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# On the $k$ edge-disjoint 2-hop-constrained paths polytope

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**Abstract**

The $k$ edge-disjoint 2-hop-constrained paths problem consists in finding a minimum cost subgraph such that between two given nodes $s$ and $t$ there exist at least $k$ edge-disjoint paths of at most 2 edges. We give an integer programming formulation for this problem and characterize the associated polytope.
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## 1. Introduction

Given a graph $G=(N,E)$ with $s,t \in N$, a 2-$st$-path in $G$ is a path between $s$ and $t$ of length at most 2, where the length of a path is the number of its edges (also called *hops*). Given a function $c: E \to \mathbb{R}$ which associates a cost $c(e)$ to each edge $e \in E$, the *$k$ edge-disjoint 2-hop-constrained paths problem* ($k$HPP) is to find a minimum cost subgraph such that between $s$ and $t$ there exist at least $k$ edge-disjoint 2-$st$-paths.

In this paper, we give an integer programming formulation for the $k$HPP and discuss its associated polytope. In particular, we give a minimal complete linear description of that polytope.

The $k$HPP arises within the framework of survivable network design problems. Indeed, basic requirements, like the 2-edge connectivity for example, are often not sufficient to guarantee an effective survivable network. In fact, for some types of networks (like ATM and IP networks), a higher level of connectivity is required. Also hop-constrained paths are needed to assure the quality of the (re)routing.

Moreover, the $k$HPP can be seen as a special case of the more general problem when more than one

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pair of terminals is considered. This is the case, for instance, when several commodities have to be routed in the network. Thus an efficient algorithm for solving the $k$HPP would be useful to solve (or produce upper bounds for) this more general problem.

Despite these interesting applications, we do not have any knowledge of a previous study of the $k$HPP. Huygens et al. [6] have already investigated the case where $k=2$ and the bound $L$ on the length of the paths is 2 or 3. They present a complete and minimal linear description of its associated polytope. There has been however a considerable amount of research on many related problems. In [3], Dahl and Johannessen consider the 2-path network design problem which consists of finding a minimum cost subgraph connecting each pair of terminal nodes by at least one path of length at most 2. In [1], Dahl considers the hop-constrained path problem, that is the problem of finding between two distinguished nodes $s$ and $t$ a minimum cost path with no more than $L$ edges when $L$ is fixed. He gives a complete description of the dominant of the associated polytope when $L \leqslant 3$. Thus this hop-constrained path problem corresponds to the special case $k=1$ in $k$HPP. A main idea in the completeness proof in [1] for the case $L=2$ turns out to be applicable to the case of a general $k \geqslant 1$ (when $L=2$). This is the basis for our completeness result stated in Theorem 6 in Section 3. Dahl and Gouveia [2] consider the directed hop-constrained path problem. They describe valid inequalities and characterize the associated polytope when $L \leqslant 3$.

Given a graph $G=(N,E)$ and an edge subset $F \subseteq E$, the $0-1$ vector $x^F \in \mathbb{R}^E$, such that $x^F(e)=1$ if $e \in F$ and $x^F(e)=0$ otherwise, is called the *incidence vector* of $F$. The convex hull of the incidence vectors of the solutions to the $k$HPP on $G$, denoted by $P_k(G)$, will be called the $k$HPP *polytope*. Given a vector $w \in \mathbb{R}^E$ and an edge subset $F \subseteq E$, we let $w(F)=\sum_{e \in F} w(e)$. For two node subsets $W_1, W_2 \subset N$, we note $[W_1, W_2]$ the set of edges having one node in $W_1$ and the other in $W_2$. If $W_1=\{w_1\}$, we will write $[w_1, W_2]$ for $[\{w_1\}, W_2]$. If $W \subset N$ is a node subset of $G$, we denote $N \backslash W$ by $\overline{W}$. The set of edges that have only one node in $W$ is called a *cut* and denoted by $\delta(W)$. We will write $\delta(v)$ for $\delta(\{v\})$. A cut $\delta(W)$ such that $s \in W$ and $t \in \overline{W}$ will be called an *st-cut*.

If $x^F$ is the incidence vector of the edge set $F$ of a solution to the $k$HPP, then clearly $x^F$ satisfies the following inequalities:

$$x(\delta(W)) \geqslant k \quad \text{for all } st\text{-cut } \delta(W), \tag{1}$$

$$1 \geqslant x(e) \geqslant 0 \quad \text{for all } e \in E. \tag{2}$$

Inequalities (1) will be called *st-cut inequalities* and inequalities (2) *trivial inequalities*.

In [1], Dahl introduces a class of inequalities valid for the dominant of the hop-constrained path problem. For the special case of $L=2$, they are as follows.

Let $V_0, V_1, V_2, V_3$ be a partition of $N$ such that $s \in V_0$, $t \in V_3$ and $V_i \neq \emptyset$ for $i=1,2$. Let $T$ be the set of edges $e=uv$ where $u \in V_i$, $v \in V_j$ and $|i-j|>1$. Then the inequality

$$x(T) \geqslant 1$$

is valid for the 2-path polyhedron.

Using the same partition, this inequality can be generalized in a straightforward way to the $k$HPP polytope as

$$x(T) \geqslant k. \tag{3}$$

The set $T$ is called a 2-*path-cut* and a constraint of type (3) is called a 2-*path-cut inequality*. See Fig. 1 for an example of a 2-path-cut inequality with $V_0=\{s\}$ and $V_3=\{t\}$.

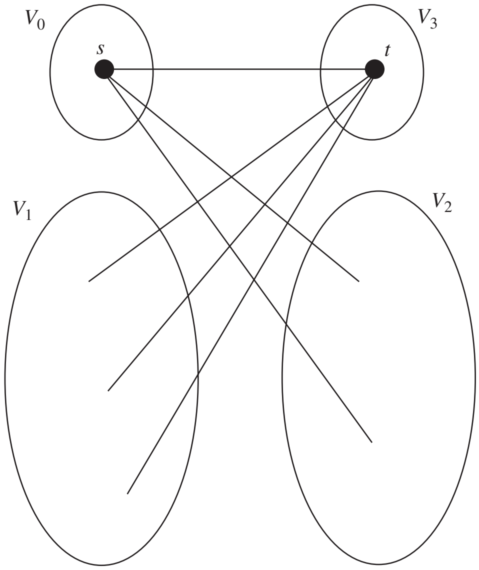


Fig. 1. Support graph of a 2-path-cut inequality.

Note that the 2-path-cut $T$ intersects each 2-*st*-path in exactly one edge. Let $E_1$ be the set of edges involved in a 2-*st*-path in $G$. Thus, $E_1$ consists of the edges in $[s,t]$ and $[s,v]$, $[v,t]$ for all those nodes $v$ for which $G$ contains these edges. Let $G_1=(N,E_1)$ be the subgraph of $G$ induced by $E_1$.

Observe that it is equivalent to consider the *k*HPP on $G$ and on $G_1$. More precisely, an optimal solution in $G$ will consist of an optimal solution in $G_1$, plus the edges in $E\backslash E_1$ of negatives costs, if any.

Also, it is not hard to see that $T$ (in $G$) corresponds to the *st*-cut $\delta(V_0 \cup V_1)$ in $G_1$. Therefore, we will consider the inequalities

$$x(\delta_{G_1}(W)) \geqslant k \quad \text{for all } W \subset N, \;\; s \in W, \;\; t \notin W, \qquad (4)$$

where $\delta_{G_1}(W)$ stands for the cut induced by $W$ in $G_1$. Clearly, inequalities (4) dominate inequalities (1) and (3).

Let $Q_k(G)$ be the solution set of the system given by inequalities (2), (4). In the next section, we show that inequalities (2), (4), together with the integrality constraints, give an integer programming formulation for the *k*HPP. In Section 3, we study the *k*HPP polytope, $P_k(G)$, and show that $P_k(G)=Q_k(G)$. In Section 4, we discuss the polynomial time solvability of the problem and give some concluding remarks.

## 2. Formulation

In this section, we show that the trivial inequalities and inequalities (4), together with the integrality constraints, suffice to formulate the *k*HPP as a $0-1$ linear program. To this aim, we first give a lemma. Its proof can be found in [6].

**Lemma 1.** *Let $G=(N,E)$ be a graph, and $s$, $t$ two nodes of $N$. Suppose that there do not exist $k$ edge-disjoint 2-st-paths in $G$, with $k \geqslant 2$. Then there exists a set of at most $k-1$ edges that intersects every 2-st-path.*

**Theorem 2.** *Let $G=(N,E)$ be a graph and $k \geqslant 2$. Then the kHPP is equivalent to the integer program*

$$\text{Min}\left\{cx;\; x \in Q_k(G),\; x \in \{0,1\}^E\right\}.$$

**Proof.** To prove the theorem, it is sufficient to show that every $0-1$ solution $x$ of $Q_k(G)$ induces a solution of the *k*HPP. Let us assume the contrary. Suppose that $x$ does not induce a solution of the *k*HPP. If $x$ does not satisfy an *st*-cut inequality, then clearly one of inequalities (4) is not satisfied. So suppose that $x$ satisfies the *st*-cut and trivial constraints. We will show that $x$ necessarily violates at least one inequality (4) which corresponds to a 2-path-cut constraint. Let $G_x$ be the subgraph induced by $x$. As $x$ is not a solution of the problem, $G_x$ does not contain $k$ edge-disjoint 2-*st*-paths. It then follows, by Lemma 1, that there exists a set of at most $k-1$ edges in $G_x$ that intersects every 2-*st*-path. Consider the graph $\tilde{G}_x$ obtained from $G_x$ by deleting these edges. Obviously, $\tilde{G}_x$ does not contain any 2-*st*-path. We claim that $\tilde{G}_x$ contains at least one *st*-path of length at least 3. In fact, as $x$ is a $0-1$ solution and satisfies the *st*-cut inequalities, $G_x$ contains at least $k$ edge-disjoint *st*-paths. Since at most $k-1$ edges were removed from $G_x$, at least one path remains between $s$ and $t$ in $\tilde{G}_x$. However, since $\tilde{G}_x$ does not contain a 2-*st*-path, that path must be of length at least 3.

Now consider the partition $V_0,\ldots,V_3$ of $N$, with $V_0=\{s\}$, $V_i$ the set of nodes at distance $i$ from $s$ in $\tilde{G}_x$, for $i=1,2$, and $V_3=N\backslash(\bigcup_{i=0}^{2} V_i)$, where the distance between two nodes is the length of a shortest path between these nodes. Since there does not exist a 2-*st*-path in $\tilde{G}_x$, it is clear that $t \in V_3$. Moreover, as, by the claim above, $\tilde{G}_x$ contains an *st*-path of length at least 3, the sets $V_1$, $V_2$ are nonempty. Furthermore, no edge of $\tilde{G}_x$ is a chord of the partition (that is, an edge between two sets $V_i$ and $V_j$, where $|i-j|>1$). In fact, suppose that there exists an edge $e=v_iv_j \in [V_i,V_j]$ with $|i-j|>1$ and $i<j$. Then $v_j$ is at distance $i+1$ from $s$, a contradiction.

Thus, the edges deleted from $G_x$ are the only edges that may be chords of the partition in $G_x$. In consequence, if $T$ is the set of chords of the partition in $G$, then $x(T) \leqslant k-1$. But this implies that the 2-path-cut inequality induced by $T$, and hence the corresponding inequality (4), are violated by $x$. $\square$

## 3. Facets and completeness

In this section, we will show that inequalities (2), (4) completely describe the polytope $P_k(G)$. In order to give a minimal system for this polytope, we first study when these inequalities are facet defining. We suppose familiarity with polyhedral theory. For specific details, the reader is referred to [7].

### 3.1. Facets

We first establish the dimension of $P_k(G)$. An edge $e \in E$ will be called *essential* if $e$ belongs either to an *st*-cut of cardinality $k$, or to a 2-path-cut of cardinality $k$. Let $E^*$ denote the set of essential edges. Thus, $P_k(G-e)=\emptyset$ for all $e \in E^*$. We have the following theorem, which is easily seen to be true.

**Theorem 3.** $\dim(P_k(G)) = |E| - |E^*|$.

Throughout this section, $G=(N,E)$ is a complete graph with $|N| \geqslant k+2$, which may contain multiple edges. Hence any *st*-cut and 2-path-cut of $G$ contains at least $k+1$ edges. Therefore, by Theorem 3, $P_k(G)$ is full dimensional.

**Theorem 4.** (i) *Inequality* $x(e) \leqslant 1$ *defines a facet of* $P_k(G)$ *for all* $e \in E$.

(ii) *Inequality* $x(e) \geqslant 0$ *defines a facet of* $P_k(G)$ *if and only if* $|N| \geqslant k+3$, *or* $|N|=k+2$ *and* $e$ *does not belong to either an st-cut or a* 2*-path-cut with exactly* $k+1$ *edges*.

**Proof.** (i) As $|N| \geqslant k+2$, and $P_k(G)$ is full dimensional, $E_f = E\backslash\{f\}$ is a solution for the *k*HPP for every $f \in E\backslash\{e\}$. Hence the sets $E$ and $E_f$ for $f \in E\backslash\{e\}$ constitute a family of $|E|$ solutions of *k*HPP. Moreover, their incidence vectors satisfy $x(e)=1$, and are affinely independent.

(ii) Suppose first that $|N| \geqslant k+3$. Then $G$ contains $k+2$ node-disjoint 2-*st*-paths (an edge of $[s,t]$ and $k+1$ paths of the form $(s,u,t)$, $u \in N\backslash\{s,t\}$). Hence any edge set $E\backslash\{f,g\}$, $f,g \in E$, contains at least $k$ paths among these $k+2$ 2-*st*-paths. Consider the sets

$$E_f = E\backslash\{e,f\} \quad \text{for all } f \in E\backslash\{e\}.$$

By the above remark, these sets induce solutions of *k*HPP. Now, it is easy to see that the incidence vectors of $E\backslash\{e\}$ and $E_f$, $f \in E\backslash\{e\}$, all satisfy $x(e)=0$ and are affinely independent.

Now suppose that $|N|=k+2$. If $e$ belongs to an *st*-cut $\delta(W)$ (resp. a 2-path-cut $T$) with $k+1$ edges, then $x(e) \geqslant 0$ is redundant with respect to the inequalities

$$\begin{aligned} &x(\delta(W)) \geqslant k \quad (\text{resp. } x(T) \geqslant k),\\ &-x(f) \geqslant -1 \quad \text{for all } f \in \delta(W)\backslash\{e\}\\ &\qquad\qquad\qquad (\text{resp. } f \in T\backslash\{e\}),\end{aligned}$$

and hence, cannot be facet defining.

If $e$ does not belong to neither an *st*-cut nor a 2-path-cut with $k+1$ edges, then the edge sets $E\backslash\{e\}$ and $E_f$, $f \in E\backslash\{e\}$, introduced above, are still solutions for the problem. Moreover, their incidence vectors satisfy $x(e)=0$ and are affinely independent. □

**Theorem 5.** *Constraints* (4) *define facets for* $P_k(G)$.

**Proof.** Let us denote inequality (4) by $ax \geqslant \alpha$, and let $bx \geqslant \beta$ be a facet defining inequality of $P_k(G)$ such that

$$\{x \in P_k(G);\, ax=\alpha\} \subseteq \{x \in P_k(G);\, bx=\beta\}.$$

We will show that $b=\rho a$ for some $\rho>0$. Let $T=\delta_{G_1}(W)$, $W_1 = W\backslash\{s\}$ and $W_2=\overline{W}\backslash\{t\}$. Let $\overline{E}=E\backslash T$. As $|N| \geqslant k+2$, $|W_1|+|W_2| \geqslant k$. So consider $k$ nodes $v_1,\dots,v_k \in W_1 \cup W_2$. Suppose that $v_1,\dots,v_q \in W_1$ and $v_{q+1},\dots,v_k \in W_2$ for some $0 \leqslant q \leqslant k$. Let $e_i \in [v_i,t]$ for $i=1,\dots,q$ and $e_i \in [s,v_i]$ for $i=q+1,\dots,k$. Let

$$F_1 = \{e_1,\dots,e_k\} \cup \overline{E}.$$

It is clear that $F_1$ induces a solution of the *k*HPP. Let $e \in T\backslash\{e_1,\dots,e_k\}$. If $e$ is parallel to one of the edges $e_i$, say $e_1$, then clearly $F_1'=(F_1\backslash\{e_1\})\cup\{e\}$ still induces a solution for the problem. Since $ax^{F_1}=ax^{F_1'}=\alpha$, we get $b(e_1)=b(e)$. This implies that

$$b(f)=\rho_i \quad \text{for every } f \text{ parallel to } e_i \quad \text{for some } \rho_i \in \mathbb{R} \quad \text{for } i=1,\dots,k. \tag{5}$$

If $e$ is not parallel to any $e_j$, then $F_j'=(F_1\backslash\{e_j\})\cup\{e\}$ induces a solution for the *k*HPP, for $j=1,\dots,k$. As an edge of $[s,t]$ is such an edge, this together with (5) implies that, for some $\rho \in \mathbb{R}$,

$$b(e)=\rho \quad \text{for every edge } f \in T. \tag{6}$$

Now we shall show that $b(f)=0$ for all $f \in \overline{E}$. Suppose $f \in [s,W_1]$. If $f$ is not incident to any node among $v_1, \ldots, v_q$, then $F_1 \backslash \{f\}$ induces a solution of the problem, and hence, $b(e) = 0$. If $f \in [s, v_i]$ for some $1 \leqslant i \leqslant q$, then let

$$\tilde{F}_1 = (F_1 \backslash \{f, e_i\}) \cup \{g\},$$

where $g$ is an edge of $[s, t]$. It is easy to see that $\tilde{F}_1$ still induces a solution of the $k$HPP. As, by (6), $b(e_i) = b(g)$, it follows that $b(f) = 0$. Similarly, we can show that $b(f) = 0$ for all $f \in [W_2, t]$. If $f \in [W_1, W_2] \cup E(W_1) \cup E(W_2)$, then obviously $F_1 \backslash \{f\}$ is a solution of the problem, and hence, we obtain that $b(f) = 0$.

Thus we have that

$$\begin{array}{ll} b(e) = \rho & \text{if } e \in T, \\ b(e) = 0 & \text{if not.} \end{array}$$

Since $ax \geqslant \alpha$ is not a trivial inequality, we have that $\rho > 0$, and hence, that $b = \rho a$. $\square$

### 3.2. The polytope $P_k(G)$

We now consider the polytope $P_k(G)$ and we give a complete linear description of this polytope. As mentioned in the Introduction we may use an idea in [1] which was used to find a complete linear description of the dominant of $P_1(G)$. The idea is based on the fact that the only edges $e \in E$ that can lie in a 2-*st*-path are those in $[s, t]$, and $[s, v]$, $[v, t]$ for $v \neq s, t$, that is, the edges of $E_1$. Thus, essentially, the remaining edges play no role. Our proof uses this reduction combined with a well-known result on edge-disjoint paths.

A linear system $Ax \leqslant b$ is *totally dual integral* (TDI) if the minimum in the LP duality relation $\max \{c^T x : Ax \leqslant b\} = \min \{y^T b : y^T A = c^T, y \geqslant 0\}$ has an integral optimal solution for all integral $c$ such that the minimum is finite. In what follows, we give a TDI system that characterizes $P_k(G)$.

**Theorem 6.** *The system given by* (2) *and* (4) *completely describes the polytope* $P_k(G)$. *Moreover, this system is TDI.*

**Proof.** Observe first that $P_k(G)$ is the product of the polytope $P_k(G_1)$, with $G_1 = (N, E_1)$ as defined in the Introduction, and $[0, 1]^{E \backslash E_1}$. Moreover, in $G_1$ *every* *st*-path is a 2-*st*-path. Thus, $P_k(G_1)$ equals the solution set of the system

$$\begin{array}{lll} x(\delta_{G_1}(W)) \geqslant k & \text{for all } W \subseteq N, \quad s \in W, \quad t \notin W, \\ 0 \leqslant x(e) \leqslant 1 & \text{for all } e \in E. \end{array}$$

This is a direct consequence of a well-known result on edge-disjoint paths (a recent reference is [8, p. 204]). Moreover, the system is TDI. The theorem now follows by noting that the TDI property extends to the product of the two polytopes (this is immediate from the definition of TDI). $\square$

From Theorems 4–6 we have the following.

**Corollary 7.** *If* $G = (N, E)$ *is complete and* $|N| \geqslant k + 2$, *a minimal system describing* $P_k(G)$ *is the following*.

$$\begin{array}{ll} x(\delta_{G_1}(W)) \geqslant k & \textit{for all } W \subset N, \ s \in W, \ t \notin W, \\ x(e) \leqslant 1 & \textit{for all } e \in E, \\ x(e) \geqslant 0 & \textit{for all } e \in E \textit{ satisfying condition} \\ & \quad \text{(ii) } \textit{of Theorem } 4. \end{array}$$

## 4. Solvability and concluding remarks

The separation problem for a system of inequalities consists in verifying whether a given solution $x^* \in \mathbb{R}^E$ satisfies the system and, if not, in finding an inequality of the system that is violated by $x^*$. The separation problem for inequalities (4) can be solved in polynomial time using any polynomial max-flow algorithm (e.g., [5]). Therefore, the $k$HPP can be solved in polynomial time using a cutting plane algorithm. Note that, if the graph has no parallel edges, the problem can also be solved polynomially by enumerating the (at most) $|N| - 1$ different 2-*st*-paths in $G$ and picking the $k$ of these paths with smallest cost.

Also note that the polynomial cutting plane algorithm can be used for solving the node-disjoint case, that is to find a minimum cost subgraph containing at least $k$ node-disjoint 2-*st*-paths. In fact, for this problem, we can suppose that the underlying graph does not contain multiple edges. In consequence, two 2-*st*-paths are node-disjoint if and only if they are edge-disjoint. Therefore, the system given in Theorem 7 is also a minimal description of the associated polytope.

A natural extension of the $k$HPP is to consider paths of length at most $L$ where $L$ is a fixed integer. The case studied in this paper corresponds to the case where $L = 2$. Inequalities (3) can be easily extended to the case $L \geqslant 3$ (see, [1]). Moreover, as it is shown in [4], they can be separated in polynomial time when $L \leqslant 3$. In [6], it is shown that these inequalities and the *st*-cut inequalities, together with the trivial inequalities, completely describe the corresponding polytope when $k = 2$ and $L = 2, 3$. It would be interesting to study the polytope associated with the $k$HPP when $L = 3$, and see whether these inequalities suffice to describe the polytope in this case.

Another generalization to consider is when $k$ edge-disjoint paths of length at most $L$ are required relatively to several demands. Once again, the results obtained here will initiate our approach to this new problem. For example, the formulation can be derived directly by writing the previous constraints for each demand together. Our goal is to determine new classes of facet defining inequalities for this more general problem in order to solve it with an efficient Branch and Cut algorithm.

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