text{ is acyclic}\})$, and the minimal dicyclic sets of G are nothing but the circuits of $(E, \mathcal{I})$.

In this paper we give a complete characterization of the minimal dicyclic sets of a digraph, that is, we reveal the structure of the digraphs induced by the vertex sets of minimal dicyclic sets. Despite the graph-theoretical nature of our result, its motivation comes from a new linearization technique based on dicyclic sets for a binary quadratic program associated with finding an induced subgraph of G maximizing an arc-weight function. The obtained binary linear program only considers arc variables but may have an exponential number of dicycle-set based constraints whose separation problem is showed to be polynomial-time solvable.

Related works have been done by Cornaz and Fonlupt [3] and Cornaz and Mahjoub [4] on the maximum biclique and maximum bipartite induced subgraph problems, respectively. Although their approach is quite similar to the one presented in this paper, the structure and characterization of the minimal dicyclic sets are much harder to achieve and require specific development.

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The paper is organized as follows. The next section illustrates the connection between the minimal dicyclic sets and a novel linearization technique for the maximum arc-weight induced subgraph problem. In Section 3 we present some structural properties of the dicyclic sets that are later used, in the following section, to completely characterize the minimal dicycle sets.

We conclude this introduction with some definitions and notation, which have been mainly taken from [11]. We will denote a graph (undirected or directed) by $G = (V, E)$ where V is the *vertex* set and E is the *edge* set when G is undirected or the *arc* set when G is directed. An edge with end vertices u and v will be denoted by uv, while an arc with head u and tail v is represented as the ordered pair (u, v). For any $S \subseteq V$, let $E[S]$ denote the set of edges (arcs) incident with the vertices in S, and $G[S] = (S, E[S])$ the subgraph of G *induced* by S. If $F \subseteq E$, then the set $V(F)$ is composed of all the vertices of G incident with the edges (arcs) in F.

Throughout the paper, the graphs we consider are loopless. A path (dipaths) in G then is identified by its sequence of distinct edges (arcs). If the first and last vertices of a path (dipath) $P \subseteq E$ coincide, P then is called a *cycle* (*dicycle)* in G. A subgraph of G is *acyclic* if it has no cycles (dicycles). An acyclic subgraph of an undirected graph will sometimes be referred to as a *forest*.

Given a vector $x \in \mathbb{R}^A$ where A is a set, $x(B)$ will denote $\sum_{a\in B} x_a$ for any subset B of A.

2. Motivation

In this section, we let $G = (V, E)$ be a (directed or undirected) graph and $c \in \mathbb{Q}^E$ be a weight vector associated with the edges (arcs) of G. We consider the problem of finding a vertex-induced acyclic subgraph of G with maximum weight, that is, so that the sum of the weights of the selected edges (arcs) is maximum. This problem is NP-hard even for its maximum-size version (i.e., with unit weights) as shown by the next proposition.

Proposition 1. *The problem of determining a maximum-size vertex-induced acyclic subgraph of G is* NP-*hard.*

Proof. If G is undirected, the reduction is from the NP-hard maximum stable set problem. Let $G = (V, E)$ be an undirected graph and let $\alpha(G)$ be the maximum size of a stable set of G. Construct a graph $\tilde{G}$ from G as follows: First each edge is replaced by two parallel edges, and then, for each vertex v, a new vertex v' and an edge vv' are created. It can be easily seen that the maximum-size of an induced forest of $\tilde{G}$ is equal to $\alpha(G)$.

The proof is similar if G is a digraph except that $\tilde{G}$ is a digraph obtained by replacing each edge of G by two opposite arcs. ■

Let $\mathcal{C}(G)$ be composed of all the cycles (dicycles) of G, that is,

$$\mathcal{C}(G) := \{C \subseteq E : C \text{ is a cycle (dicycle) of } G\}.$$

In general when considering vertex variables, this NP-hard problem can be recorded as the following quadratic binary program

$$(\mathrm{QB}) \begin{cases} \max & x^T W x \\ \text{s.t.} & \displaystyle\sum_{v\in V(C)} x_v \le |C| - 1 \quad \text{for } C \in \mathcal{C}(G), \\ & x \in \{0, 1\}^V, \end{cases}$$

where W is a $V \times V$ matrix defined for the undirected case by $W_{uv} := \frac{1}{2}\sum_{uv\in E} c_{uv}$, and for the directed one by $W_{uv} := \frac{1}{2}\sum_{(u,v)\in E} c_{(u,v)}$. Observe that the continuous relaxation of this formulation may be extremely weak. If G is a graph with two vertices u and v and parallel edges (opposite arcs), then $\bar{x}_u = \bar{x}_v = \frac{1}{2}$ is optimal for the continuous relaxation of (QB) and the objective-function value is $\frac{1}{4}\sum_{e\in E} c_e$, although the optimal value of (QB) is zero.

A classical approach for solving quadratic binary problems consists of introducing additional variables in order to obtain equivalent reformulations as mixed-integer linear programs. The so-called *level-1 of the Reformulation-Linearization Technique (RLT1)* of Sherali and Adams [1] appears to be one of the most standard linearization techniques and it provides tighter linearizations than the classical ones of Glover and Woolsey [6] and Glover [5]. Other linearization approaches related to RLT1 were considered by Lovász and Schrijver [10], Balas, Ceria, and Cornuéjols [2], and Lasserre [8], yet possibly maybe for the one of Lasserre, they are not tighter than RLT1. (See Laurent [9] for a comparison of these linearizations.)

To apply RLT1 to our problem, we consider the *vertex-cycle (vertex-dicycle) inequalities*

$$x(V(C)) \le |C| - 1 \quad \text{for } C \in \mathcal{C}(G), \tag{1}$$

together with the *box constraints*

$$0 \le x_i \le 1 \quad \text{for } i \in V. \tag{2}$$

Let $P(G)$ be the polytope of $\mathbb{R}^V$ defined by (1) and (2). RLT1 linearization is obtained by first performing the reformulation operation where each constraint of $P(G)$ is multiplied by x_i and by $1 - x_i$ (for each $i \in V$), and then by linearizing $x_i x_j$ into y_{ij} (for each unordered pair $\{i, j\}$ of distinct vertices of V) and x_i^2 into x_i (for each $i \in V$).

To illustrate this linearization technique, consider, for instance, the following (QB) problem

$$\begin{array}{lll} \max & x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_1x_5 & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 \le 4, & \\ & 0 \le x_i \le 1 & \text{for } 1 \le i \le 5, \\ & x \in \{0,1\}^5. & \end{array} \tag{3}$$

Observe that problem (3) corresponds to problem (QB) associated with G being a cycle or dicycle on the five vertices $\{1, \ldots, 5\}$ and c being the all-one vector of $\mathbb{R}^5$. It is immediate to see that the optimum of (3) is 3. The linear program produced by RLT1 for (3) is

$$\begin{array}{llllll} \max & y_{12} + y_{23} + y_{34} + y_{45} + y_{15} & & & & \\ \text{s.t.} & 3x_i - 4 + \sum_{j \in \{1,2,3,4,5\}} x_j \le & \sum_{j \in \{1,2,3,4,5\} \setminus \{i\}} y_{ij} & \le 3x_i & \text{for } 1 \le i \le 5, \\ & x_i + x_j - 1 \le & y_{ij} & \le x_i, x_j & \text{for } 1 \le i < j \le 5, \\ & & x \in [0,1]^5, & & \\ & & y \in [0,1]^{10}. & & \end{array} \tag{4}$$

The last operation (i.e., linearizing x_i^2 into x_i) truly strengthens the formulation. In fact, the projection of RLTn (the formulation obtained after repeating n times such reformulation-linearization operations to $P(G)$, where n is the dimension of x) onto the x-space is nothing but the convex hull of the integer points of $P(G)$ [12]. The projection of RLTn onto the y-space therefore is integer as well. Of course, it is not easy to obtain RLTn in general. The last inequalities of (4) are the result of the RLT1 linearization on the box inequalities (2), and are called the *product constraints*. These inequalities exactly are the ones that Glover and Woolsey [6] added in their linearization approach for (QB) while replacing, in the objective function, each cross-product x_ix_j by variable y_{ij}.

Let VIAS(G) denote the convex hull of the incidence vectors of the edge (arc) set of the vertex-induced acyclic subgraphs of G. The projection of (4) onto the y-space is not equal to VIAS(G). Indeed, it admits the feasible solution $y^1 \in \mathbb{R}^{10}$ defined as

$$\begin{array}{l} y^1_{13} = y^1_{14} = y^1_{24} = y^1_{25} = y^1_{35} = \dfrac{1}{3}, \\ y^1_{ij} = \dfrac{2}{3} \quad \text{for all the other unordered pairs } \{i,j\} \text{ of } \{1, \ldots, 5\}, \end{array}$$

that gives an objective-function value of $\frac{10}{3}$. An inequality of interest for this small example is in fact $y_{12} + y_{23} + y_{34} + y_{45} + y_{15} \le 3$, since it cuts off point y^1. It can be easily seen that this inequality belongs to the family of *cycle (dicycle) inequalities*, that is, it is a valid inequality for VIAS(G) of the type

$$\sum_{e \in C} y_e \le |C| - 2 \quad \text{for } C \in \mathcal{C}(G). \tag{5}$$

The validity of (5) follows directly from the fact that removing at least one vertex from a cycle (dicycle) implies to remove at least two edges (arcs). Let $\mathcal{P}_3(G)$ be composed of the triplets of edges (arcs) forming a path (undirected dipath) or a cycle (undirected dicycle) of G, that is,

$$\mathcal{P}_3(G) := \{(e,f,g) \in E^3 : f \text{ adjacent with both } e \text{ and } g\}.$$

Recall that an undirected dipath (dicycle, respectively) of a digraph G is a path (cycle, respectively) in the underlying undirected graph of G [11]. Let $Q^1(G)$ be the polytope of $[0,1]^E$ defined by the cycle (dicycle) inequalities (5) and the *3-path inequalities*

$$y_e - y_f + y_g \le 1 \quad \text{for } (e,f,g) \in \mathcal{P}_3(G). \tag{6}$$

Inequalities (6) guarantee that any integral vector of $Q^1(G)$ corresponds to the set of edges (arcs) of a vertex-induced acyclic subgraph of G. Notice that inequality (6) $y_{ij} - y_{jk} + y_{kl} \le 1$ can be obtained by projecting down onto the y-space the polyhedron defined by the inequalities $x_j + x_k - y_{jk} \le 1$, $y_{ij} \le x_j$, and $y_{kl} \le x_k$ of the RLT1 linearization.

The maximum-weight vertex-induced acyclic subgraph problem can thus be formulated as the integer linear program

$$\max\{c^T y : y \in Q^1(G) \cap \mathbb{Z}^E\}.$$

Observe that if G contains two parallel edges (opposite arcs) e and f, inequalities (5) enforce that $y_e = y_f = 0$ for any vector pf $Q^1(G)$.

Proposition 2. *Formulation $Q^1(G)$ is stronger than the formulation obtained by the classical linearization of Glover and Woolsey* [6].

Proof. Let $y \in Q^1(G)$ and let $x \in \mathbb{R}^V$ be defined as $x_i := \max\{y_e : e \in \delta(i)\}$ for all $i \in V$, where $\delta(i)$ is composed of all the edges (arcs) of G incident with vertex i. Since y satisfies the 3-path inequalities (6), (x, y) satisfies the product constraints. We now prove that all the vertex-cycle inequalities (1) are satisfied by (x, y) when G is a digraph. (The proof is similar if G is undirected.)

Let C be a dicycle of G. For any arc (i, j) in C, the product constraint is $x_i + x_j - y_{ij} \le 1$. By summing up all the product constraints associated with the arcs of C and the vertex-cycle inequality $y(C) \le |C| - 2$, one gets $2x(V(C)) \le 2|C| - 2$. Consequently $Q^1(G)$ is contained in the projection onto the y-space of Glover and Woolsey's formulation. ■

The number of inequalities (5) may be exponential in the size of the graph. One hence must focus on the complexity of the *separation problem* for this family of inequalities. The *separation problem* for a family of inequalities $\mathcal{I}$ is defined as follows: Given a vector $y \in \mathbb{R}^E$, find an inequality in $\mathcal{I}$ violated by y or prove that none exists. In the next proposition, we state that the separation problem for (5) is polynomial-time solvable, which implies that so is optimizing a linear function over $Q^1(G)$ [7].

Proposition 3. *Let $y \in [0, 1]^E$. The separation problem for inequalities (5) can be solved in polynomial time.*

Proof. Any inequality $y(C) \le |C| - 2$ can be rewritten as $\sum_{e \in C}(1 - y_e) \ge 2$. The separation problem for inequalities (5) then is equivalent to finding a minimum-cost cycle (dicycle) in G with respect to the edge (arc) cost vector $\mathbf{1} - y$. ($\mathbf{1}$ is the all-one vector in $\mathbb{R}^E$.) Since $y \in [0, 1]^E$, this problem can be solved in polynomial time, since it reduces, for each edge (arc) of G, to computing a shortest path between its extremities (from its head to its tail). ■

Let us now look at another, but close, example. Consider the following (QB) problem

$$\begin{array}{lll} \max & x_1x_6 + x_2x_3 + x_4x_5 & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 \le 4, & \\ & 0 \le x_i \le 1 & \text{for } 1 \le i \le 6, \\ & x \in \{0, 1\}^6. & \end{array} \tag{7}$$

Observe that we have simply added a unit-weight pending edge $e = 16$ (arc, no matter its orientation is), and dropped some of the weights to zero. It is not hard to see that the optimum of (7) now equals 2. The linear program produced by RLT1 for the continuous relaxation of (7) is

$$\begin{array}{lrclll} \max & & y_{16} + y_{23} + y_{45} & & & \\ \text{s.t.} & 3x_i - 4 + \sum_{j \in \{1,2,3,4,5\}} x_j \le & \sum_{j \in \{1,2,3,4,5\} \setminus \{i\}} y_{ij} & \le 3x_i & \text{for } 1 \le i \le 5, \\ & 4x_6 - 4 + \sum_{j \in \{1,2,3,4,5\}} x_j \le & \sum_{j \in \{1,2,3,4,5\}} y_{6j} & \le 4x_6, & \\ & x_i + x_j - 1 \le & y_{ij} & \le x_i, x_j & \text{for } 1 \le i < j \le 6, \\ & & x \in [0, 1]^6, & & \\ & y \in [0, 1]^{15}. & & & \end{array} \tag{8}$$

The first and the third families of constraints in (8) are already present in (4). Again the projection of (8) onto the y-space is not equal to VIAS(G). Indeed, it admits the following feasible solution $y^2 \in \mathbb{R}^{15}$ defined as

$$\begin{array}{l} y^2_{24} = y^2_{25} = y^2_{34} = y^2_{35} = \dfrac{1}{3}, \\ y^2_{ij} = \dfrac{2}{3} \quad \text{for all the other unordered pairs } \{i, j\} \text{ of } \{1, \dots, 6\}, \end{array}$$

that gives an objective-function value of $\frac{7}{3}$. The interesting inequality for this second small example is in fact $y_{16} + y_{23} + y_{45} \le 2$, since it cuts off point y^2. Let $\mathcal{D}(G)$ be composed of all the sets D of edges (arcs) of E such that $G[V(D)]$ contains a cycle (dicycle), that is,

$$\mathcal{D}(G) := \{D \subseteq E : G[V(D)] \text{ is not acyclic}\}.$$

The elements in $\mathcal{D}(G)$ are the dicyclic sets of digraph G as introduced in Section 1, and similarly are called the *cyclic set* of G if G is undirected. It can be easily seen that this inequality belongs to the family of *cyclic-set (dicyclic-set) inequalities*, that is, it is a valid inequality for VIAS(G) of the type

$$y(D) \le |D| - 1 \quad \text{for } D \in \mathcal{D}(G). \tag{9}$$

The validity of (9) is straightforward from the definition of the cyclic (dicyclic) sets of G and, of course, (9) can be restricted to only the (inclusionwise) minimal elements of $\mathcal{D}(G)$.

Let $Q^2(G)$ be the polytope $[0, 1]^E$ defined by the cyclic-set (dicyclic-set) inequalities (9) and the 3-path inequalities (6). The maximum-weight vertex-induced acyclic subgraph problem can thus be formulated as the integer linear program

$$\max\{c^T y : y \in Q^2(G) \cap \mathbb{Z}^E\}.$$

Observe again that if y belongs to $Q^2(G)$, inequalities (9) then imply that $y_e = y_f = 0$ for any parallel edges (opposite arcs) e and f of G.

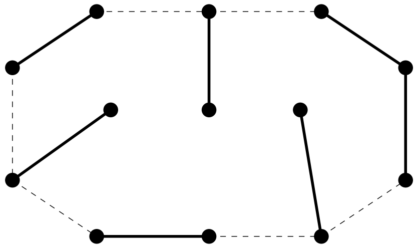

Fig. 1. A graph induced by a minimal subset of edges D so that $G[V(D)]$ has a cycle.

Proposition 4. *Formulation $Q^2(G)$ is stronger than the formulation obtained by the classical linearization of Glover and Woolsey* [6].

Proof. We proceed as in the proof of Proposition 2, that is, we consider a vector $y \in Q^2(G)$ and define a vector $x \in \mathbb{R}^V$ as $x_i := \max\{y_e : e \in \delta(i)\}$ for all $i \in V$. We only consider the case of an undirected graph G, the digraph case being similar.

Let C be any cycle of G. Let F be the set of arcs e of E such that $y_e = x_v$ for some vertex v of C. Since $G[V(F)]$ contains cycle C, there must exist a minimal subset D of F so that $G[V(D)]$ contains cycle C. For any arc e in D, the subgraph $G[V(D \setminus \{e\})]$ is acyclic, and there thus exists a vertex of C, say $\phi(e)$, that is not in $G[V(D \setminus \{e\})]$. Since e must be the only arc of D incident with $\phi(e)$, the mapping $\phi : D \leftrightarrow V_\phi$ then is one-to-one between some subset V_ϕ of $V(C)$ and D. Consequently $|V(C) \setminus V_\phi| = |C| - |D|$, and therefore

$$\bar{x}(V(C)) = \bar{x}(V_\phi) + \bar{x}(V(C) \setminus V_\phi) = \bar{y}(D) + \bar{x}(V(C) \setminus V_\phi) \leq \bar{y}(D) + |C| - |D| \leq |C| - 1,$$

where the last inequality follows from $\bar{y}(D) \leq |D| - 1$. So $Q^2(G)$ is contained in the projection onto the y-space of Glover and Woolsey's formulation. ■

As for the cycle (dicycle) inequalities (5), the number of cyclic-set (dicyclic-set) inequalities (9) may be exponential in the size of G. Later on, we prove that the separation problem for inequalities (9) is polynomial-time solvable by adapting the separation algorithm of [4] that is based on the characterization of the minimal cyclic (dicyclic) sets of G. So before presenting this separation algorithm, Theorem 1 below gives a characterization of the minimal cyclic sets of an undirected graph G (having at least three edges, otherwise it would be easy to obtain a characterization by enumeration). In essence, Theorem 1 states that the minimal cyclic sets of an undirected graph are as in Fig. 1, where the solid lines correspond to the edges of a cyclic set D and the dashed lines to the edges of $G[V(D)]$ not in D.

Theorem 1. *Let $G = (V, E)$ be an undirected simple graph and let be an edge subset $D \subseteq E$ with $|D| \geq 3$. Then D is an (inclusionwise) minimal cyclic set of G if and only if*

(i) *every component of the partial subgraph (V, D) is a path of length at most 2,*

and the induced subgraph $G[V(D)]$ contains a cycle C such that

(ii) *every edge in $D \setminus C$ has one vertex incident with two edges in $C \setminus D$ and the other vertex is of degree one in $G[V(D)]$,*
(iii) *every edge of $G[V(D)]$ which is not in D must be an edge of the cycle C.*

Proof. *Sufficiency:* By (ii)–(iii), cycle C is the only cycle of $G[V(D)]$. By (i)–(ii), every edge e of D has at least one vertex v in C which is incident to no other edge in D, and hence v does not belong to $G[V(D \setminus \{e\})]$. It follows that $G[V(D')]$ is a forest for every proper subset $D' \subset D$.

Necessity: Let $D = \{e_1, e_2, \ldots, e_k\}$ with $k \geq 3$ so that $G[V(D)]$ has a cycle and $G[V(D')]$ is a forest for every proper subset $D' \subset D$. It suffices to prove that $G[V(D)]$ has exactly one cycle, since the necessity of (i)–(iii) then comes straightforwardly from the minimality of D.

In order to prove that $G[V(D)]$ has exactly one cycle, we prove that, given an edge of D and two cycles of $G[V(D)]$, this edge must be a chord of at least one of the two cycles. This would lead indeed to a contradiction because, if an edge $e \in D$ were a chord of a cycle C, then C could be split into two cycles C_1, C_2 the intersection of which could be $\{e\}$; so e is a chord of neither C_1 nor C_2.

Suppose that $G[V(D)]$ contains at least two cycles C and C'. We know that for $i = 1, 2, \ldots, k$, there exists two (not necessarily distinct) vertices v_i and v_i' of e_i such that v_i belongs to C and v_i' to C'. Moreover if $i \neq j$ then $v_i \neq v_j$ and $v_i' \neq v_j'$. Suppose that e_3 is a chord neither of C nor of C'. There then is a path P of C linking v_1 and v_2 which does not contain v_3, and a path P' of C' linking v_1' and v_2' which does not contain v_3'. However this is impossible since the vertices spanned by P and P' form a cycle which contains no vertex of e_3. ■

The characterization of the minimal dicyclic sets of a digraph turns out to be more challenging, and then is postponed to the next two sections. At this point in the motivation, let us say that Theorem 2 of Section 4 holds true. Roughly speaking, a minimal dicyclic set D of a digraph resembles those in Figs. 2 and 3 of Section 3. We now are set to prove that the separation problem for the cyclic-set (dicyclic-set) inequalities (9) can be solved in polynomial-time solvable.

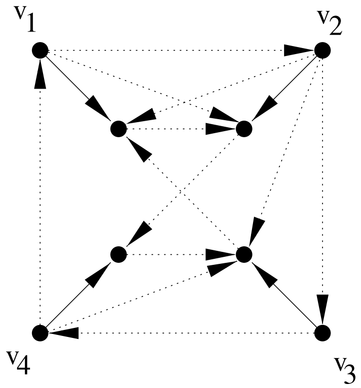

Fig. 2. Digraph induced by a minimal dicyclic set with two vertex-disjoint dicycles.

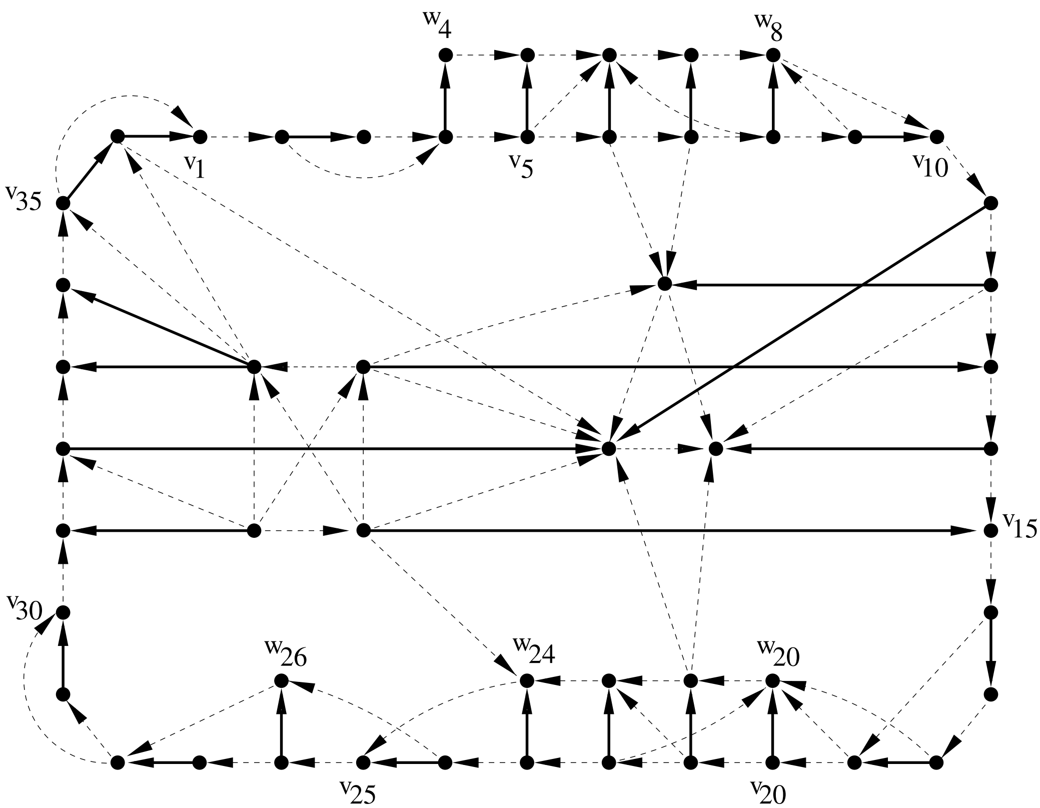

Fig. 3. Digraph induced by a minimal dicyclic set.

Proposition 5. *Let $y \in [0, 1]^E$. The separation problem for inequalities (9) can be solved in polynomial time.*

Proof. Let $z \in \mathbb{Q}_+^E$. With any vertex v of V, we associate a weight $w_v := \min\{z_e : e \in \delta(v)\}$. Given a subset C of E and a subset C' of C, a vertex of $V(C)$ is *saturated* if it belongs to $V(C')$, otherwise it is *unsaturated*. The weight

$$w(C) := \min_{C' \subseteq C} \quad z(C') + \sum_{\substack{v \in V(C) \\ v \text{ unsaturated}}} w_v$$

of C is the minimum over all $C' \subseteq C$ of the z-value of C' plus the weight of the unsaturated vertices of $V(C)$. A *minimum-weight cycle (dicycle) of* (G, z) is a cycle (dicycle) C minimizing $w(C)$.

Observe that a vector $y \in [0, 1]^E$ satisfies the inequalities (9) if and only if the vector $z = \mathbf{1} - y$ satisfies

$$z(D) \geq 1 \quad \text{for } D \in \mathcal{D}(G),$$

that is, D is a cyclic (dicyclic) set of G. Hence, separating the constraints (9) amounts to find a cyclic (dicyclic) set D minimizing its value $z(D)$. If G is undirected, it is straightforward from Theorem 1 that the minimum z-value of a cyclic set D is equal to the minimum weight $w(C)$ of a cycle C of G. If G is a digraph, Theorem 2 of Section 4 then implies that the z-value of a dicyclic set D is equal to the minimum weight $w(C)$ of a dicycle C of G.

We now first focus on the case of G being a digraph. The polynomial separation algorithm of [4] is based on the following transformation $(\tilde{G}, \tilde{z})$ of the weighted digraph (G, z) in two steps:

1. For each vertex v of G, create four vertices $v_{lu}, v_{ru}, v_{ls}, v_{rs}$ in $\tilde{G}$ together with three 0-weight arcs (v_{lu}, v_{ru}), (v_{lu}, v_{rs}) and (v_{ls}, v_{ru}), and one arc (v_{ls}, v_{rs}) with weight w_v;
2. For each arc (u, v) of G, create one 0-weight arc (u_{rs}, v_{ls}) and one arc (u_{ru}, v_{lu}) with weight $z_{(u,v)}$.

We distinguish two types of arcs in $\tilde{G}$, the *vertex arcs* which are the arcs created at step 1, and the *true arcs* created at step 2. In the subscripts of the created vertices and arcs, "l" and "r" stand for left and right, and "s" and "u" stand for saturated and unsaturated. So every vertex arc of $\tilde{G}$ goes from a left vertex to a right vertex, and every true arc goes from a right vertex to a left vertex. Observe that any dicycle $\tilde{C}$ of $\tilde{G}$ alternates between vertex arcs and true arcs which correspond to the alternated vertex and arc sequence of a dicycle C in G. In fact, it is proved in [4] that there is a one-to-one mapping between the dicycles $\tilde{C}$ of $\tilde{G}$ and the dicycles C of G, and that $\tilde{z}(\tilde{C}) = w(C)$ for any dicycle $\tilde{C}$ of $\tilde{G}$ and its corresponding dicycle C in G. Thus a minimum $\tilde{z}$-value dicycle of $(\tilde{G}, \tilde{z})$ corresponds to a minimum-weight dicycle of (G, z), and, since finding a minimum $\tilde{z}$-value dicycle is polynomial, so is the separation of (9) in the directed case.

If G is undirected, we let $(\tilde{G}, \tilde{z})$ be the weighted graph obtained from the weighted digraph on V by considering two opposite arcs (u, v) and (v, u) with value z_{uv} for each edge uv of G. In order to find a minimum-weight cycle of G we proceed as follows: for each edge uv of G, we remove both true arcs (v_{ru}, u_{lu}) and (v_{rs}, u_{ls}) in $\tilde{G}$, and we look for a minimum $\tilde{z}$-value dicycle of $\tilde{G}$ containing either the true arc (u_{ru}, v_{lu}) or the true arc (u_{rs}, v_{ls}). ■

From a strict combinatorial optimization point of view, we have identified two families of valid inequalities that strengthen Sherali-Adams linearization [12] of (QB), and we have observed that [4] implies that they can be separated in polynomial time. Moreover, we have noticed that both families of inequalities give in fact stronger formulations than the classical linearization of Glover and Woolsey [6]. From a structural point of view, we have obtained a good characterization of the cyclic set of an undirected graph. Now that the pertinence of describing the structure of the minimal dicyclic set has been motivated, the next two sections are devoted to achieving such a characterization.

3. Preliminaries, definition and examples

Throughout, an arc (u, v) will simply be denoted by uv, and vu denotes the arc (v, u).

The characterization of the minimal dicycle sets D with a small cardinality is easy, so we will assume that $|D| \geq 3$. Clearly, D is a minimal dicyclic set of G if and only if it is a minimal dicyclic set of $G[V(D)]$. Hence for the sake of conciseness, we assume that

$$\text{every vertex of } G \text{ is incident with an arc of } D. \tag{10}$$

Since D has at least three arcs, G is loopless and it has no opposite arcs $e = uv$, $e' = vu$. Also, we can assume that G has no parallel arcs $e = uv = e'$ since such arcs play the same role for our task.

Both Figs. 2 and 3 show a digraph G induced by (the vertex set of) minimal dicyclic sets D where the solid arcs correspond to the arcs in D and those in dotted lines correspond to the arcs in $E \setminus D$. The essence of our characterization is that any minimal dicyclic set falls either in the case of Fig. 2 or in that of Fig. 3.

In order to describe both cases in detail we will introduce some definitions all along this section. Most of the definitions are related to the case of Fig. 3 which is the more complicated.

Throughout the rest of the paper $C \subseteq E$ is a dicycle of G with vertex sequence $v_1, v_2, \ldots, v_n$ where $n \geq 3$ and all the vertices are distinct. Dicycles, dipaths and undirected dipaths will be considered as arc sets but described by their vertex sequence. All the vertices, in particular the extremities, of our paths and dipaths must be distinct vertices. When referring to Fig. 2, C is the dicycle $v_1, \ldots, v_4$. In Fig. 3, C is given by the vertex sequence $v_1, \ldots, v_{36}$ (the names v_i are represented in the figure for $i = 1, 5, 10, \ldots, 35$).

Let $D \subseteq E$ and suppose that

$$\text{every arc of } D \text{ is incident with a vertex of } C. \tag{11}$$

The dicycle C being given, let

- $S := \{s \in V \setminus V(C)\colon s \text{ is the tail of an arc of } D\}$,
- $T := \{t \in V \setminus V(C)\colon t \text{ is the head of an arc of } D\}$.

The dicycle C induces the *external-partition* (S, T) if (S, T) defines a partition of $V \setminus V(C)$, and if furthermore, no vertex in S belongs to a dipath linking two vertices of C. If C induces the external-partition (S, T), we let

- $W := \{w \in T\colon w \text{ is an internal vertex of a dipath linking two vertices of } C\}$.

For instance in Fig. 2, C induces indeed an external-partition with $S = W = \emptyset = C \cap D$. In Fig. 3, C induces an external-partition as well with $|S| = 4$, $|T| = 13$, $|W| = 10$, and $|T \setminus W| = 3$ (in the middle of the picture, the four vertices on the left are those of S and the three vertices on the right those of $T \setminus W$). Notice that $G[T]$ is acyclic in Fig. 3 but not in Fig. 2.

For two disjoint subsets V_1 and V_2 of V, we denote by $\delta(V_1, V_2)$ the set of all arcs of G having their tail in V_1 and their head in V_2. From these definitions and $|D| \geq 3$ we directly remark that:

$$\delta(V(C), S) = \delta(T, S) = \delta(T \setminus W, V(C)) = \delta(T \setminus W, W) = \emptyset, \tag{12}$$

and obviously $G[S \cup T]$ has no arc in D as a consequence of (11).

In the following we introduce the definition of an obstruction which allows us to describe the structure of the partial digraph $(V, C \cup D)$. A *2-dipath* is a set of two arcs forming a dipath. An *out-star* (*in-star*, respectively) is a set of at least two arcs with the same tail (head, respectively) called a *center*.

Definition 1. Let $G = (V, E)$ be a graph and $D \subseteq E$ satisfy (10)–(11) with a dicycle C inducing the external-partition (S, T) of $V \setminus V(C)$. Then (D, C) forms an *obstruction of* G if each (weak) component of the partial subgraph (V, D) is either an (out- or in-) star, a 2-dipath, or an arc so that

(a) each out-star (in-star, respectively) has its center in S (in $T \setminus W$, respectively);
(b) each 2-dipath belongs to C;
(c) no arc in D is a chord of C.

In both Figs. 2 and 3, (D, C) is an obstruction.

In the following we introduce the concepts of short-chords and out-ladders which allow us to describe how the arcs of $E \setminus (C \cup D)$ take place. Throughout the rest of the paper, given two vertices v_i and v_j of C, we denote by P_{ij} the dipath of C going from v_i to v_j. We write $i \leq i' \leq j$ whenever $v_{i'}$ belongs to P_{ij}. (Subscripts will always be reduced to modulo n.)

Definition 2. Let (D, C) be an obstruction of G. An arc $e = (v_i, v_j) \in E \setminus D$ is a *short-chord* of C if $j = i + 2$ and either (a),(b) or (c) holds, or $j = i + 3$ and (c) holds, where

(a) (v_i, v_{i+1}) is the only arc in D of $P_{i-1\,i+2}$,
(b) (v_{i+1}, v_{i+2}) is the only arc in D of $P_{i\,i+3}$,
(c) (v_i, v_{i+1}) and (v_{j-1}, v_j) are the two only arcs in D of $P_{i-1\,j+1}$.

In Fig. 3, (v_2, v_4) is a short-chord of type (a), (v_{28}, v_{30}) is a short-chord of type (b), (v_{35}, v_1) is a short-chord of type (c) with $j = i + 2$, and (v_{16}, v_{19}) is a short-chord of type (c) with $j = i + 3$.

Given an obstruction (D, C) with external-partition (S, T) and a vertex $w \in W$, then w is incident with exactly one arc of D whose tail is a vertex of C called the *mate of* w. In the following, since C has vertex sequence $v_1, \ldots, v_i, \ldots, v_n$, we denote by w_i every vertex in W with mate v_i.

Definition 3. Let (D, C) be an obstruction of G. A maximal dipath L of $G[W]$ induces an *out-ladder* if it satisfies the following conditions:

(a) its vertex sequence has the form $w_i, w_{i+1}, \ldots, w_j$,
(b) (v_{j+1}, v_{j+2}) is the only arc in D of $P_{i-1\,j+3}$,
(c) (w_j, v_{j+2}) (in $E \setminus D$) is the only arc from L to C,
(d) every arc $(v_{i'}, w_{j'})$ from C to L is so that either
 $i - 1 \leq i' \leq j + 1$ and $i \leq j' \leq i' + 1$, or
 $(v_{i-3}, v_{i-2}) \notin D$, $(v_{i-2}, v_{i-1}) \in D$, $i' = i - 2$, and $j' = i$.

In Fig. 3, the three dipaths with vertex sequence $w_4, \ldots, w_8$, vertex sequence $w_{20}, \ldots, w_{24}$, and vertex sequence w_{26} induce out-ladders.

An arc $e \in E \setminus D$ *belongs to an out-ladder* if it is an arc of L, or an arc between L and C satisfying the conditions of Definition 3. For instance, (v_5, w_6), (v_8, w_6) and (v_9, w_8) belong to the out-ladder induced by $w_4, \ldots, w_8$ as well as all arcs of the dipath $w_4, \ldots, w_8$.

4. Structure of minimal dicyclic sets

This section is devoted to proving our main theorem which gives a complete characterization of minimal dicyclic sets.

An obstruction (D, C) is called a *good obstruction* if every arc e in $E \setminus (C \cup D)$ satisfies one of the three following conditions

(a) at least one extremity of e is in $S \cup (T \setminus W)$, or
(b) e is a short-chord, or
(c) e belongs to an out-ladder.

Observe that in (a), the tail of e is in S or the head of e is in $T \setminus W$, non-exclusively. We are ready now to state and prove the main result of this paper.

Theorem 2. *Let $G = (V, E)$ be a graph and let $D \subseteq E$ with $V(D) = V$. Then D is a (inclusionwise) minimal dicyclic set of G if and only if there is a dicycle C of G with external-partition (S, T) such that (D, C) is a good obstruction satisfying either*

(i) *T induces a dicycle C' with $|C'| = |C| = |D|$ and $S = W = \emptyset$, or*
(ii) *$G[V \setminus V(C)]$ is acyclic.*

Proof. *Sufficiency:* Let $D' \subset D$ be a strict subset of D. Since (D, C) is an obstruction, then the cycle C cannot belong to $G[V(D')]$. To see the sufficiency of (i) first observe that in this case C and C' are the only dicycles of G. Since $|C'| \geq |C|$, the arcs in D must be pairwise disjoint. Thus C' is not present in $G[V(D')]$. It follows that D is a minimal dicyclic set.

To see the sufficiency of (ii), assume there exists a dicycle C' in $G[V(D')]$. First, we remark that any dicycle of G is only composed of vertices of $V(C) \cup W$. Hence, we can delete all the arcs of $E \setminus D$ which are incident with a vertex in $S \cup T \setminus W$. Similarly, we can delete all the chords of C, since we can replace (v_i, v_j) by P_{ij}, if C' contains (v_i, v_j). Now, if C' contains an arc $e \in E \setminus D$ of an out-ladder induced by a dipath L so that $e = (v_{i'}, w_{j'})$ runs from C to L, then C' must contain the sequence $v_{i'}, w_{j'}, w_{j'+1} \dots, w_j, v_{j+2}$ and the latter can be replaced by $P_{i'j+2}$. Similarly, if $e = (w_j, v_{j+2})$ is the arc from L to C or if $e = (w_{j'}, w_{j'+1})$ is in L, then it can be removed. So we can remove all the arcs in $G[W]$, and then all the arcs in $E \setminus C$. Hence $C' = C$ which is impossible.

Necessity: Assume that D is a minimal dicyclic set, so recall that

(α) $G = G[V(D)]$ contains a dicycle,
(β) for every dicycle C of G and for every arc e of D, there exists a vertex v of C incident with e and with no other arc in D.

Given a dicycle C, let $\rho(C)$ denote the number of arcs in $D \setminus C$ whose heads are vertices of C. Choose a dicycle C of G minimizing $\rho(C)$ and maximizing $|C \cap D|$, in this order of priority. Clearly, every arc in D has at least one extremity in C. Let us denote by S (T, respectively) the set of vertices in $V \setminus V(C)$ being the tail (head, respectively) of some edge in $D \setminus C$. Obviously, $V \setminus V(C) = S \cup T$.

Now let P be a dipath from v_i to v_j such that v_i and v_j are the only vertices in $V(C)$ of P. Then

$$\text{the last arc of } P \text{ cannot be in } D \quad (\text{in particular, no chord of } C \text{ is in } D). \tag{13}$$

To see that (13) holds, let C' be the dicycle formed by the arcs in $P \cup P_{ji}$. Suppose that the last arc of P is in D. Let w be the last internal vertex along the dipath P so that w is the head of an arc in D. Such a vertex w exists since otherwise $\rho(C) > \rho(C')$. Let v_k be the first vertex along the dipath P_{jj-1} being the tail of an arc (v_k, w) in D. Such an arc exists because w is a node of T. Let C'' be the dicycle formed by the arcs of P_{jk}, the arc (v_k, w) and the arcs of P from w to v_j. Since $\rho(C) \leq \rho(C'')$, there is an arc (v_l, w) in D where v_l is distinct from v_k. This contradicts the minimality of D since C'' belongs to $G[V(D \setminus \{(v_l, w)\})]$.

One consequence of (13) is that

$$C \text{ induces the external-partition } (S, T) \text{ of } V \setminus V(C). \tag{14}$$

Indeed, first, if there exists some vertex w in $S \cap T$, then there is a dipath v_i, w, v_j consisting of two arcs in D, which is impossible. Second, if there is some vertex s in S belonging to dipath $v_i, \dots, s, \dots, v_k$, then there is a dipath $v_i, \dots, s, v_j$ so that $(s, v_j) \in D$, which is impossible by (13).

Remark that if $G[S]$ has a dicycle C', then by (14), we have $\rho(C') = 0$. But since S is nonempty, $\rho(C) > 0$, which is impossible. So, one has

$$G[S] \text{ is acyclic.} \tag{15}$$

Let W is the set of vertices outside C belonging to some dipath P from v_i to v_j. By (14), $W \subseteq T$. We have also that

$$\text{each vertex of } W \text{ is the head of exactly one arc of } D. \tag{16}$$

Let w be a vertex of W and suppose for contradiction that w is the head of two arcs e_1 and e_2 in D. Since there is a dipath from w to a vertex of C, then there are two dicycles C_1, C_2 such that $e_i \in C_i$ for $i = 1, 2$. Note that, by (β) both tails of e_1 and e_2 must belong to both dicycles C_1 and C_2, which is impossible.

Now we can prove that

$$(D, C) \text{ forms an obstruction.} \tag{17}$$

To see that (17) is true, let Q be the arc-set of a (weak) component of (V, D). Clearly, the minimality of D implies that Q can contain neither a triangle nor an undirected path with more than two arcs. If Q has a path u, v, w, then by (β), both u and w are in $V(C)$. If the path u, v, w is in fact a dipath, by (13), the internal vertex v must belong to C. Moreover since C has no chord, no arc in D can be incident with v. So Q is either an arc, a dipath belonging to C, an out-star, or an in-star. (Recall that stars have at least two arcs). If Q is an out-star, its center is clearly in S. By (16), no in-star S has its center in W. Thus (17) holds.

Let us prove the following lemma which will be useful.

Lemma 1. *Given a dipath P from v_i to v_j, such that v_i and v_j are the only vertices in $V(C)$ of P, and an arc $e \in D$, then*

(a) *if e is in P_{ij}, then it is incident with either v_i or v_j;*

(b) *if e is incident with one internal vertex of P_{ij}, then its other extremity is in P (its head);*
(c) *the vertex v_i (v_j, respectively) is incident with at most one arc in D;*
(d) *if e is in P, then it is the first arc of P and P has no other arc in D, moreover, (v_{j-1}, v_j) is the only arc in D of $P_{i-1,j+1}$.*

Proof. If (a) does not hold, then e is incident with no vertex of the dicycle C' formed by the arcs in $P \cup P_{ji}$; this contradicts (β). If (b) does not hold, since e is not a chord of C, then again e is disjoint from C'; impossible. So (a) and (b) hold. Assume now that (c) does not hold. Since v_i is incident with two arcs in D, then by (17), (v_{i-1}, v_i) and (v_i, v_{i+1}) both belong to D. Now, by (β), v_{i+1} must be in C', hence $j = i + 1$. It follows that P has at least one internal vertex w. So there is an arc $(v_k, w) \in D$, where $v_k \in V(C)$. Clearly, v_k is distinct from v_{i-1}, v_i, v_{i+1}. Let P' be the subdipath of P from w to v_j. Yet v_{i-1} is not in the dicycle created with P', P_{jk} and (v_k, w); one has a contradiction with (β). Thus (c) holds. Since P has no internal arc in D, then (13) implies that the only possible arc in D of P is the first arc. Suppose that the first arc of P is in D. Let w be the last internal vertex of P and let C' be the dicycle formed by the arc $(v_{i'}, w) \in D$, the arc $(w, v_j) \in E \setminus D$, and $P_{ji'}$. Since $\rho(C') = \rho(C)$, then $P_{i'j}$ has at least one arc in D since otherwise $|C \cap D| < |C' \cap D|$. Since (a) and (c) hold, it follows that (v_{j-1}, v_j) is the only arc in D of $P_{i-1,j+1}$. So (d) holds and the proof of the lemma is complete. □

Remark that in Lemma 1, the arc $e \in D$ may be incident with one internal vertex of P and having no extremity in P_{ij} for instance $P = v_8, w_6, w_7, w_8, v_{10}$ and $e = (v_7, w_7)$ in Fig. 3. Of course, such a situation occurs only if $|C' \cap D| < |C \cap D|$.

Applying Lemma 1.(a)–(c) to a path P which is a chord of C (in $E \setminus D$), one directly obtains that

$$\text{every chord of } C \text{ is a short-chord.} \tag{18}$$

A consequence of Lemma 1(d) is that

$$\text{if } W \neq \emptyset, \quad \text{then } |C \cap D| \geq 1. \tag{19}$$

Indeed, if there is a vertex $w \in W$, then there is a dipath $v_i, w, \ldots, v_j$. Since the first arc is in D, Lemma 1.(d) implies that $C \cap D$ is non-empty.

Proposition 6. *If G has a dicycle C' vertex-disjoint from C, then Case* (i) *of the theorem holds.*

Proof. By (β), every arc e of D has one extremity in C, the other in C', and all the arcs of D are pairwise disjoint. So $|D| \geq 3$ and, more generally, any two vertex-disjoint dicycles have necessarily the same cardinality. Therefore, as a chord creates a smaller dicycle, C and C' are chordless. Since we know that $C \cap D = \emptyset$, (19) implies $W = \emptyset$. If neither S nor T is empty, then there are two arcs of D, one from C to C' and the other from C' to C. But it creates a dicycle C'' having no vertex incident with arcs in D; a contradiction with (β). So either S or T is empty, and by (15), $S = \emptyset$. □

Finally the proof of necessity follows from this last proposition.

Proposition 7. *Any maximal dipath of $G[W]$ induces an out-ladder. In particular, the digraph $G[W]$ is a union of vertex-disjoint dipaths.*

Proof. Let $w_{i_0}, w_{i_1}, \ldots, w_{i_{|L|}}$ be the sequence of vertices of a maximal-length path L in $G[W]$. Since by (16), there is a unique arc $e_{i_l} \in D$ whose head is w_{i_l}, we let v_{i_l} be the tail of e_{i_l}, that is in other words, the mate of w_{i_l}.

Let $j := i_{|L|}$. Since $w_j \in W$, there is an arc (w_j, v_k) in $E \setminus D$. By Lemma 1.(d) with the dipath v_j, w_j, v_k, then (v_{k-1}, v_k) is the only arc in D of P_{j-1k+1}. By Lemma 1.(b), v_{k-1} is the only internal vertex of P_{jk}, so $k = j + 2$. The same argument can be applied to the dipath $v_{i_{|L|-1}}, w_{i_{|L|-1}}, w_j, v_k$ in order to prove that $i_{|L|-1} = j - 1$. Then it follows by induction that $i_{|L|-l} = j - l$ for $l = 0, \ldots, |L|$. So the vertex sequence of L has the form $w_i, w_{i+1}, \ldots, w_j$, and, moreover, (v_{j+1}, v_{j+2}) is the only arc in D of P_{i-1j+3}. So Definition 3.(a)–(b) holds.

Suppose that $v_{j'}$ is the head of an arc in $E \setminus D$ whose tail is a vertex $w_{i'}$ of L. Consider the dipath $v_{i'}, w_{i'}, v_{j'}$. By Lemma 1.(d), $j' = j + 2$. By Lemma 1.(b), $i' = j$. So Definition 3.(c) holds.

Let $(v_{i'}, w_{j'}) \in E \setminus D$ be an arc from C to L and let P be the dipath $v_{i'}, w_{j'}, w_{j'+1} \ldots, w_j, v_{j+2}$. Note that $j + 1 \leq j + 2 \leq i$. Suppose first that $i - 1 \leq i' \leq j + 1$ and that if $i' = j + 1$, then $j + 1 \neq i - 2$. So, if $i' + 2 \leq j' \leq j$, then Lemma 1.(b) is contradicted by the internal vertex $v_{i'+1}$ of P. Hence we can assume that $j + 1 \leq i' \leq i - 1$. Since all internal vertices of P have their mate in $P_{j'j}$, then, by Lemma 1.(a)–(b), $P_{i'j+2}$ can have neither internal arcs in D, nor internal vertices outside $P_{j'j}$. It follows that $j' = i$. Moreover, $i' = i - 2$ and $(v_{i-1}, v_i) \in D$. Hence, $(v_{i-2}, v_{i-1}) \notin D$ by Lemma 1.(c). □

The proof of Theorem 2 then is complete. ■

References

[1] W. Adams, H. Sherali, A tight linearization and an algorithm for zero-one quadratic programming problems, Manage. Sci. 32 (10) (1986) 1274–1290.
[2] E. Balas, S. Ceria, G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed 0–1 programs, Math. Program. 58 (3) (1993) 295–324.
[3] D. Cornaz, J. Fonlupt, Chromatic characterization of biclique covers, Discrete Math. 306 (2006) 495–507.
[4] D. Cornaz, A.R. Mahjoub, The maximum induced bipartite subgraph problem with edge weights, SIAM J. Discrete Math. 21 (2007) 662–675.
[5] F. Glover, Improved linear integer programming formulations of nonlinear integer problems, Manage. Sci. 22 (1975) 455–460.
[6] F. Glover, E. Woolsey, Converting the 0–1 polynomial programming problem to a 0–1 linear program, Oper. Res. 22 (1974) 180–182.
[7] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 169–197.

[8] J.-B. Lasserre, An explicit exact SDP relaxation for nonlinear 0–1 programs, in: Integer Programming and Combinatorial Optimization, Utrecht, 2001, in: LNCS, vol. 2081, Springer, Berlin, 2001, pp. 293–303.
[9] M. Laurent, A comparison of the Sherali–Adams, Lovász–Schrijver, and Lasserre relaxations for 0–1 programming, Math. Oper. Res. 28 (3) (2003) 470–496.
[10] L. Lovász, A. Schrijver, Cones of matrices and set-functions and 0–1 optimization, SIAM J. Optim. 1 (2) (1991) 166–190.
[11] A. Schrijver, Combinatorial Optimization: Polyhedral and Efficiency, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
[12] H. Sherali, W. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero–one programming problems, SIAM J. Discrete Math. 3 (1990) 411–430.