THE MULTI-TERMINAL VERTEX SEPARATOR PROBLEM: POLYHEDRAL ANALYSIS AND BRANCH-AND- CUT

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THE MULTI-TERMINAL VERTEX SEPARATOR PROBLEM: POLYHEDRAL ANALYSIS AND BRANCH-AND-CUT

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ABSTRACT

Let $G = (V \cup T, E)$ be an undirected graph such that V is a set of vertices, E a set of edges and T a set of terminal vertices. The *Multi-terminal vertex separator problem* consists in partitioning V into $k+1$ subsets $\{S, V_1, ..., V_k\}$ minimizing the size of S and such that there is no edge between two subsets V_i and V_j and each subset V_i contains exactly one terminal. Set S is called a *separator*. In this paper, we show that this problem is NP-complete. We discuss the problem from a polyhedral point of view. We describe some valid inequalities and characterize when they define facets. Using this we develop a Branch-and-Cut algorithm.

Keywords: Combinatorial optimization, Polyhedral approach, Branch-and-Cut, Complexity, Vertex separator problem.

1 INTRODUCTION:

Let $G = (V \cup T, E)$ be a simple graph with V a set of vertices, E a set of edges and T a set of *terminal* vertices. The multi-terminal vertex separator problem, MTVSP for short, consists in finding the smallest subset $S \subseteq V$ called a *separator* such that the graph induced by $V \setminus S$ contains k disjoint components and each component contains exactly one terminal. This problem is a variant of the *vertex separator problem* that consists in partitioning V into $k+1$ subsets $S, V_1, \dots, V_k$ in such a way that S is minimum and there is no edge between two subsets V_i and V_j. The MTVSP is equivalent to finding the smallest node subset S such that each path between two terminals intersects S. The MTVSP has applications in different areas like VLSI conception, linear algebra, connectivity problems and parallel algorithms. Many variants of the vertex separator problem have been studied [2] [3] [4]. In [2] Balas and Suza studied the following problem. Given a simple graph $G = (V, E)$ and an integer $\beta(n)$ with $n = |V|$, partition V into three subsets A, B and C such that $|C|$ is minimum, no vertex in A is incident to a vertex in B and $\max\{|A|, |B|\} \leq \beta(n)$. In [5] authors studied another variant of the problem. Let $G = (V, E)$ be a simple graph and $a, b \in V$ two terminal nodes. The problem here is to partition V into three subsets A, B and C minimizing $|\delta(C)|$ such that no vertex in A is incident to a vertex in B and $a \in A, b \in B$. This problem can be solved in polynomial time. It is equivalent to a minimum cut problem in a transformed graph.
The paper is organized as follows, in Section 2 we discuss the complexity of the MTVSP. In Section 3

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we propose two $0-1$ linear programming formulations for the problem. In Section 4 we study the problem from polyhedral point of view and propose some valid inequalities. In Section 5 we present a branch-and-cut algorithm along with some experimental results.

We denote by $G=(V\cup T,E)$ a simple graph with V a set of vertices, T a collection of terminal vertices and E a set of edges. We denote by n the size of the set V and k the number of terminals in T. Given a vertex $v\in V\cup T$, we denote by $N(v)\subseteq(V\cup T)$ the set of vertices incident to v and by $d(v)$ the size of $N(v)$ called degree of v in G. Given a subset $R\subseteq(V\cup T)$, we denote by $N(R)\subseteq(V\cup T)$ the set of vertices incident to at least one vertex in R. Let $\delta(v)$ be the set of edges incident to v and $\delta(R)$ the set of edges having exactly one vertex in R. Let $C\in\mathbb{Z}^V$ be a vector, $C(R)$ is equivalent to $\sum_{v\in R}C(v)$. Let $H=(U,I)$ be a subgraph of G. We denote by $H\subseteq V$ the subset U and by $H\subseteq E$ the subset I. The *internal vertices* of a path are all vertices of the path except the extremities. A path having its extremities in T is called a *terminal path*. In this paper we consider the following hypotheses:

- There is no edge between two terminals, otherwise the problem has no solution.
- For each pair of terminals $t_i,t_j\in T$, we have $N(t_i)\cap N(t_j)=\emptyset$. Otherwise all vertices of $N(t_i)\cap N(t_j)$ belong to every separator.
- For each vertex $v\in V$, there is at least one path, between two terminals, containing v. Otherwise v cannot belong to a minimal separator.
- The graph G is connected.

2 COMPLEXITY ANALYSIS

In this section we consider the three-terminal vertex separator problem $(TTVSP)$. It has been shown that the $(TTVSP)$ is NP-complete [10]. In this section we give a simpler of this result using a polynomial reduction from the minimum vertex cover set problem VC. The VC problem is a well-known NP-complete problem. It consists of finding the smallest subset of vertices such that all edges have at least one vertex in it.

It is clear that the $TTVS$ problem is in NP. Let $H=(U,E')$ be a simple graph. We construct a graph $G=(V_1\cup V_2\cup V_3\cup T,E)$ from the graph H using the following operations:

- add three vertices t_1, t_2 and t_3 in T.
- for each vertex $u\in U$, add three vertices, v_1^u in V_1, v_2^u in V_2 and v_3^u in V_3
- for each vertex $u\in U$, add three edges $t_1v_1^u$, $t_2v_2^u$ and $t_3v_3^u$ in E.
- for each vertex $u\in U$, add two edges $v_1^uv_3^u$ and $v_2^uv_3^u$ in E.
- for each edge $uw\in E'$, add two edges $v_1^uv_2^w$ and $v_1^wv_2^u$ in E.

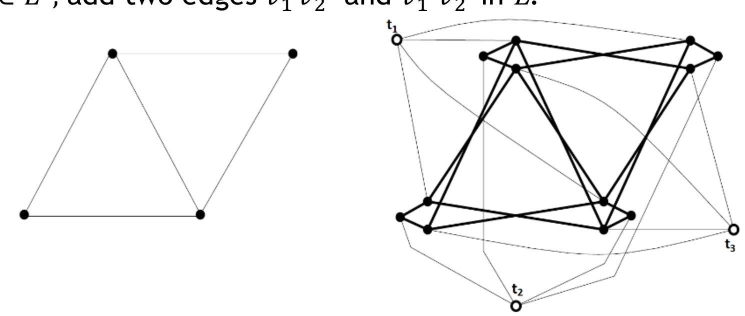

Figure 1: Graph transformation

Figure 1 illustrates the above graph transformation. Let $S\subseteq V_1\cup V_2\cup V_3$ be a separator and $v_1^u,v_2^u,v_3^u\in V$ three vertices associated with each vertex $u\in U$.

Proposition 1: For a vertex $u \in U$, if S is the smallest separator of G then either $v_1^u, v_2^u \in S$ and $v_3^u \notin S$ or $v_1^u, v_2^u \notin S$ and $v_3^u \in S$.

Proof.
If v_1^u belongs to the separator S and not v_2^u and v_3^u then there is a path from t_3 to t_2. If v_1^u and v_3^u belong together to the separator S but not v_2^u then we can replace v_3^u by v_2^u. Clearly, the separator remains the smallest since v_3^u is only incident to v_1^u, v_2^u and t_3. If v_1^u, v_2^u and v_3^u belong together to S, then $S \backslash \{v_3^u\}$ is also a separator, a contradiction with the minimality of S. ■

Proposition 2: The smallest vertex cover set in H is of size q if and only if the smallest separator in G is of size $q + |U|$.

Proof.

(⇒) Let $R \subseteq U$ be the vertex cover set of size q. If for each vertex $u \in R$, we add its corresponding vertices v_1^u and v_2^u in set S, and for each vertex $u \in U \backslash R$ we add its corresponding vertex v_3^u in set S, then S is a separator in G of size $2q - q + |U| = q + |U|$. Moreover, S is the smallest separator. (⇐) For a terminal path containing $v_1^u \in V_1$ and $v_2^v \in V_2$, either $v_1^u \in S$ or $v_2^v \in S$. We know that there is an edge between v_1^u and v_2^v if there exists an edge $uv \in E'$. It then follows that the corresponding vertices of $S \cap V_1$ represent a vertex cover set in H of cardinality q. If $v_1^u \notin S$ or $v_2^v \notin S$, then $v_3^u \in V_3$ belongs to S. So the separator is of size $2q - q + |U| = q + |U|$, and the vertex cover in H is of size q. It is clear that if S is the smallest in G, then the vertex cover is in H. ■

3 FORMULATIONS

In this section we propose two different $0-1$ linear formulations for the problem, the first one has a polynomial number of variables and constraints and uses double indices. The second has a polynomial number of variables but an exponential number of constraints.

3.1 Double indices formulation

Let $x \in \{0,1\}^{(V \cup T) \times T}$ such that:

$$x_{vt} = \begin{cases} 1, & \text{if the vertex } v \text{ belongs to the subset } V_t, \\ 0, & \text{otherwise.} \end{cases} \quad \text{for every } v \in (V \cup T), t \in T$$

$$\max \quad \sum_{v \in V} \sum_{t \in T} x_{vt}$$

$$x_{ut} + \sum_{k \in T \backslash \{t\}} x_{vk} \leq 1 \quad \forall\, (uv) \in E, \forall t \in T, \qquad (1)$$

$$\sum_{t \in T} x_{vt} \leq 1 \quad \forall v \in (V \cup T), \qquad (2)$$

$$x_{tt} = 1 \quad \forall t \in T, \qquad (3)$$

$$x_{vt} \in \{0,1\} \quad \forall t \in T, \forall v \in (V \cup T) \qquad (4)$$

3.2 Natural formulation

Let Γ be the set of terminal paths between each pair of terminals. Let $x \in \{0,1\}^V$ such that:

$$x_v = \begin{cases} 1, & \text{if the vertex } v \text{ belongs to the separator,} \\ 0, & \text{otherwise.} \end{cases} \quad \text{for every } v \in V$$

$$\begin{array}{llll} min & \sum_{v\in V} x_v & & \\ & \sum_{v\in P_{t_it_j}} x_v & \geq 1 & \forall P_{t_it_j}\in \Gamma, \forall (t_i,t_j)\in T, \quad (5)\\ & x_v & \leq 1 & \forall v\in V \quad (6)\\ & x_v & \geq 0 & \forall v\in V \quad (7)\\ & x_v & \text{integer} & \forall v\in V \quad (8)\end{array}$$

We notice that the first formulation has $(n+k)k$ variables and the second has only n variables. Inequalities (5) in this latter formulation, which are in an exponential number can be separated in polynomial time. Since the second formulation has less variables, we consider it for our analysis.

4 POLYHEDRAL ANALYSIS

For $S\subset V$, let $x^S\in\{0,1\}^V$ be the vector given by $x_v=1$ if $v\in S$ and $x_v=0$ otherwise. x^S is called the incidence vector of S. Let $P(G,T)$ be the convex hull of solutions of the above program, that is, $P(G,T)=\ conv(x\in\{0,1\}^V \mid x$ satisfies (5)).

4.1 Dimension and valid inequality

We have the following results:

Proposition 3: Polytope $P(G,T)$ is full dimensional.

Proposition 4: For $v\in V$, inequality (6) defines facet of $P(G,T)$.

Proposition 5: For a vertex $v\in V$, inequality (7) defines a facet of $P(G,T)$ if and only if, vertex v does not belong to a terminal path containing two internal vertices. ■

4.2 Path inequalities

Theorem 1: Inequality (5) associated with a path $P_{t_it_j}$ defines a facet of $P(G,T)$ if and only if:

- $P_{t_it_j}$ is inclusewise minimal, that is only two internal vertices from $P_{t_it_j}$ connected to a terminal.
- No vertex $v\notin P_{t_it_j}$ is incident to a terminal $t\notin P_{t_it_j}$ and to two vertices of $P_{t_it_j}$.
- No vertex from $P_{t_it_j}$ is incident to more than two vertices from $P_{t_it_j}$.

Proof.(⇐)

- Suppose there exists a nonterminal vertex $v\in P_{t_it_j}$, adjacent to $t_k\in T\backslash\{t_i,t_j\}$. It is clear that there exists a terminal path between t_k and t_j such that $P_{t_kt_j}\subset P_{t_it_j}$. Inequality (5) associated with $P_{t_it_j}$ can then be obtained from inequality (5) associated with $P_{t_kt_j}$ and inequalities (7) associated with each nonterminal vertex of $P_{t_it_j}\backslash P_{t_kt_j}$.
- Suppose there exists $v\in V\backslash P_{t_it_j}$ and v adjacent to $t_k\in T\backslash\{t_i,t_j\}$ and to two vertices of $P_{t_it_j}$. The following inequality is valid for $P(G,T)$: $x(P_{t_it_j})+x_v\geq 2$. This is obtained by chvátal-gomory procedure on inequalities (5) induced by the paths $P_{t_it_j}, P_{t_it_k}$ and $P_{t_jt_k}$. Inequality (5) associated with the path $P_{t_it_j}$ can be obtained from the above inequality and inequality (6) of the vertex v.
- If there exists a vertex $v\in P_{t_it_j}$ incident to more than two vertices of $P_{t_it_j}$, this means that there exists a terminal path $P'_{t_it_j}$ such that $P'_{t_it_j}\subset P_{t_it_j}$. Inequality (5) associated with $P_{t_it_j}$ can then be obtained from inequality (5) of $P'_{t_it_j}$ and inequalities (7) associated with vertices of $P_{t_it_j}\backslash P'_{t_it_j}$.

(⇒)Denote by $ax \geq \alpha$ inequality (5). Let $bx \geq \beta$ be an inequality that defines a facet of $P(G,T)$. We suppose that $\{x \in P(G,T): ax = \alpha\} \subseteq \{x \in P(G,T): bx = \beta\}$. Since $P(G,T)$ is full dimensional, we need to prove that there exists ρ such that $b = \rho a$.

For each vertex $v \in P_{t_i t_j}$, define a separator $S^v = \left(V \setminus P_{t_i t_j}\right) \cup \{v\}$. For each pair of vertices $u, v \in P_{t_i t_j}$, the incidence vectors x^{S^u} and x^{S^v} are solutions of $P(G,t)$ and satisfy inequality (5) with equality. Hence, $ax^{S^u} = ax^{S^v}$, and hence $bx^{S^u} = bx^{S^v}$. Therefore:

$$b(u) = b(v) = \rho \quad \text{for all } u, v \in P_{t_i t_j} \text{ and some scalar } \rho \in \mathbb{R}$$

For each vertex $v \notin P_{t_i t_j}$, let $u \in P_{t_i t_j}$ be a vertex adjacent to v. If there is no $u \in P_{t_i t_j}$ adjacent to v, then u would represent any vertex of $P_{t_i t_j}$. Consider the separator $S_v^u = S^u \setminus \{v\}$. The incidence vectors $x^{S_v^u}$ and x^{S^u} are solutions of $P(G,T)$ and satisfy inequality (5) with equality. Hence, $ax^{S^u} = ax^{S_v^u}$ and hence $bx^{S^u} = bx^{S_v^u}$. This implies that: $b(v) = 0 \quad \forall v \notin P_{t_i t_j}$ ■

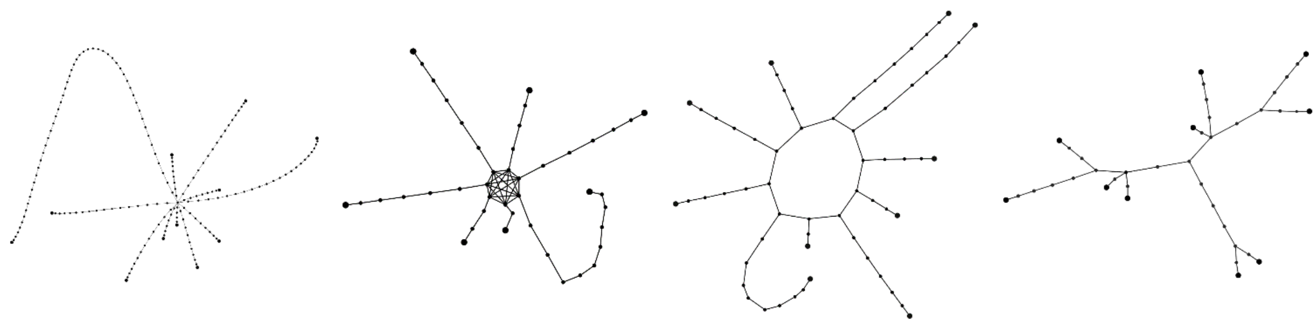

Figure 2: examples of a star tree, clique star, terminal cycle and terminal tree

4.3 Star tree inequalities

A *star tree* J is a tree given by a root vertex $v_r \in V$ and f vertex disjoint paths between v_r and f terminal vertices. Let P_t be the set of internal vertices of a path between v_r and a terminal vertex $t \in T$. Let $F_J \subseteq T$ be the leaf set of J. The following inequalities, are valid for $P(G,T)$:

$$\sum_{t \in F} x(P_t) + (f-1)x_{v_r} \geq f - 1 \qquad (9)$$

Theorem 2: Inequality (9) defines a facet of $P(G,T)$ if and only if the following hold:

- $f \geq 3$.
- No vertex $u \in J$ is incident to a terminal $t \in T \setminus F_J$.
- If two vertices u, $v \in J$ are not adjacent in the sub-graph induced by J, then $uv \notin E$.

Proof.(⇐)

- If $f = 2$, then the star tree inequality is equivalent to a path inequality associated with J.
- If a vertex $v \in P_t$ is incident to a terminal $t' \in T \setminus F_J$, then let the star tree J' with all leaves in $F_J \setminus \{t\}$. Inequality (9) can be obtained from the star tree inequality associated with J', the path inequality associated with P' and the trivial inequalities. If v_r is incident to a terminal $t' \in T \setminus F_J$, then let the a star tree J' with all leaves in $F_J \cup \{t'\}$. Inequality (9) can be obtained from the star tree inequality associated with J' and the trivial inequalities.
- Suppose there exist two vertices $u, v \in J$ not adjacent in the sub-graph induced by J and $uv \in E$.

a- $u \in P_t$ and $v \in P_{t'}$, for $t, t' \in F_J$ and $t \neq t'$. Then the following inequality:

$$\sum_{t\in F_J} x\,(P_t) + (f-2)x_{v_r} \geq f-1$$

is valid for $P(G,T)$ and dominates inequality (9). Then, inequality (9) cannot be facet defining.

b- $u, v \in P_t$, for $t \in F_J$. Let $P_{uv} \subset P_t \setminus \{u, v\}$ be the internal vertices of the path between u and v. Then the following inequality:

$$\sum_{t\in F_J} x\,(P_t) + (f-1)x_{v_r} - x(P_{uv}) \geq f-1$$

is valid for P(G,T) and dominates inequality (9). Then, inequality (9) cannot be facet defining.

(⇒) Let us inequality (9) denote by $ax \geq \alpha$. Let $bx \geq \beta$ be an inequality that defines a facet of $P(G,T)$ such that $\{x \in P(G,T): ax = \alpha\} \subseteq \{x \in P(G,T): bx = \beta\}$. Since $P(G,T)$ is full dimensional, we need to prove that there exists ρ such that $b = \rho a$. For a terminal $t \in F_J$, let a ring $Q_i^t \subset (J \setminus (P_t \cup \{v_r\}))$ be a subset of $f-1$ vertices containing exactly one vertex of each P_l for all $l \in (F_J \setminus \{t\})$, i.e., for all $l \in (F_J \setminus \{t\}), |P_l \cap Q_t| = 1$. Consider two vertices $u_1, u_2 \in J \setminus (F_J \cup \{v_r\})$. There exists two terminals $t, t' \in F_J$ and two associated rings $Q_1^t, Q_2^{t'}$ such that $u_1 \in Q_1^t$, $u_2 \in Q_2^{t'}$ and $Q_1^t \setminus \{u_1\} = Q_2^{t'} \setminus \{u_2\}$. Let $S^{Q_1^t} = (V \setminus J) \cup Q_1^t$ (resp. $S^{Q_2^{t'}} = (V \setminus J) \cup Q_2^{t'}$). Clearly $S^{Q_1^t}$ and $S^{Q_2^{t'}}$ are two separators.

The incidence vectors $x^{S^{Q_1^t}}$ and $x^{S^{Q_2^{t'}}}$ satisfy inequality (9) with equality. Hence, $ax^{S^{Q_1^t}} = ax^{S^{Q_2^{t'}}}$. Therefore $bx^{S^{Q_1^t}} = bx^{S^{Q_2^{t'}}}$ implying that $b\,(u_1) = b\,(u_2)$. As u_1 and u_2 are arbitrary, we deduce that:

$$b\,(u) = b\,(v) = \rho \qquad \forall u, v \in J \setminus (F_J \cup \{v_r\}) \text{ and a scalar } \rho \in \mathbb{R}$$

Set $S^0 = (V \setminus J) \cup \{v_r\}$. Clearly, S^0 is a separator, and its incidence vector satisfy inequality (9) with equality. Hence, $ax^{S^{Q_1^t}} = ax^{S^0}$ and therefore, $bx^{S^{Q_1^t}} = bx^{S^0}$. This yields:

$$b\,(v_r) = \sum_{v\in Q_1^t} b\,(v) = (f-1)\rho$$

For a vertex $w \notin J$, set $\bar{S}_w = S^{Q_1^t} \setminus \{w\}$. Clearly, $\bar{S}_w$ is a separator of G and its incidence vector satisfies inequality (9) with equality. Hence $ax^{\bar{S}_w} = ax^{S^{Q_1^t}}$. Therefore $bx^{\bar{S}_w} = bx^{S^{Q_1^t}}$. This implies that:

$$b\,(w) = 0 \quad \forall w \notin J$$

From the above equalities, it follows that $b = \rho a$. ■

4.4 Clique star inequalities

A *clique star* is a graph defined by a clique induced by $K_f \subset V$ and f vertex disjoint paths from each vertex of K_f to a terminal $t \in T$. Let $P_i^{v_i}$ be the internal vertices of the path from v_i to a terminal vertex $t_i \in T$. The following inequality is valid for $P(G,T)$:

$$\sum_{v_i\in K_f} x\left(P_i^{v_i}\right) + x(K_f) \geq f-1$$

4.5 Terminal cycle inequalities

Let $C \subseteq V$ be a cycle and $Q \subseteq C$ be a vertex subset of C of size f. A *terminal cycle* L is a graph given by C and f vertex disjoint paths between vertices of Q to f terminals of T. The following inequality is valid for $P(G,T)$:

$$x(C) \geq \lceil \frac{f}{2} \rceil$$

Theorem 3: A terminal cycle inequality defines a facet of $P(G,T)$ if and only if the following hold:

- f is odd.
- If two vertices $u, v \in Q$ are not incident in L then $(uv) \notin E$.
- If there is a vertex w not in L, adjacent to all vertices of $Q' \subseteq Q$, then there exists a vertex cover in C of size $\lceil \frac{f}{2} \rceil$ that contains at least $|Q'| - 1$ vertices of Q'. ■

4.6 Terminal tree inequalities

A terminal tree R is a tree induced by $R \subseteq V$ in G such that all the leaf vertices are terminals. For a vertex $v \in R$, let $d_R(v)$ be the number of edges that are incident to v in R. Let f_R be the number of terminals of R. For a terminal tree the following inequality is valid for $P(G,T)$:

$$\sum_{v \in R} (d_R(v) - 1)x(v) \geq f_R - 1$$

5 BRANCH-AND-CUT ALGORITHM

We developed a branch-and-cut algorithm to solve the multi-terminal vertex separator problem. The path inequalities are separated in polynomial time by an exact algorithm. However we used some heuristic algorithms to separate the valid inequalities cited before. We compare the CPU time of the branch-and-cut with the one of the commercial solver *Cplex* using the double indices formulation. We use *DIMACS graphs coloring* instances [9] by adding some terminal vertices. We use also some random graphs. In the following table n, m and k represent respectively the number of vertices, edges and terminals. Ps, St, Cs, Tt and Tc represent respectively the number of path inequalities, star tree inequalities, clique star inequalities, terminal tree inequalities and terminal cycle inequalities separated in the branch-and-cut algorithm. Nodes and Gap represent respectively the number of branching nodes and the gap given by the branch-and-cut algorithm. Finally, B&C and Cplex represent respectively the CPU Time of the branch-and-cut algorithm and Cplex in seconds.

Table 1: Numerical experimentation and comparison of results

instances	n	m	k	Ps	St	Cs	Tt	Tc	Nodes	Gap	B&C	Cplex
DSJR500	500	99258	10	94	934	903	1	255	1	0%	116.18	37.859
Anna	138	493	11	210	155	12	31	0	58	20%	**0.784**	0.972
DSJC125	125	1472	11	61	665	158	136	103	7	2%	13.245	1.272
games120	120	638	10	106	1421	136	6	40	31	11%	14.701	0.933
David	87	406	10	137	1220	54	42	1	137	31%	9.641	1.223
Jean	80	254	6	34	11	7	0	0	5	4%	**0.143**	0.172
Huck	74	391	9	155	81	0	1	0	36	13%	0.584	0.31
miles250	128	387	15	474	617	40	0	6	426	28%	6.722	1.195
myciel5	47	236	7	39	23	0	3	6	1	0%	0.376	0.12
myciel6	95	755	11	176	1354	0	3	14	17	7%	13.413	5.188
myciel7	192	2360	11	177	344	0	0	77	1	0%	**4.609**	11.013
myciel7	192	2360	17	286	987	0	369	69	1	0%	38.996	23.767
Queen8_ 8	64	728	8	79	83	8	81	15	1	0%	**0.382**	0.398
Queen8 _12	96	1368	11	56	199	73	171	19	1	0%	2.136	1.895

Queen10 _ 10	100	2940	11	106	223	18	40	26	5	1%	**1.880**	1.932
Queen12_ 12	144	5192	16	126	483	134	150	23	33	7%	**7.673**	8.697
Queen14_ 14	196	8372	18	163	600	410	6	81	12	2%	**22.8**	33.074
Queen16_ 16	256	12640	16	125	84	90	62	34	1	0%	**2.78**	62.87
Queen16_ 16	256	12640	20	386	246	152	201	48	4	0.4%	**15.529**	83.827
Random_1000	1000	275308	4	13	1	0	0	0	1	0%	**20.528**	979.788
Random_1200	1200	396283	10	45	35	15	0	2	1	0%	**148.569**	-
Random_1300	1300	465918	15	187	70	70	26	10	1	0%	**517.499**	-
Random_1500	1500	619257	4	6	2	1	0	1	1	0%	**75.633**	3780.41
Random_1800	1800	891586	15	111	38	26	39	14	1	0%	**719.949**	-
Random_2000	2000	1100866	4	27	10	9	0	5	1	0%	**243.373**	-
Random_2100	2100	1213802	10	66	29	21	0	0	1	0%	**588.63**	-
Random_2300	2300	1456690	15	107	37	29	32	7	1	0%	**1077.519**	-
Random_3000	3000	2478761	4	18	6	5	1	1	1	0%	**697.077**	-
Random_3300	3300	2998740	10	66	29	10	0	1	1	0%	**1618.887**	-
Random_3800	3800	3978688	15	192	25	24	25	10	1	0%	**3039.716**	-

The bold numbers represent the cases where the branch-and-cut is better than Cplex. We can see that when the number of vertices and edges is small, both of Cplex and branch-and-cut are efficient. For huge random graphs, when the density and the number of terminals are high, Cplex cannot solve instances. We can notice that Cplex is not efficient with high density graphs.

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