SIAM J. DISCRETE MATH.
Vol. 21, No. 3, pp. 662–675

THE MAXIMUM INDUCED BIPARTITE SUBGRAPH PROBLEM WITH EDGE WEIGHTS*

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Abstract. Given a graph $G = (V, E)$ with nonnegative weights on the edges, the maximum induced bipartite subgraph problem (MIBSP) is to find a maximum weight bipartite subgraph $(W, E[W])$ of G. Here $E[W]$ is the edge set induced by W. An edge subset $F \subseteq E$ is called independent if there is an induced bipartite subgraph of G whose edge set contains F. Otherwise, it is called dependent. In this paper we characterize the minimal dependent sets, that is, the dependent sets that are not contained in any other dependent set. Using this, we give an integer linear programming formulation for MIBSP in the natural variable space, based on an associated class of valid inequalities called dependent set inequalities. Moreover, we show that the minimum dependent set problem with nonnegative weights can be reduced to the minimum circuit problem in a directed graph, and can then be solved in polynomial time. This yields a polynomial-time separation algorithm for the dependent set inequalities as well as a polynomial-time cutting plane algorithm for solving the linear relaxation of the problem. We also discuss some polyhedral consequences.

Key words. induced bipartite subgraph, edge weight, minimal dependent set, separation algorithm, polytope

AMS subject classifications. 05C85, 90C27, 90C57

DOI. 10.1137/060650015

1. Introduction. A graph is called *bipartite* if its node set can be partitioned into two nonempty sets V_1 and V_2 such that no two nodes in V_i are linked by an edge, for $i = 1, 2$. Let $G = (V, E)$ be a graph. A subgraph (W, F) of G is said to be *induced* if F is the set of edges having both endnodes in W. Given w, a function that associates with each edge $e \in E$ a nonnegative weight $w(e)$, the *maximum induced bipartite subgraph problem* (MIBSP) is to find an induced bipartite subgraph with maximum weight.

An edge subset $F \subseteq E$ is called *independent* if there is an induced bipartite subgraph of G with edge set $B \subseteq E$ such that $F \subseteq B$. Otherwise, it is called *dependent*. In this paper we characterize the minimal dependent sets of a graph $G = (V, E)$. Using this, we give a 0-1 linear programming formulation for MIBSP in the natural variable space, based on an associated class of valid inequalities called dependent set inequalities. We also show that the minimum dependent set problem with nonnegative weights can be reduced to the minimum circuit problem in a directed graph and can then be solved in polynomial time. This yields a polynomial-time separation algorithm for the dependent set inequalities as well as a polynomial-time cutting plane algorithm for solving the linear relaxation of the problem.

To the best of our knowledge, MIBSP has not been considered before in the literature. However, the maximum bipartite subgraph problem has been extensively investigated. Here, given a graph $G = (V, E)$ and weights on the edges of G, the

*Received by the editors January 15, 2006; accepted for publication (in revised form) March 26, 2007; published electronically August 29, 2007.
http://www.siam.org/journals/sidma/21-3/65001.html

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problem is to find a bipartite subgraph (not necessarily induced) with maximum weight. In [3] Barahona, Grötschel, and Mahjoub describe several classes of facet defining inequalities of the associated bipartite subgraph polytope. They also present some methods with which new facet defining inequalities of that polytope can be constructed from known ones.

A graph is said to be weakly *bipartite* if the bipartite subgraph polytope coincides with the polytope given by the trivial inequalities and the so-called odd cycle inequalities. Grötschel and Pulleyblank [18] showed that the bipartite subgraph problem can be solved in polynomial time in that class of graphs. Barahona showed that planar graphs [1] and graphs G that contain two nodes which cover all the odd-cycles of G [2] belong to that class of graphs. In [10] Fonlupt, Mahjoub, and Uhry generalize these results by showing that the graphs noncontractible to K_5 are weakly bipartite. Recently Guenin [19] gave a characterization for that class of graphs.

The closely related MIBSP with node weights has also been studied. Here we suppose that the weights are associated with the nodes of the graph, and the problem is to determine an induced bipartite subgraph with maximum weight. This problem has applications to the via-minimization problem which arises in the design of integrated circuits and printed circuit boards [6], [11]. In [4] Barahona and Mahjoub study the polytope $\mathrm{BP}(G)$ associated with this problem. They exhibit some basic classes of facet defining inequalities for $\mathrm{BP}(G)$ and describe several lifting methods. In [5] they study a composition technique for $\mathrm{BP}(G)$ in the graphs which are decomposable by one- and two-node cutsets. Fouilhoux and Mahjoub [12] (see also Fouilhoux [11]) study the polytope $\mathrm{BP}(G)$. They describe new classes of facet defining inequalities and discuss separation procedures. Using this, they develop a branch-and-cut algorithm for the problem and present some computational results. In [13] Fouilhoux and Mahjoub consider the via-mimization problem and show that this can be reduced to the MIBSP with appropriate node weights. Further applications of the MIBSP with node weights to the via-minimization problem and DNA sequencing are also discussed in [11].

A related work has been done by Cornaz and Fonlupt [7] on the maximum biclique problem. (A biclique is the edge set of a complete bipartite (not necessarily induced) subgraph). Although the MIBSP and the maximum biclique problem are different, this paper gives rise to some structural relations between the minimal dependent sets associated to both problems.

The paper is organized as follows. In the following section we give some notation, definitions, and preliminary results. In section 3 we study the dependent sets and give a characterization for these sets. In section 4 we show that the minimum dependent set problem with nonnegative weights can be reduced to the minimum odd circuit problem and can then be solved in polynomial time. In section 5 we discuss some polyhedral consequences and give some concluding remarks.

2. Definitions, notation, and preliminary results.

2.1. Definitions and notation. Throughout the paper we consider only simple graphs and digraphs. We will denote a graph by $G = (V, E)$, where V is the *node set* and E is the *edge set*. An edge with endnodes u and v will be denoted by uv. For $W \subseteq V$, we let $E[W]$ denote the set of edges having both nodes in W. The graph $G[W] = (W, E[W])$ is the subgraph of G induced by W. If $F \subseteq E$, we let $V(F)$ denote the set of nodes incident to edges of F, and $G(F) = (V(F), F)$. Note that $G(F) = G[V(F)]$ holds if and only if $G(F)$ is an induced subgraph of G.

We denote a directed graph (or digraph) by $D = (V, A)$, where V is the node set and A the arc set of D. An arc with initial node u and terminal node v will be denoted by uv. (Note that $uv \neq vu$ for digraphs.)

A path in G (resp., D) is an alternate sequence of nodes and edges (resp., arcs) $P = v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $k \geq 1$, all the nodes v_i are distinct, and $e_i = v_i v_{i+1} \in E$ (resp., $e_i = v_i v_{i+1} \in A$) for $i = 1, 2, \ldots, k$. The nodes v_1 and v_{k+1} are the *extremities* of P, and we will say that P links v_1 and v_{k+1} (resp., v_1 to v_{k+1}). The integer k is called the *length* of P, and P is said to be *even* (*odd*) if k is even (odd). If $v_1 = v_{k+1}$, P is called a *cycle* (resp., *circuit*). An edge linking two nonconsecutive nodes of a path (cycle) P is called a *chord* of P. A chordless cycle is also called a *hole*. Given a path P, we let $E(P)$ (resp., $A(P)$) and $V(P)$ denote the sets of edges (resp., arcs) and nodes of P, respectively.

Given a vector $x \in \mathbb{R}^E$ and $T \subseteq E$, we let $x(T)$ denote $\sum_{e \in T} x(e)$. Bipartite graphs have the following property.

REMARK 2.1. *A graph is bipartite if and only if it does not contain an odd cycle.*

2.2. Signed digraphs. A *signed* digraph consists of a digraph $D = (V, A)$ and a subset $\Sigma \subseteq A$ of arcs called *signed arcs*. The arcs in $A \setminus \Sigma$ are said to be *unsigned*. Given a signed digraph $D = (V, A)$, a circuit is said to be *odd* if it contains an odd number of signed arcs. Note that if $\omega \in \mathbb{R}^A$ is a weight vector, then finding a minimum weight odd circuit in D reduces to finding a minimum weight odd circuit in an unsigned digraph. In fact, for this, it suffices to replace every unsigned arc $uv \in A \setminus \Sigma$ by a path u, uw, w, wv, v, where w is a new node, and associate to the new arcs uw, wv the weight $\frac{\omega(uv)}{2}$. Moreover, finding a minimum weight odd circuit in a digraph reduces to a shortest path problem [17] (see also [16]). As the weights are nonnegative, it can then be solved in polynomial time, using, for instance, Dijkstra's algorithm [9].

2.3. Independent sets. Given a graph $G = (V, E)$, we let $\mathcal{B}(G)$ denote the set of the edge sets of the induced bipartite subgraphs of G, i.e.,

$$\mathcal{B}(G) = \{B \subseteq E : \ G(B) = G[V(B)] \text{ and } G(B) \text{ is bipartite}\}.$$

Hence the MIBSP is equivalent to

$$\text{maximize } \{\omega(B) : B \in \mathcal{B}(G)\}.$$

Given a graph $G = (V, E)$, a node subset $W \subseteq V$ is called a *stable set* if $E[W] = \emptyset$. The *stable set problem* in G consists in finding a stable set of maximum cardinality. Note that the stable set problem can be reduced to the MIBSP. In fact, consider the graph $\bar{G} = (\bar{V}, \bar{E})$ obtained from G by adding a universal node (a node adjacent to all the other nodes of G) and associate with the edges of $\bar{E}$ the weight $\omega(e) = 1$ if $e \in \bar{E} \setminus E$ and $\omega(e) = 0$ if not. It is easy to see that an optimum solution of the MIBSP in $\bar{G}$ with respect to weight vector ω corresponds to a maximum cardinality stable set in G. This implies that the MIBSP is NP-hard. The maximum cardinality MIBSP is to find a set in $\mathcal{B}(G)$ with maximum cardinality. In what follows we shall show that the maximum cardinality MIBSP is also NP-hard, which implies that MIBSP is strongly NP-hard.

PROPOSITION 2.2. *The maximum cardinality MIBSP is NP-hard.*

Proof. We show that the stable set problem in a graph $G = (V, E)$ reduces to the maximum cardinality MIBSP. As the former problem is NP-hard [14], the latter is also NP-hard.

Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the graph obtained from $G = (V, E)$ by considering a copy $G' = (V', E')$ of G and adding all the possible edges between V and V', that is,

$\tilde{V} = V \cup V'$ and $\tilde{E} = E \cup E' \cup \{vv' : v \in V,\ v' \in V'\}$. Note that for every stable set $S \subseteq V$ of G and its copy $S' \subseteq V'$ of G', $\tilde{G}[S \cup S']$ is a complete bipartite graph. Thus for every maximum cardinality solution $B \in \mathcal{B}(\tilde{G})$ and every stable set $S \subseteq V$ of G, we have that $|B| \geq |S|^2$. In particular $|B| \geq |S^*|^2$, where S^* is a maximum stable set of G.

Now let $B \in \mathcal{B}(\tilde{G})$ be of maximum cardinality and $\tilde{S} \subseteq \tilde{V}$ a maximum cardinality stable set of $\tilde{G}$ such that every node $v \in \tilde{S}$ is incident to an edge in B. Obviously, either $\tilde{S} \subseteq V$ or $\tilde{S} \subseteq V'$. Thus $|\tilde{S}| \leq |S^*|$. As $|B| \leq |\tilde{S}|^2$, it follows that $|B| \leq |S^*|^2$. Consequently, $|B| = |\tilde{S}|^2 = |S^*|^2$. □

Denote by $\mathcal{I}(G)$ the set of the independent sets of G, i.e.,

$$\mathcal{I}(G) = \{I \subseteq E :\ \exists\ B \in \mathcal{B}(G),\ I \subseteq B\}.$$

Obviously, $\mathcal{B}(G) \subseteq \mathcal{I}(G)$. However, in general we have that $\mathcal{B}(G)$ is a strict subset of $\mathcal{I}(G)$. For instance, consider the graph G consisting of a path of three edges (e_1, e_2, e_3). Clearly, the edge subset $I = \{e_1, e_3\}$ is independent. However, $G(I) \neq G[V(I)]$, and hence $I \notin \mathcal{B}(G)$. Also note that, since the weights are nonnegative,

$$\max\{\omega(\mathcal{I})\ :\ \mathcal{I} \in \mathcal{I}(G)\} = \max\{\omega(B)\ :\ B \in \mathcal{B}(G)\}.$$

Therefore the MIBSP is equivalent to finding a maximum weight independent set in G. Moreover, we have the following which is a direct consequence of Remark 2.1.

Lemma 2.3. *Given an edge set $I \subseteq E$, $I \in \mathcal{I}(G)$ if and only if $G[V(I)]$ contains no odd cycle.*

In what follows we will denote by $\mathcal{C}(G)$ the set of the minimal dependent sets of G, i.e.,

$$\mathcal{C}(G) = \{C \subseteq E :\ C \notin \mathcal{I}(G) \text{ and } C' \in \mathcal{I}(G)\ \forall\ C' \subset C\}.$$

We have the following.

Lemma 2.4. *Given an edge set $C \subseteq E$, $C \in \mathcal{C}(G)$ if and only if*

(i) *there exists at least one odd cycle in $G[V(C)]$, and*

(ii) *for every odd cycle Q of $G[V(C)]$ and every edge $f \in C$, there exists a node $v_f \in V(Q)$ such that f is the unique edge of C incident to v_f.*

Proof.

Necessity.

(i) This follows from Lemma 2.3.

(ii) Let Q be an odd cycle of $G[V(C)]$ and f an edge of $C \in \mathcal{C}(G)$. If the statement does not hold, it is not hard to see that $V(Q) \subseteq V(C \setminus \{f\})$. But this implies that $C \setminus \{f\}$ is dependent, contradicting the minimality of C.

Sufficiency. By Lemmas 2.4(i) and 2.3, we have that $C \notin \mathcal{I}(G)$. Now suppose that C is not minimal. Then there exists an edge $f = uv \in C$ such that $C' = C \setminus \{f\} \notin \mathcal{I}(G)$. By Lemma 2.3, this implies that $G[V(C')]$ contains an odd cycle, say Q, and hence $V(Q) \subseteq V(C')$. Moreover, by Lemma 2.4(ii), it follows that one of the nodes of f, say v, belongs to $V(Q)$ and is not incident to any edge in C'. But this implies that $v \in V(Q) \setminus V(C')$, a contradiction. □

Figure 1 shows a subgraph which is induced by the node set of a minimal dependent set. The dependent set is presented by bold lines. We can remark that the subgraph contains an odd cycle, and that for every edge f of the dependent set, there is a node of the cycle such that f is the only edge incident to it.

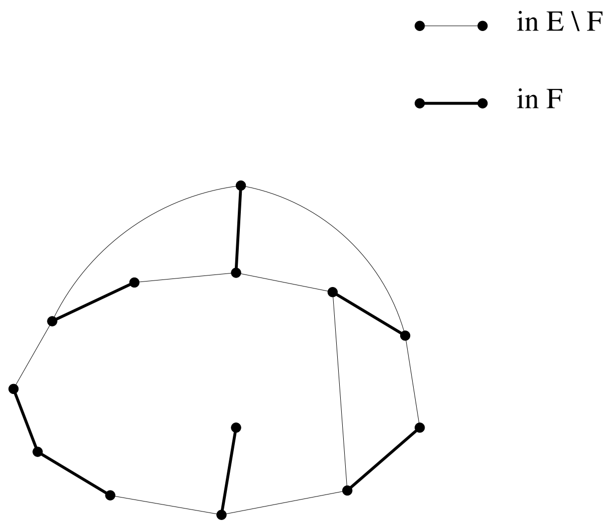

FIG. 1. *A subgraph induced by a minimal dependent set.*

3. Minimal dependent sets. The characterization of the minimal dependent sets, given by Lemma 2.4, is not strong enough to obtain certain polyhedral results for the MIBSP which we will present in the next sections. In this section we give a stronger characterization of the minimal dependent sets. This will be given in Theorem 3.2. To this end, we first introduce some definitions.

Let $F \subseteq E$ and Q be an odd cycle of G. A node $v \in V(Q)$ is said to be *unsaturated* with respect to F and Q if v is not incident to any edge of $F \cap E(Q)$; otherwise v is said to be *saturated*.

DEFINITION 3.1. *Given an edge set $F \subseteq E$, we say that F induces an obstruction with respect to an odd cycle Q if Q is an odd cycle of $G[V(F)]$ and if conditions* (1) *and* (2) *below are satisfied.*

(1) *Every edge $f \in F \setminus E(Q)$ is of the form $f = vw$ where $v \in V(Q)$, $w \in V \setminus V(Q)$, and there is no edge in $F \setminus \{f\}$ adjacent to f.*

(2) *Every edge in $F \cap E(Q)$ is adjacent to at most one edge of F.*

Figure 2 shows an obstruction induced by an edge set F with respect to the odd cycle on seven edges. We can remark here that F does not correspond to a minimal dependent set. The edges e, f induce a minimal dependent set.

Let $F \subseteq E$ be an edge set. And suppose that F induces an obstruction with respect to an odd cycle

$$Q = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

of $G[V(F)]$, where $v_{k+1} = v_1$. Let $e \in E[V(F)] \setminus (F \cup E(Q))$. Edge e is called a *diagonal* (with respect to F and Q) if there is $i \in \{1, \ldots, k\}$ such that $e = v_i v_{i+3}$, the edges e_i, e_{i+2} are in F, and the edges $e_{i-1}, e_{i+1}, e_{i+3}$ are in $E \setminus F$ (the indices are taken modulo k). And edge e is called a *forward* (resp., *backward*) *wing* (with respect to F and Q) if there is $i \in \{1, \ldots, k\}$ and a node $w \in V \setminus V(Q)$ such that $e = wv_i$ with $e_i, wv_{i+2} \in F$ (resp., $wv_{i-2}, e_{i-1} \in F$), and $e_{i-1}, e_{i+1}, e_{i+2} \in E \setminus F$ (resp., $e_{i-3}, e_{i-2}, e_i \in E \setminus F$). An edge is called a *wing* if it is either a forward or a backward wing.

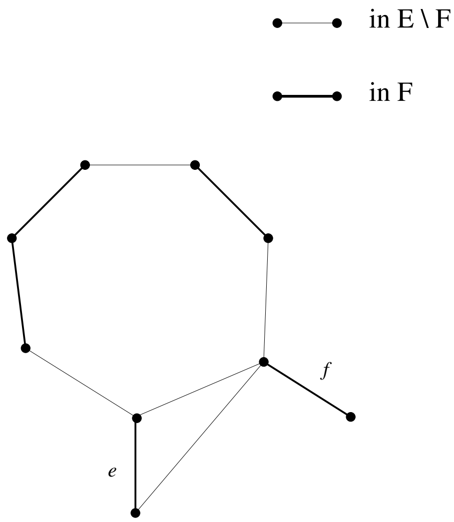

FIG. 2. *An obstruction induced by F.*

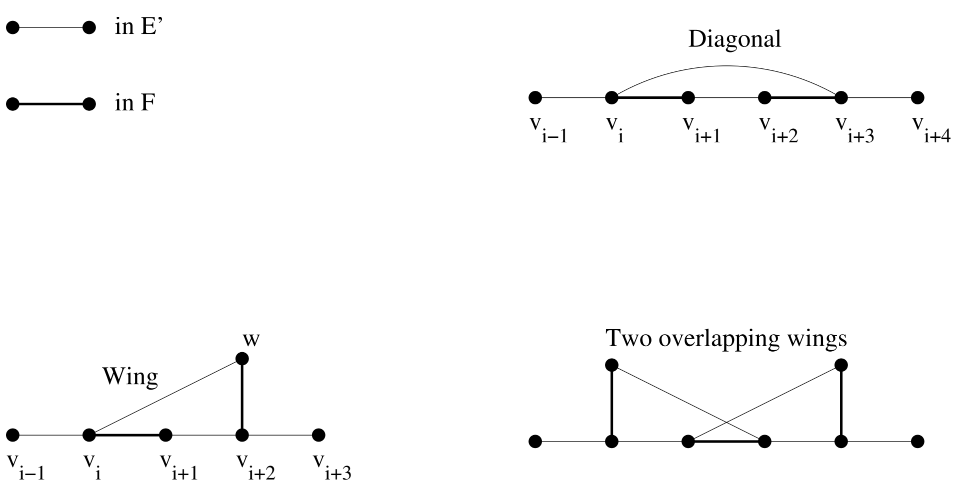

FIG. 3. *Wings and diagonals.*

We say that two wings wv_i and $w'v_j$ *overlap* if $v_iv_j \in F$. Note that if two wings overlap, then necessarily one is forward and the other is backward (see Figure 3 for an illustration).

The following theorem gives a characterization of the set of minimal dependent sets of G.

THEOREM 3.2. *Let $G = (V, E)$ be a graph and $F \subseteq E$ an edge subset of E. Then F is a minimal dependent set if and only if F induces an obstruction with respect to an odd cycle Q such that*

(i) *every edge of $E[V(F)] \setminus (F \cup E(Q))$ is either a diagonal or a wing, and*
(ii) *no wings overlap.*

Proof.

Necessity. Suppose $F \in \mathcal{C}(G)$. Let $W = V(F)$. By Lemma 2.3, $G[W]$ contains an odd cycle. Let

$$Q = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1},$$

with $v_{k+1} = v_1$, be an odd cycle of $G[W]$ such that $|F \cap E(Q)|$ is maximum. Let $F_1 = F \cap E(Q)$, $F_2 = F \setminus F_1$, and $E' = E[W] \setminus (F \cup E(Q))$. Note that by Lemma 2.4(ii), no edge of F_1 is adjacent to an edge of F_2.

For the rest of the proof, we will need to consider paths of $G[W]$ with extremities in $V(Q)$ and internal nodes in $W \setminus V(Q)$. If P is a path of $G[W]$ linking two nodes v_i and v_j of $V(Q)$ such that $E(P) \cap E(Q) = \emptyset$ and none of the internal nodes of P belongs to $V(Q)$, we let $P_1 = v_j, e_{j+1}, \ldots, e_{i-1}, v_i$ and $P_2 = v_i, e_i, \ldots, e_{j-1}, v_j$ (where the indices are modulo k) denote the edge-disjoint paths of Q between v_i and v_j. Note that $Q = P_1 \cup P_2$. We also denote by $Q_1 = P \cup P_1$ and $Q_2 = P \cup P_2$ the cycles obtained by adding P to Q. Note that Q_1 and Q_2 are of opposite parity. We will suppose, without loss of generality, that Q_1 is odd and Q_2 is even.

By Lemma 2.4(ii), for every edge $f \in F$, there exists a node $v_f \in V(Q)$ such that f is the only edge of F incident to v_f. By a similar argument, there is also a node $v_f^1 \in V(Q_1)$ such that f is the only edge of F incident to v_f^1 (v_f^1 and v_f may be the same). We have the following claims.

Claim 1. *Let P be a path of $G[W]$ linking two nodes v_i and v_j of $V(Q)$ whose internal nodes belong to $W \setminus V(Q)$. Let $f_2, f_2' \in F_2$ and $e \in E'$. Then the following cases cannot occur:*

(a) *$P = v_i, f_2, v_j$,*

(b) *$P = v_i, f_2, w, f_2', v_j$ with $w \in W \setminus V(Q)$, or*

(c) *$P = v_i, f_2, w, e, w', f_2', v_j$ with $w, w' \in W \setminus V(Q)$.*

Proof. Assume that P is of type (a), (b) or (c). Notice that if $f \in F \cap E(P_2)$, then $v_f^1 = v_i$ or v_j. Also note that v_i and v_j are both incident to an edge of $F \setminus E(P_2)$. Then it follows that $F \cap E(P_2) = \emptyset$. Since $|F \cap E(P)| \geq 1$, $|F \cap E(Q)| < |F \cap E(Q_1)|$. But this contradicts the maximality of $|F \cap E(Q)|$. ∎

Claim 2. *F induces an obstruction with respect to Q.*

Proof. Let $f = wv_f \in F \setminus E(Q)$. (Recall that v_f is the node of $V(Q)$ such that f is the only edge of F incident to it.) If $w \in V(Q)$, then w, f, v_f is a path of $G[W]$. But this is impossible by Claim 1(a). So suppose that $w \notin V(Q)$. If there is an edge $f' = wv_{f'} \in F$, then $P = v_f, f, w, f', v_{f'}$ is a path of $G[W]$, contradicting Claim 1(b). In consequence, f is adjacent to no edge in F, and thus condition (1) of Definition 3.1 is satisfied.

Now let $f = uv_f$ be an edge of $F \cap E(Q)$. Since F satisfies condition (1) of Definition 3.1, f is adjacent to no edge in F_2. Moreover, we have that v_f is incident to no edge in F. Hence f is adjacent to at most one edge of $F \cap E(Q)$. Therefore condition (2) of Definition 3.1 is satisfied. ∎

Claim 3. *Every edge of E' is incident to a node of Q.*

Proof. Suppose that for an edge $e = ww'$ of E', we have $\{w, w'\} \cap V(Q) = \emptyset$. Since $w, w' \in W \setminus V(Q)$, there exist two edges $f = wv_f$ and $f' = w'v_{f'}$ of F. Note that $v_f \neq v_{f'}$. Hence $v_f, f, w, e, w', f', v_{f'}$ is a path of $G[W]$, which contradicts Claim 1(c). ∎

Claim 4. *Let $e \in E'$, $f \in F$, and $v_i, v_j \in V(Q)$.*

(1) *If $P = v_i, e, w, f, v_j$ is a path of $G[W]$, then e is a forward wing.*

(2) *If $P = v_i, f, w, e, v_j$ is a path of $G[W]$, then e is a backward wing.*

Proof. We prove (1), the proof of (2) is similar. As Q_2 is even, the path P_2 must contain an even number of edges, and hence $|E(P_2)| \geq 2$. Moreover, as by Claim 2 F

induces an obstruction with respect to Q, f is the only edge of F incident to v_j (w). In consequence, v_j is unsaturated. Moreover, v_j is the unique unsaturated node of P_2. Indeed, if v was a further unsaturated node of P_2, then there must exist an edge, say f', of F_2 incident to v. As by Claim 2 F induces an obstruction with respect to Q, $v^1_{f'} \in V \setminus V(Q)$. (Recall that $v^1_{f'}$ is the node of $V(Q_1)$ such that f' is the only edge of F incident to it.) Thus $v^1_{f'} = w$. But this is impossible since f is (the only edge of F) incident to w. In consequence, e_i is the unique edge of F in P_2 and thus $|E(P_2)| = 2$. Therefore e_{i-1}, e_{j-1}, e_j are not in F, and hence e is a forward wing. ◻

CLAIM 5. *Let $e \in E'$. If $P = v_i, e, v_j$ is a path of $G[W]$ with $v_i, v_j \in V(Q)$, then e is a diagonal.*

Proof. As $|P|$ is odd and Q_2 is even, P_2 must be odd, and therefore $|E(P_2)| \geq 3$. Also P_2 contains no unsaturated nodes. In fact, if P_2 contains an unsaturated node v, then there must exist $f \in F$ incident to v such that $v^1_f \in V(P_1)$. But this contradicts Claim 1(a). In consequence, two consecutive edges of $E(P_2)$ cannot both be in E'. From Lemma 2.4(ii), it then follows that e_i and e_{j-1} are the only edges of F in $E(P_2)$ and that $|E(P_2)| = 3$. We also have that e_{i-1} and e_j are not in F, $j = i + 3$, and $e_{i+1} \notin F$. Thus e is a diagonal. ◻

CLAIM 6. *No wings overlap.*

Proof. Suppose that there are two wings $e = wv_{i+1}$ and $e' = w'v_i$ that overlap. Note that $w, w' \in W \setminus V(Q)$, $w \neq w'$, and $e_i = v_iv_{i+1} \in F$. The path P' with edge set $E(P') = \{wv_{i-1}, e, e_i, e', w'v_{i+2}\}$ has three edges in F. And the path P'' of Q with edge set $E(P'') = \{e_{i-1}, e_i, e_{i+1}\}$ has only one edge in F. Note that both P' and P'' are odd. Hence the cycle $\tilde{Q}$ obtained from Q by replacing P'' by P' is odd. As $V(\tilde{Q}) \subseteq W$ and $|E(\tilde{Q}) \cap F| > |E(Q) \cap F|$, we have a contradiction. ◻

By Claim 2, F induces an obstruction with respect to Q. If $e \in E'$, then by Claim 3, e belongs to a path P of the form either v_i, e, w, f, v_j or v_i, f, w, e, v_j or v_i, e, v_j with $v_i, v_j \in V(Q)$, $f \in F$, and $w \in W \setminus V(Q)$. It then follows by Claims 4 and 5 that e is either a wing or a diagonal. Moreover, by Claim 6, no wings overlap.

Sufficiency. Suppose that F induces an obstruction with respect to an odd cycle Q satisfying (i) and (ii). By Lemma 2.3, F is a dependent set. We will show that $F' = F \setminus \{f\}$ is an independent set for every $f \in F$. Let $W' = V(F')$ and let us set as before $Q = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ with $v_1 = v_{k+1}$. First remark that if f is an edge of $E(Q)$, as F induces an obstruction with respect to Q, at most one edge of F is adjacent to f. Thus Q cannot be a subgraph of $G[W']$. If f links a node not in $V(Q)$ to an unsaturated node, say v, of $V(Q)$, then v is not a node of $G[W']$ and again Q is not a subgraph of $G[W']$.

Now assume, by contradiction, that $F \setminus \{f\}$ is not in $\mathcal{I}(G)$. By Lemma 2.3, $G[W']$ contains an odd cycle, say D. Suppose that D contains an edge $e = v_iv_{i+3}$ which is a diagonal with respect to F and Q. Thus $e_i, e_{i+2} \in F$ and $e_{i-1}, e_{i+1}, e_{i+3} \notin F$. Also, since F is an obstruction, e_i and e_{i+2} are the only edges of F incident to v_i and v_{i+3}, respectively. In consequence, f can be neither e_i nor e_{i+2}. Now if we replace in D e by the path $v_i, e_i, v_{i+1}, e_{i+1}, v_{i+2}, e_{i+2}, v_{i+3}$, we get a new cycle in $G[W']$ which does not contain e and which is still odd. We can reiterate this procedure until we get an odd cycle in $G[W']$, still denoted by D, without diagonals with respect to F and Q.

Suppose that $V(D)$ contains a node $w \notin V(Q)$. As $w \in V(F')$, there is an edge, say g', belonging to $F_2 \cap E(D)$ incident to w. By condition (1) of Definition 3.1, this edge is the only edge of F incident to w. Consequently, there must exist an edge $g \in E(W) \setminus ((F \cup E(Q)) \cap E(D))$ incident to w. By our hypothesis, g is then a wing with respect to Q. Suppose, without loss of generality, that $g = wv_i$ is a forward wing. Hence $g' = wv_{i+2} \in E(D)$. Also note that e_i is the only edge of F incident to v_i, which implies that $f \neq v_i v_{i+1}$. If we replace in D the path v_i, g, w, g', v_{i+2} by $v_i, e_i, v_{i+1}, e_{i+1}, v_{i+2}$, we get an odd cycle in $G[W']$. Moreover, this new cycle does not contain the wing g. So, if we reiterate this procedure, we get an odd cycle D in $G[W']$ which contains neither a diagonal nor a wing with respect to F and Q and whose nodes are all in $V(Q)$. But this implies that D contains only edges of Q, which contradicts the fact that Q is not a subgraph of $G[W']$. $\square$

4. Finding a minimum dependent set. In this section we consider the problem of finding a minimum dependent set in a graph with nonnegative weights. Using the characterization of the minimal dependent sets given in section 3, we will show that this problem reduces to the minimum odd circuit problem in a signed directed graph, and can then be solved in polynomial time. Some polyhedral and algorithmic consequences of this result will be discussed in the next section.

Let $G = (V, E)$ be a graph and $(c(e),\ e \in E)$ a nonnegative weight vector associated with the edges of E. In what follows we are going to construct from G a signed digraph $D = (U, A)$. For convenience we will use the following notation.

For every node u of G, $c(u)$ will denote the minimum weight of an edge incident to u, and e_u a minimum weight edge incident to u. That is, $c(e_u) = c(u)$. Given a minimal dependent set $F \in \mathcal{C}(G)$ of G, we let

$$Q = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

denote the odd cycle of the obstruction induced by F. (Such a cycle exists by Theorem 3.2.) And we suppose that the sequence $e_1, \ldots, e_k$ follows the clockwise order. We say that a node $v_i \in V(Q)$ is *left-saturated* (resp., *right-saturated*) if e_{i-1} (resp., e_i) is in F; otherwise v_i is said to be *left-unsaturated* (resp., *right-unsaturated*). Note that, since by Definition 3.1 $E(Q) \setminus F \neq \emptyset$, Q has at least one left-unsaturated node and one right-unsaturated node. If v_i is unsaturated (i.e., left- and right-unsaturated), we denote by f_i the unique edge in F incident to v_i. A node v is said to be a *left-node* (resp., *right-node*) if v is either left-saturated or left-unsaturated (resp., right-saturated or right-unsaturated).

Now we define the signed digraph $D = (U, A)$ (see Figure 4 for an illustration).

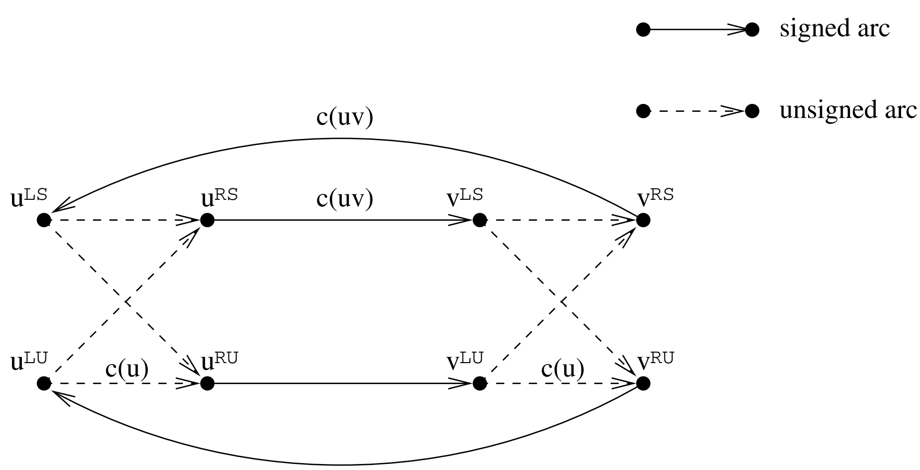

FIG. 4. *The subdigraph of D corresponding to an edge uv of G.*

- For every node $u \in V$, consider four nodes $u^{\mathtt{LS}}, u^{\mathtt{RS}}, u^{\mathtt{LU}}, u^{\mathtt{RU}}$ which correspond to the four possible states of u: left-saturated, right-saturated, left-unsaturated, and right-unsaturated. And consider four unsigned arcs: $u^{\mathtt{LS}}u^{\mathtt{RS}}$, $u^{\mathtt{LS}}u^{\mathtt{RU}}, u^{\mathtt{LU}}u^{\mathtt{RS}}$ with weight 0 and $u^{\mathtt{LU}}u^{\mathtt{RU}}$ with weight $d(u^{\mathtt{LU}}u^{\mathtt{RU}}) = c(u)$.
- For every edge $uv \in E$ consider four signed arcs: $u^{\mathtt{RU}}v^{\mathtt{LU}}, v^{\mathtt{RU}}u^{\mathtt{LU}}$ with weight $d(u^{\mathtt{RU}}v^{\mathtt{LU}}) = d(v^{\mathtt{RU}}u^{\mathtt{LU}}) = 0$ and $u^{\mathtt{RS}}v^{\mathtt{LS}}, v^{\mathtt{RS}}u^{\mathtt{LS}}$ with weight $d(u^{\mathtt{RS}}v^{\mathtt{LS}}) = d(v^{\mathtt{RS}}u^{\mathtt{LS}}) = c(uv)$.

Observe that the tail of an unsigned arc is always a left-node and the head is always a right-node. Also note that any sequence of arcs in D alternates between signed and unsigned arcs. In consequence, any circuit of D has an even number of arcs.

We let Σ denote the set of signed arcs and $U = A \setminus \Sigma$ the set of unsigned arcs. So a circuit Q^D of D is odd if $|A(Q^D) \cap \Sigma|$ is odd.

Let Q^D be a minimum weight odd circuit of D with respect to the weight vector d. Let F be a minimum weight dependent set of G with respect to the weight vector c. In what follows we are going to show that $d(Q^D) = c(F)$. As a consequence, we can compute minimum dependent sets by calculating minimum weight odd circuits in an auxiliary graph.

First we show that $c(F) \leq d(Q^D)$. Let F' be the set of edges uv of G such that either

(i) $u^{\mathtt{RS}}v^{\mathtt{LS}}$ is an arc of Q^D,
(ii) $v^{\mathtt{RS}}u^{\mathtt{LS}}$ is an arc of Q^D, or
(iii) the edge uv is the minimum weight edge e_u incident to u and $u^{\mathtt{LU}}u^{\mathtt{RU}}$ is an arc of Q^D.

The way we defined the cost vector d yields $c(F') \leq d(Q^D)$. Let $w_1, \dots, w_q$ be the sequence of nodes of Q^D (where the indices are modulo q). Then the sequence $u_1, \dots, u_{q'}$ of nodes of G, obtained by taking the node u_i if w_i is either $u_i^{\mathtt{LS}}$, $u_i^{\mathtt{RS}}$, $u_i^{\mathtt{LU}}$, or $u_i^{\mathtt{RU}}$ (note that $q' \leq q$), induces a subgraph H of G whose edges correspond to the signed arcs of Q^D. Since Q^D is odd, H contains an odd cycle Q'. Since Q' is a cycle of the graph $G[V(F')]$, by Lemma 2.3, F' is dependent. As F is chosen minimum, $c(F) \leq c(F')$ and therefore

$$c(F) \leq d(Q^D).$$

Now we show that $c(F) \geq d(Q^D)$. Since c is nonnegative we can assume that F is minimal. By Theorem 3.2, F induces an obstruction with respect to an odd cycle Q.

Let $P_1 = v_i, e_i, \dots, v_{j-1}, e_{j-1}, v_j$ be a path of Q such that all the edges of P_1 are in F and $e_{i-1}, e_j \notin F$. The node v_i is left-unsaturated and right-saturated, the nodes $v_{i+1}, \dots, v_{j-1}$ are left- and right-saturated, and v_j is left-saturated and right-unsaturated. In the digraph D, the path P_1 corresponds to a path P_1^D with node set $V(P_1^D) = \{v_i^{\mathtt{LU}}, v_i^{\mathtt{RS}}, v_{i+1}^{\mathtt{LS}}, v_{i+1}^{\mathtt{RS}}, \dots, v_{j-1}^{\mathtt{LS}}, v_{j-1}^{\mathtt{RS}}, v_j^{\mathtt{LS}}, v_j^{\mathtt{RU}}\}$. The arc set of P_1^D is $A(P_1^D) = \{a_i, \sigma_i, \dots, a_{j-1}, \sigma_{j-1}, a_j\}$, where $a_i, \dots, a_j \in U$ are unsigned arcs with weight 0 and σ_l is a signed arc with cost $d(\sigma_l) = c(e_l)$ for $l = i, \dots, j-1$. Thus

$$\begin{aligned} d(P_1^D) &= d(a_i) + d(\sigma_i) + \cdots + d(a_{j-1}) + d(\sigma_{j-1}) + d(a_j) \\ &= c(e_i) + \cdots + c(e_{j-1}) \\ &= c(P_1). \end{aligned}$$

Let $P_2 = v_i, e_i, \dots, v_{j-1}, e_{j-1}, v_j$ be a path of Q such that no edge of P_2 is in F. The node v_i is right-unsaturated, the nodes $v_{i+1}, \dots, v_{j-1}$ are left- and right-unsaturated, and v_j is left-unsaturated. In D, there is a path P_2^D with node set

$V(P_2^D) = \{v_i^{\mathtt{RU}}, v_{i+1}^{\mathtt{LU}}, v_{i+1}^{\mathtt{RU}}, \dots, v_{j-1}^{\mathtt{LU}}, v_{j-1}^{\mathtt{RU}}, v_j^{\mathtt{LU}}\}$. The arc set of P_2^D is $A(P_2^D) = \{\sigma_i, a_{i+1}, \sigma_{i+1}, \dots, a_{j-1}, \sigma_{j-1}\}$, where $a_l \in U$ is an unsigned arc with cost $c(v_l)$ for $l = i+1, \dots, j-1$ and $\sigma_i, \dots, \sigma_{j-1}$ are signed arcs with cost 0.

Thus

$$\begin{aligned} d(P_2^D) &= d(\sigma_i) + d(a_{i+1}) + d(\sigma_{i+1}) + \cdots + d(a_{j-1}) + d(\sigma_{j-1}) \\ &= c(v_{i+1}) + \cdots + c(v_{j-1}) \\ &\leq \sum_{l=i+1,\dots,j-1} c(f_l). \end{aligned}$$

(Recall that f_l is the unique edge of F incident to v_l.) Observe now that Q decomposes into paths of types P_1 and P_2. The associated paths of D of types P_1^D and P_2^D form a circuit R^D of D whose weight $d(R^D)$ is less than or equal to $c(F)$. Since $|\Sigma \cap A(R^D)| = |E(Q)|$, R^D is an odd circuit of D. Then $d(Q^D) \leq d(R^D)$, and therefore

$$c(F) \geq d(Q^D).$$

So we can state the following theorem.

THEOREM 4.1. *The minimum dependent set problem with nonnegative weights can be solved in polynomial time.*

5. Polyhedral consequences and concluding remarks. Given a graph $G = (V, E)$, let $\mathrm{IBSP}(G)$ be the convex hull of the incidence vectors of the edge sets of induced bipartite subgraphs of G.

Let $\mathcal{P}(G)$ be the polyhedron given by

$$0 \leq x(e) \leq 1 \qquad \forall\, e \in E, \tag{1}$$

$$x(C) \leq |C| - 1 \qquad \forall\, C \in \mathcal{C}(G). \tag{2}$$

Obviously, inequalities (1) and (2) are valid for $\mathrm{IBSP}(G)$. Constraints (1) are called the *trivial inequalities*. Constraints (2) will be called the *dependent set inequalities*.

Moreover, we have that MIBSP is equivalent to the integer program

$$\max\ \{wx\ :\ x \in \mathcal{P}(G),\ x \text{ integer}\}.$$

The *separation problem* for a class of inequalities is to decide whether a given vector $\bar{x} \in \mathbb{Q}^E$ satisfies the inequalities and, if not, to find an inequality that is violated by $\bar{x}$.

Given a vector $\tilde{x} \in \mathbb{R}_+^E$, let $\bar{x} \in \mathbb{R}_+^E$ such that $\bar{x}(e) = 1 - \tilde{x}(e)$ for all $e \in E$. Clearly, there is an inequality of type (2) violated by $\tilde{x}$ if and only if the minimum weight of a dependent set with respect to $\bar{x}$ is less than 1. It thus follows by Theorem 4.1 that the separation problem associated with inequalities (2) is solvable in polynomial time. From [15] we then have the following corollary.

COROLLARY 5.1. *The problem*

$$\max\ \{wx\ :\ x \in \mathcal{P}(G)\}$$

can be solved in polynomial time.

A natural question that may be posed is to characterize the graphs for which the polytope $\mathcal{P}(G)$ is integral. As it will turn out, these graphs are precisely the bipartite graphs.

PROPOSITION 5.2. *$\mathcal{P}(G)$ is integral if and only if G is bipartite.*

Proof. If $G = (V, E)$ is bipartite, then $\mathcal{P}(G)$ is given by the trivial inequalities, and hence any extreme point of $\mathcal{P}(G)$ is integer.

Now suppose that $G = (V, E)$ is not bipartite, and let $Q = v_1, e_1, v_2, e_2, \ldots, v_{2k+1}, e_{2k+1}, v_{2k+2}$, where $v_{2k+2} = v_1$, be an odd cycle of G. We can assume that Q is a hole. Consider the solution $\bar{x} \in \mathbb{R}^E$ given by

$$\bar{x}(e) = \begin{cases} \frac{k}{k+1} & \text{if } e \in E(Q), \\ 0 & \text{if not.} \end{cases}$$

Let $C_i = \{e_i, e_{i+2}, \ldots, e_{i+2k}\}$ for $i = 1, \ldots, 2k+1$ (the indices are modulo $2k+1$). Note that $|C_i| = k+1$. By Theorem 3.2, the C_i's are in $\mathcal{C}(G)$. We also have that $\bar{x}$ satisfies the system

$$\begin{aligned} x(C_i) &= |C_i| - 1 && \text{for } i = 1, \ldots, 2k+1, \\ x(e) &= 0 && \forall e \in E \setminus E(Q). \end{aligned}$$

Furthermore it is not hard to see that $\bar{x}$ is the unique solution of that system. Hence $\bar{x}$ is an extreme point of $\mathcal{P}(G)$. ☐

In contrast with the classical bipartite subgraph problem, the linear relaxation of the MIBSP does not seem to be quite strong. As it has been shown by Guenin [19], for the former problem, the trivial and the cycle inequalities suffice to describe the bipartite subgraph polytope in a large class of graphs (the weakly bipartite graphs) which contains, for instance, planar graphs and bipartite graphs. However, for the MIBSP, as shown by Proposition 5.2, the graphs for which the trivial and the dependent set inequalities completely describe IBSP(G) are reduced to the bipartite graphs. We can also notice that any inequality that is valid for the bipartite subgraph polytope is also valid for the IBSP(G). These inequalities are not, however, so strong for the MIBSP. To see this, consider, for instance, a clique (W, T) on p nodes in a graph G. The inequality $x(T) \le \lfloor \frac{p}{2} \rfloor \lceil \frac{p}{2} \rceil$ is valid for the bipartite subgraph polytope on G and is facet defining [3], whereas, any solution for the MIBSP can take at most one edge from T.

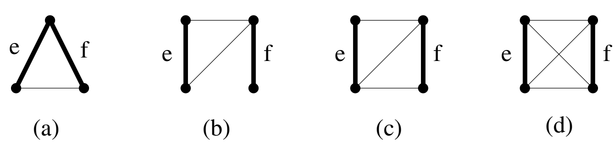

FIG. 5. *The minimal dependent sets of size* 2.

These negative observations motivated us to investigating new valid inequalities for IBSP(G). By Theorem 3.2, $\{e, f\} \in \mathcal{C}(G)$ if and only if the subgraph induced by the endnodes of e, f is one of the four graphs of Figure 5. Let us consider from G an auxiliary graph $A(G)$ whose nodes correspond to the edges of G and such that two nodes e, f are linked by an edge if and only if $\{e, f\} \in \mathcal{C}(G)$. Remark that any independent set of G is a stable set of $A(G)$. (Note that the converse is not true.) Hence the so-called clique and odd cycle inequalities of the stable set polytope of $A(G)$ (see [20]) given by

(3) $\quad x(K) \le 1 \quad$ for every clique K of $A(G)$,

(4) $\quad x(C) \le \dfrac{|C| - 1}{2} \quad$ for every odd cycle C of $A(G)$

are valid for IBSP(G). Notice that the edge set of a clique of G corresponds to a clique of $A(G)$. (Note that the converse is not true.) As it will turn out, the separation problem for inequalities (3) is NP-hard. The separation problem for inequalities (4) can be solved in polynomial time (see, for instance, [17]).

PROPOSITION 5.3. *The separation problem for inequalities* (3) *is NP-hard.*

Proof. We use a reduction from the maximum clique problem. Let $G = (V, E)$ be a graph. Add a node u and the edges uv for each $v \in V$ to obtain the graph $\tilde{G} = (\tilde{V}, \tilde{E})$. Let $\tilde{x} \in \mathbb{R}^{\tilde{E}}$ given by

$$\tilde{x}(e) = \begin{cases} 1/k & \text{if } e \in \tilde{E} \setminus E, \\ 0 & \text{if } e \in E. \end{cases}$$

Clearly, there is a clique K in $A(\tilde{G})$ with $\tilde{x}(K) > 1$ if and only if there is a clique of size $k+1$ in G. ∎

Let $G = (V, E)$ be a (nonbipartite) graph and let Q and C_i be as defined in the proof of Proposition 5.2, $i = 1, \ldots, 2k+1$. Observe that e belongs to $k+1$ different C_i's for each $e \in E(Q)$. As the C_i's are dependent sets in G, the following inequalities are valid for IBSP(G):

$$x(C_i) \le k \qquad \text{for } i = 1, \ldots, 2k+1.$$

By summing these inequalities, we obtain the inequality

$$(k+1)x(E(Q)) \le k(2k+1).$$

Therefore the inequalities

$$x(E(Q)) \le |E(Q)| - 2, \qquad \text{for every odd cycle } Q \text{ of } G, \tag{5}$$

are valid for IBSP(G). Inequalities (5) also arise naturally since any independent set of G uses at most $|V(Q)| - 1$ nodes (and thus at most $|E(Q)| - 2$ edges) of Q. Note that inequalities (5) are different from the inequalities induced by the odd cycles of $A(G)$. (If, for instance, $G = (V, E)$ is an odd cycle with edge set, say $E = \{e_1, \ldots, e_5\}$, then $A(G)$ has no edge, while G produces the inequality $x(E) \le 3$ of type (5) which is facet defining.) Inequalities (5) will be called *cycle inequalities*.

PROPOSITION 5.4. *The separation problem for inequalities* (5) *can be solved in polynomial time.*

Proof. Let $\bar{x}$ be a vector associated with the edges of G. We may suppose that $\bar{x}$ satisfies the trivial inequalities. Let $y \in \mathbb{R}^E$ such that $y(e) = 1 - \bar{x}(e)$ for all $e \in E$. An inequality (5) is violated by $\bar{x}$ if and only if $y(E(Q)) < 2$. Thus the separation problem for inequalities (5) reduces to finding a minimum odd cycle in G with respect to the weight vector y. As $y(e) \ge 0$ for all $e \in E$, this can be done in polynomial time as shown in [18]. ∎

It would be interesting to determine when the dependent, cycle, and clique inequalities are facet defining for the polytope IBSP(G).

The approach presented in this paper can be adapted to handle the maximum induced forest and the maximum induced acyclic subgraph problems (see [8]).

Acknowledgments. We would like to thank the anonymous referees for their constructive comments. We are grateful to Hervé Kerivin for many comments that have improved the presentation.

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