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On a Composition of Independence Systems by Circuit Identification*

R. EULER

Laboratoire d'Informatique de Brest (LIBR), Université de Bretagne Occidentale, 6 Avenue Le Gorgeu, 29287 Brest Cedex, France

AND

A. R. MAHJOUB

Department of Statistics, King Saud University, Riyadh, Saudi Arabia

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The composition of general bipartite subgraph respectively acyclic subdigraph independence systems and in particular of their associated polyhedra by the identification of a pair of 3-cycles resp. 2-dicycles together with its implications for an algorithmic treatment has been the central subject of recent papers. We generalize this kind of composition within the framework of independence systems having a certain exchange property with respect to one of their circuits, and extend it to the case of independence systems associated with K_3-covers of a graph. We discuss its implications for associated polyhedra, totally dual integral linear systems describing these as well as related optimization problems. As a special result we obtain that the K_3-cover problem is polynomially solvable in graphs not contractible to $K_5 - e$. Particular attention is also given to independence systems, which are linearly relaxable (with respect to their circuits), i.e., for which the circuit inequalities $x(C) \leqslant |C| - 1$ together with the trivial inequalities $0 \leqslant x_e \leqslant 1$ are sufficient to describe $P(\mathscr{I})$, the convex hull of the incidence vectors of members of $\mathscr{I}$.

1. INTRODUCTION

The concept of an *independence system* has turned out to be suitable to embrace a large variety of combinatorial structures. It is defined to be an ordered pair $(E, \mathscr{I})$ consisting of a finite ground set E and a nonempty

* This paper was written while both authors were at Artemis (IMAG), University of Grenoble, St. Martin d'Hères, France.

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system $\mathcal{I}$ of subsets of E, which satisfies the following condition of "subclusiveness":

$$J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}.$$

We call the members of $\mathcal{I}$ *independent* sets and those of $2^E \setminus \mathcal{I}$ *dependent* sets, 2^E denoting the power set of E. A maximal independent set is a *base* and a minimal dependent set a *circuit* of $(E, \mathcal{I})$. The corresponding set systems are denoted by $\mathcal{B}$ and $\mathcal{C}$. Throughout the following we restrict ourselves to *normal* independence systems, i.e., those for which $\{e\} \in \mathcal{I}$ for all $e \in E$. The *restriction* $(X, \mathcal{I}^X)$ of $(E, \mathcal{I})$ to $X \subseteq E$ is the independence system with $\mathcal{I}^X = \mathcal{I} \cap 2^X$. With any independence system we may associate an optimization problem, the *independent set problem* (*ISP*) as follows:

> Given real weights w_e on the elements of E find an independent set I which has maximum total weight $w(I) = \sum_{e \in I} w_e$.

In general, this problem is NP-hard but if we restrict ourselves to special classes of independence systems, as for instance matroids, 2-matroid-intersections, or those given by matchings in a graph, the resulting problem is polynomially solvable provided one can test in polynomial time whether a given set I is independent or not. Very often the validity of these algorithms could be shown by using linear programming tools such as complementary slackness or the duality theorem. Necessary for such an approach is an explicit knowledge of the associated polytope $P(\mathcal{I})$, defined to be the convex hull of the incidence vectors of all members of $\mathcal{I}$. Whereas the pioneering results have been obtained by establishing linear systems sufficient to describe $P(\mathcal{I})$ for selected classes of independence systems, recent efforts have been made to investigate independence systems, whose associated polytopes can be described in a straightforward way. One such class is given by independence systems $(E, \mathcal{I})$, for which the circuit inequalities $x(C) \leqslant |C| - 1$, $C \in \mathcal{C}$, and the trivial inequalities $0 \leqslant x_e \leqslant 1$, $e \in E$, are sufficient to describe $P(\mathcal{I})$. In such a case we will speak of a *linearly relaxable* independence system. By definition, examples are given by bipartite subgraph systems in weakly bipartite graphs (cf. Grötschel and Pulleyblank [8]) or acyclic subdigraph systems in weakly acyclic digraphs (cf. Grötschel, Jünger, and Reinelt [7]). Both of these are nontrivial examples since the first class contains as special case bipartite subgraph systems over graphs not contractible to K_5 (cf. Fonlupt *et al.* [6]) and the second class contains acyclic subdigraph systems over graphs not contractible to $K_{3,3}$ (cf. Barahona and Mahjoub [3]). Another well known independence system of this type is that given by the stable sets in a bipartite graph.

Recently, the composition of independence systems arising from bipartite subgraphs of graphs (cf. Fonlupt *et al.* [6]) or from acyclic subdigraphs of digraphs (cf. Barahona and Mahjoub [3]) has been studied. For the first class a method has been presented, which determines a bipartite subgraph of maximum weight in a graph G obtained by composing two given graphs G_1 and G_2 in a specific way. Moreover, a system of linear inequalities (in short, a *linear system*) sufficient to describe the associated polytope $P(\mathscr{I})$ has been shown to be obtainable from two such systems describing $P(\mathscr{I}_1)$ and $P(\mathscr{I}_2)$. For the special case of a "2-sum" $P(\mathscr{I})$ could even be characterized by a minimal such system. Reference [6] also contains a discussion of the implications for weakly bipartite graphs. Acyclic subdigraph systems in digraphs have been studied much along the same line in [3].

We point to the fact that in both cases an "exchange property with respect to a circuit" is satisfied, which we are going to study in full detail in Section 2. It was especially this property that gave us the motivation to write this paper. Its organization is similar to that of [3, 6]. Section 2 is devoted to a discussion of the "exchange property" and, in particular, the composition of independence systems having this property with respect to one of its circuits. Section 3 contains our results on the composition of associated polyhedra. Algorithmic aspects are treated in Section 4, and our results on the K_3-cover problem in graphs are presented in Section 5. Results on minimal linear systems sufficient to describe the polyhedra and on totally dual integral such systems are summarized in the final section.

2. Composing Independence Systems by Circuit Identification

We are now going to introduce independence systems, whose systems of circuits share a certain "exchange property" with respect to one of its circuits. Given an independence system $(E, \mathscr{C})$ and a circuit $C \in \mathscr{C}$, which we call a *distinguished circuit*, we say that *$\mathscr{C}$ has the exchange property with respect to C*, if the following condition holds:

> For all $C_1, C_2 \in \mathscr{C}$ such that $C \cap C_1$ and $C \cap C_2$ constitute a partition of C there exists a circuit $C_3 \in \mathscr{C}$ that is contained in $(C_1 \cup C_2) \backslash C$. (1)

Using a simple counting argument it can be shown that this property is shared by elementary cycles of odd cardinality in a graph with respect to any such cycle as the distinguished one. A similarly easy argument based on the orientations of edges can be used to verify the property for elementary dicycles and any distinguished one in a digraph.

For general stable set independence systems over graphs we have

PROPOSITION (2.1). *Let $G=[V,E]$ be a finite, undirected graph, let $K_i=[V_i,E_i]$ for $i=1,\ldots,m$ denote its connected components, and let $(V,\mathscr{I})$ be the independence system given by all stable sets of G with E as its system of circuits. Then E satisfies condition* (1) *with respect to all of its edges if and only if the components K_i induce complete bipartite graphs.*

Proof. If K_i does not contain an odd elementary cycle let (V_i^1, V_i^2) be a partition of V_i such that every edge from E_i connects an element of V_i^1 to one of V_i^2. If the nodes $v_1, v_2 \in V_i^1, V_i^2$, respectively, are not connected by such an edge, by connectedness of K_i and the circuit-exchange property we obtain a contradiction. If K_i does contain an odd elementary cycle C of length say $2k+1$, $k \geqslant 2$, then again by the circuit-exchange property applied to a halfcycle of C having 4 nodes we obtain an elementary cycle of length $2k-1$ and repeated application shows the existence of a triangle in K_i, a contradiction to our circuit-exchange property and the fact that the independence systems we are considering are normal. ∎

A further example arises from K_3-*covers* in a graph $G=[V,E]$, i.e., those subsets F of E which contain at least one edge from every triangle in G. Clearly, the collection of all (triangle-free) sets $E\backslash F$ induces an independence system over E which we denote by $(E, \Delta(G))$ and whose system $\mathscr{C}$ of circuits is exactly the collection of edge sets of triangles in G. Clearly, $\mathscr{C}$ has the exchange property with respect to all triangles. For a comprehensive study of K_i-covers, related optimization problems, and polyhedra see Conforti *et al.* [4]. Also note that triangle-free graphs play a role in finding a maximum weight clique in certain classes of graphs (cf. Balas *et al.* [1]).

Assume now we are given two independence systems $(E_1, \mathscr{I}_1)$, $(E_2, \mathscr{I}_2)$, both having property (1) with respect to $C_1 \in \mathscr{C}_1$, $C_2 \in \mathscr{C}_2$ as distinguished circuits, respectively, such that the following conditions hold:

there exist partitions (S, T) of C_1 $[(U, V)$ of $C_2]$ such that for every circuit $C \in \mathscr{C}_1 \backslash C_1$ $[C \in \mathscr{C}_2 \backslash C_2]$ the set $C \cap C_1$ $[C \cap C_2]$ is either empty or coincides with S or T $[U$ or $V]$; (2)

the sets S and U (T and V) have the same cardinality. (3)

Now let $\mathscr{C}'_2$ be the family obtained from $\mathscr{C}_2$ by identifying the elements of U (V) with those of S (T) and consider the independence system $(E_3, \mathscr{C}_3)$ defined as

$$E_3 := E_1 \cup (E_2 \backslash C_2),$$

$$\mathscr{C}_3 := \mathscr{C}_1 \cup \mathscr{C}'_2 \cup \mathscr{C}'_3,$$

where

$$\mathscr{C}_3' = \{((C \cup D)\backslash C_1) : C \in \mathscr{C}_1\backslash C_1, D \in \mathscr{C}_2'\backslash C_1, C_1 \subseteq C \cup D\}.$$

It is easily seen that $(E_3, \mathscr{C}_3)$ again has properties (1) and (2) with respect to C_1. However, is $(E_3, \mathscr{C}_3)$ a proper generalization of the bipartite subgraph independence system over the 2-sum of two given graphs?

To see this consider the 2-sum of two graphs $G_1 = [V_1, E_1]$, $G_2 = [V_2, E_2]$ with respect to the edges e_1, e_4 as illustrated in Fig. (2.2). The elementary odd cycles in this 2-sum are of 3 types:

(i) an odd elementary cycle in G_1;

(ii) an odd elementary cycle in G_2;

(iii) the union of the edge set of an odd elementary path P_o (connecting the endpoints of e_1) in G_1 and that of an even elementary path P_e (connecting the endpoints of e_4) in G_2, or vice versa.

Now, if P_o is such an odd path in G_1, its edge set plus the edge e_1 forms an even elementary cycle, which is not a circuit of the associated bipartite subgraph independence system. If, however, we add edges e_2, e_3 to form a triangle with e_1, we can complete P_o to an odd elementary cycle. Moreover, if $(E_1, \mathscr{C}_1)$ denotes the bipartite subgraph independence system over the modification of G_1, it can be verified that $(E_1, \mathscr{C}_1)$ satisfies conditions (1) and (2) with respect to the triangle $\{e_1, e_2, e_3\}$. A similar reasoning holds for G_2 and two new edges e_5, e_6. Finally, we form the 3-sum of the two modified graphs by identifying the 2 triangles (cf. Fig. (2.2)) and we observe that the collection of odd elementary cycles in this 3-sum

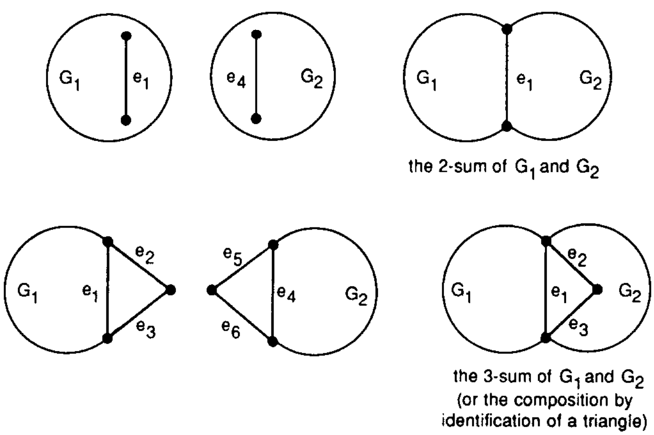

FIGURE (2.2)

corresponds exactly to the collection of circuits as defined by our composition. For further details we refer the reader to [5].

The same idea of identifying two triangles can be used for the composition of two independence systems $(E_1, \Delta(G_1))$, $(E_2, \Delta(G_2))$ arising from G_1 and G_2. Note that in this case the system $\mathscr{C}_3'$ will be empty. The approach undertaken by Barahona and Mahjoub in [3] for the acyclic subdigraph independence system is exactly along the same line.

To conclude this section we would like to formulate a result, which is of a more technical nature and which will be useful for many of the proofs to follow. For this let $(E, \mathscr{I})$ be an independence system having $\mathscr{C}$ as a system of circuits. Moreover, let $C \in \mathscr{C}$ and (S, T) be a partition of C such that (1) and (2) hold with respect to $(E, \mathscr{I})$, C, and (S, T). Then we have

LEMMA (2.3). *Any base of* $(E, \mathscr{I})$ *contains* $|C| - 1$ *many elements of* C.

Proof. Suppose there is a base $B \in \mathscr{B}$ containing less than $|C| - 1$ many elements from C.

Case 1. $S \subseteq B$ or $T \subseteq B$. Suppose $S \subseteq B$, i.e., $|B \cap T| < |T| - 1$ by supposition. Since B is a base, there is a circuit in $B \cup \{e\}$ for every $e \in E \setminus B$. So let $e_o \in T \setminus B$ and C' be a circuit contained in $B \cup \{e_o\}$. Then C' contains at most $|T| - 1$ many elements from C, a contradiction to property (2) with respect to C and (S, T).

Case 2. $S \nsubseteq B$ and $T \nsubseteq B$. Let $e_1 \in S \setminus B$ and $e_2 \in T \setminus B$. If now $(B \cap S) \cup \{e_1\} \neq S$ or $(B \cap T) \cup \{e_2\} \neq T$, we can use the same argument as in Case 1. Therefore, we may suppose that

$$(B \cap S) \cup \{e_1\} = S \qquad \text{and} \qquad (B \cap T) \cup \{e_2\} = T.$$

Since B is a base, there are two circuits C' and C'' such that

$$C' \subseteq B \cup \{e_1\} \qquad \text{(and, moreover, } S = C' \cap C\text{)},$$
$$C'' \subseteq B \cup \{e_2\} \qquad \text{(and, moreover, } T = C'' \cap C\text{)}.$$

By property (1) there exists a circuit C_3 in $(C' \cup C'') \setminus C$, a contradiction to the independence of B. ∎

3. COMPOSING ASSOCIATED POLYHEDRA

The main subject of this section will be the composition of polyhedra associated with independence systems as they have been studied in the last section. It will turn out that in particular the operation of "mixing" circuits in Section 2 has a straightforward counterpart in terms of linear inequalities. First of all let us introduce some basic notions from polyhedral theory.

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of a finite number of halfspaces in $\mathbb{R}^m$, i.e., sets of points of the form $\{x \in \mathbb{R}^m : a^T x \leqslant a_0\}$, where $a \in \mathbb{R}^m \setminus \{0\}$ and $a_0 \in \mathbb{R}$. If P is bounded or, equivalently, the convex hull of finitely many points we speak of a *polytope*. A linear inequality $a^T x \leqslant a_0$ with $a \in \mathbb{R}^m \setminus \{0\}$, $a_0 \in \mathbb{R}$, is said to be *valid* for P, if $P \subseteq \{x \in \mathbb{R}^m : a^T x \leqslant a_0\}$. A subset F of P is called a *face* of P, if there exists an inequality $a^T x \leqslant a_0$ valid for P such that $F = \{x \in P : a^T x = a_0\}$. We also say that $a^T x \leqslant a_0$ *defines* F. A face F is called *proper*, if $F \neq P$. A proper, nonempty face maximal (minimal) with respect to set-inclusion is called a *facet* (*extreme point* or *vertex*) of P. Since we only deal with full-dimensional polytopes, any such P has a representation of the form $P = \{x \in \mathbb{R}^m : Ax \leqslant b\}$, where $Ax \leqslant b$ is unique up to positive multiples. We will particularly deal with $P(\mathscr{I})$, the convex hull of incidence vectors of independent sets in a given independence system $(E, \mathscr{I})$. For this and throughout the following an inequality of the type $x_e \geqslant 0$ or $x_e \leqslant 1$, $e \in E$, will be called a *trivial inequality*. Finally, expressions of the form $\sum_{e \in C} x_e$ will be abbreviated by $x(C)$.

So let $(E, \mathscr{I})$ be an independence system, $\mathscr{C}$ its system of circuits, $C \in \mathscr{C}$, and (S, T) a partition of C such that conditions (1) and (2) hold with respect to $(E, \mathscr{C})$, C, and (S, T). The following lemma is the first of a sequence of lemmata, leading to the main result of this section:

LEMMA (3.1). *If $a^T x \leqslant a_0$ is a nontrivial facet-defining inequality for $P(\mathscr{I})$, then $a_e \geqslant 0$ for all $e \in E$.*

Proof. Suppose $a_e < 0$ for some $e \in E$. Since $a^T x \leqslant a_0$ is different from the inequality $x_e \geqslant 0$, there exists an independent set $I \in \mathscr{I}$ such that $e \in I$ and $a^T x^I = a_0$. Now define I' to be $I \setminus \{e\}$. Clearly, $I' \in \mathscr{I}$. However, $a^T x^{I'} > a_0$, which contradicts the validity of $a^T x \leqslant a_0$ for $P(\mathscr{I})$. ∎

The next lemma concerns the *support* $s(a)$ of certain inequalities $a^T x \leqslant a_0$, which is defined to consist of all those elements $e \in E$, for which $a_e \neq 0$.

LEMMA (3.2). *The only facet-defining inequality for $P(\mathscr{I})$ having a support containing C is $x(C) \leqslant |C| - 1$. For any other nontrivial such inequality having support s the set $s \cap C$ is either empty or coincides with S or T, and the coefficients in $s \cap C$ have all the same value.*

Proof. Let us first show that the inequality $x(C) \leqslant |C| - 1$ defines a facet of $P(\mathscr{I})$. Clearly, it is valid for $P(\mathscr{I})$. Furthermore, by Lemma (2.3), for every $e \in E \backslash C$ there exists a base B_e of $(E, \mathscr{I})$ that contains e and $|C|-1$ many elements of C. We define

$$I_e = \begin{cases} C \backslash \{e\}, & \text{if } e \in C, \\ B_e \cap (C \cup \{e\}), & \text{if } e \in E \backslash C. \end{cases}$$

It is clear that for all $e \in E$ the sets I_e are independent, that their incidence vectors are linearly independent, and that they satisfy $x(C) \leqslant |C|-1$ with equality. Second, suppose that there is a facet-defining inequality for $P(\mathscr{I})$, say $a^T x \leqslant a_0$, which is different from $x(C) \leqslant |C|-1$ and whose support contains C. From Lemma (3.1) we know that $a_e \geqslant 0$ for all $e \in E$. We now claim that any independent set I, whose incidence vector x^I satisfies $a^T x \leqslant a_0$ with equality, also contains $|C|-1$ many elements of C. Suppose this is not the case for a particular I. Then by Lemma (2.3) there is a base B that contains I and at least one other element of $C \backslash I$. But since $a_e > 0$ for all $e \in C$ we obtain that $a^T x^B > a^T x^I = a_0$, a contradiction. This proves our claim. It now follows that the inequalities $a^T x \leqslant a_0$ and $x(C) \leqslant |C|-1$ define the same facet of $P(\mathscr{I})$, a contradiction. This completes the proof of the first part of our assertion.

For the second part let $a^T x \leqslant a_0$ be a facet-defining inequality for $P(\mathscr{I})$, different from $x(C) \leqslant |C|-1$, and having support $s(a)$. Suppose that $\varnothing \neq [s(a) \cap S] \neq S$, where $|S| \geqslant 2$ can be assumed, and let $e_0 \in s(a) \cap S$. Since $a^T x \leqslant a_0$ is nontrivial, there is an independent set I not containing e_0, whose incidence vector satisfies $a^T x^I = a_0$. By Lemma (2.3), I can be enlarged to a base B which contains $|C|-1$ many elements of C. Since $a_{e_0} > 0$, B does not contain e_0 and therefore $B \cup \{e_0\}$ must contain C.

Now let e_1 be an element in $S \backslash s(a)$. We claim that $I' := (B \backslash \{e_1\}) \cup \{e_0\}$ is independent. If this were not the case I' must contain a circuit C'. Since e_1 is not contained in C', by condition (2) it follows that $C' \cap S = \varnothing$, and thus $C' \subseteq B$, a contradiction. Now since $a_{e_0} > 0$ and $a_{e_1} = 0$, $a^T x^{I'} > a^T x^B = a_0$, a contradiction. Therefore, we can conclude that $s(a) \cap S$ is either empty or coincides with S (or T by symmetry).

By a similar reasoning we can show that the coefficients a_e for $e \in s(a) \cap S$ have all the same value. ∎

Using Lemma (3.2) the two polytopes $P(\mathscr{I}_1)$ and $P(\mathscr{I}_2)$ can be described by linear systems having the following form [*Note.* $E_i' := E_i \backslash C_i$ for $i = 1, 2$; (S, T), (U, V) are the partitions of the distinguished circuits C_1, C_2, respectively; K_1 is the index set for all those inequalities, whose support has empty intersection with C_1, K_2 is the index set when the intersection equals S, and K_3 is used for the case that the intersection equals T; the L_i's are defined similarly.]:

$$P(\mathscr{I}_1)\begin{cases}\sum_{e\in E_1'} a_e^i x_e \leqslant a_0^i & \text{for } i\in K_1 & (3.3)\\ \sum_{e\in E_1'} a_e^i x_e + x(S) \leqslant a_0^i & \text{for } i\in K_2 & (3.4)\\ \sum_{e\in E_1'} a_e^i x_e + x(T) \leqslant a_0^i & \text{for } i\in K_3 & (3.5)\\ x(C_1)\leqslant |C_1|-1 & & (3.6)\\ 0\leqslant x_e\leqslant 1 & \text{for all } e\in E_1 & (3.7)\end{cases}$$

$$P(\mathscr{I}_2)\begin{cases}\sum_{e\in E_2'} b_e^j x_e \leqslant b_0^j & \text{for } j\in L_1 & (3.8)\\ \sum_{e\in E_2'} b_e^j x_e + x(U) \leqslant b_0^j & \text{for } j\in L_2 & (3.9)\\ \sum_{e\in E_2'} b_e^j x_e + x(V) \leqslant b_0^j & \text{for } j\in L_3 & (3.10)\\ x(C_2)\leqslant |C_2|-1 & & (3.11)\\ 0\leqslant x_e\leqslant 1 & \text{for all } e\in E_2. & (3.12)\end{cases}$$

Now let $(E_3, \mathscr{I}_3)$ be the composition of $(E_1, \mathscr{I}_1)$ and $(E_2, \mathscr{I}_2)$ as defined in Section 2. In order to establish a linear system sufficient to describe $P(\mathscr{I}_3)$, we need to introduce an additional set of inequalities.

DEFINITION (3.13). Given a constraint of the type (3.4) and one of type (3.10) (resp. one of type (3.5) and (3.9)) the associated mixed constraint is given by

$$\sum_{e\in E_1'} a_e^i x_e + \sum_{e\in E_2'} b_e^j x_e \leqslant a_0^i + b_0^j - (|C_1|-1). \tag{3.14}$$

An immediate consequence is

LEMMA (3.15). *Any mixed constraint* (3.14) *is valid for* $P(\mathscr{I}_3)$.

Proof. Let I be a member of $\mathscr{I}_3$. By Lemma (2.3), I can be enlarged to a base B of $(E_3, \mathscr{I}_3)$, containing $|C_1|-1$ many elements of C_1. We define $B_i := B\cap E_i$, $i=1, 2$. Obviously, $B_i\in\mathscr{I}_i$ for $i=1, 2$. Since x^{B_1}, x^{B_2} both satisfy the constraints for $P(\mathscr{I}_1)$, $P(\mathscr{I}_2)$ used to form the mixed constraint, x^I satisfies this latter constraint. ∎

In order to establish a linear system sufficient to describe $P(\mathscr{I}_3)$ and based upon those for $P(\mathscr{I}_1)$, $P(\mathscr{I}_2)$ we will introduce two "auxiliary" independence systems $(E_3, \mathscr{I}_1^*)$, $(E_3, \mathscr{I}_2^*)$, which are defined by the following systems of circuits:

$$\mathscr{C}_1^* = \{C \in \mathscr{C}_3 : T \nsubseteq C\}, \qquad \mathscr{C}_2^* = \{C \in \mathscr{C}_3 : S \nsubseteq C\}.$$

Polyhedral descriptions of these independence systems are now going to be presented. For this let us first present some technical lemmas.

LEMMA (3.16). *Given two inequalities of type* (3.4) *and* (3.5), *i.e.*,

$$\sum_{e \in E_1'} a_e^i x_e + x(S) \leqslant a_0^i,\ i \in K_2,$$

and

$$\sum_{e \in E_1'} a_e^j x_e + x(T) \leqslant a_0^j,\ j \in K_3,$$

both valid for $P(\mathscr{I}_1)$, *the mixed inequality*

$$\sum_{e \in E_1'} (a_e^i + a_e^j) x_e \leqslant a_0^i + a_0^j - (|C_1| - 1)$$

is also valid for $P(\mathscr{I}_1)$.

Proof. In analogy to the proof of Lemma (3.15) any $I \in \mathscr{I}_1$ can be enlarged to a base B_1 of E_1 having $|C_1| - 1$ many elements in common with C_1, the distinguished circuit. The assertion follows immediately. ∎

LEMMA (3.17). *Given an equality of type* (3.4) (*resp.* (3.5)), *i.e.*,

(i) $\sum_{e \in E_1'} a_e^i x_e + x(S) \leqslant a_0^i$, $i \in K_2$, (*resp.* (ii) $\sum_{e \in E_1'} a_e^j x_e + x(T) \leqslant a_0^j$, $j \in K_3$)

the inequality

(i′) $\sum_{e \in E_1'} a_e^i x_e \leqslant a_0^i - (|S| - 1)$, $i \in K_2$, (*resp.* (ii′) $\sum_{e \in E_1'} a_e^j x_e \leqslant a_0^j - (|T| - 1)$, $j \in K_3$),

is also valid for $P(\mathscr{I}_1)$. *A similar statement holds for* $P(\mathscr{I}_2)$ *with respect to a constraint of type* (3.9) (*resp.* (3.10)).

Proof. Let I be a member of $\mathscr{I}_1$. Then by Lemma (2.3), I can be enlarged to a base B of $(E_1, \mathscr{I}_1)$ containing $|C_1| - 1$ many elements of C_1. Since x^B satisfies (i) (resp. (ii)), x^I satisfies (i′) (resp. (ii′)). ∎

Since $\{I \in \mathscr{I}_1^* : I \subseteq E_1'\} \subset \mathscr{I}_1$ and $\{I \in \mathscr{I}_1^* : I \subseteq E_2'\} \subset \mathscr{I}_2$, it follows from Lemmata (3.16) and (3.17) that the following constraints are valid for $P(\mathscr{I}_1^*)$ and hence for $P(\mathscr{I}_3)$:

$$\begin{aligned}
&\sum_{e\in E_1'} a_e^i x_e \leqslant a_0^i - (|S|-1), && i\in K_2,\\
&\sum_{e\in E_1'} a_e^j x_e \leqslant a_0^j - (|T|-1), && j\in K_3,\\
&\sum_{e\in E_2'} b_e^j x_e \leqslant b_0^j - (|U|-1), && j\in L_2,\\
&\sum_{e\in E_2'} b_e^j x_e \leqslant b_0^j - (|V|-1), && j\in L_3,\\
&\sum_{e\in E_1'} (a_e^i + a_e^j) x_e \leqslant a_0^i + a_0^j - (|C_1|-1), && i\in K_2, j\in K_3\\
&\sum_{e\in E_2'} (b_e^i + b_e^j) x_e \leqslant b_0^i + b_0^j - (|C_1|-1), && i\in L_2, j\in L_3.
\end{aligned} \tag{3.18}$$

Now, for $(E_3, \mathscr{I}_1^*)$, let P_1^* be the polyhedron defined by all constraints of type (3.3), (3.4), (3.7), (3.8), (3.9) (with U replaced by S), (3.12) (for $e\in E_2'$), (3.18), and the collection of all possible mixed constraints (3.14).

We will first show that $P(\mathscr{I}_1^*)$ equals P_1^*. By symmetry, we also obtain a linear system sufficient to describe $P(\mathscr{I}_2^*)$ and finally, we will prove that both these linear systems together with the circuit inequality $x(C_1)\leqslant |C_1|-1$ provide a complete description of $P(\mathscr{I}_3)$.

For this two more technical lemmata are required:

LEMMA (3.19). *Any inequality $x(C)\leqslant |C|-1$ for $C\in\mathscr{C}_1^*$ is redundant with respect to the inequalities defining P_1^*.*

Proof. Let C be a circuit from $\mathscr{C}_1^*$. If $C\subseteq E_1\backslash T$ or $C\subseteq E_2'\cup S$ then the associated inequality is redundant with respect to (3.3), (3.4), (3.8), (3.9):

(i) if $C\subseteq E_1'$, for any $e\in T$ there is a set $I\subseteq C$ with cardinality $|C|-1$ such that $I\cup\{e\}\in\mathscr{I}_1^*$ (otherwise we would get a contradiction to condition (1)), and then $x(C)\leqslant |C|-1$ gives rise to a constraint of type (3.3) by sequentially applying Padberg's lifting procedure (cf. [10]);

(ii) if $C\cap S_1 = S$, by Lemma (3.2) and the lifting procedure our inequality $x(C)\leqslant |C|-1$ gives rise to a constraint of type (3.4).

The case that $C\subseteq E_2'\cup S$ is treated analogously.

(iii) If $C\cap E_1\neq\varnothing$ and $C\cap E_2'\neq\varnothing$ by definition of $(E_3,\mathscr{I}_3)$, C is obtained by mixing two circuits $C'\in\mathscr{C}_1$, $C''\in\mathscr{C}_2$, whose associated inequalities give rise to facet-defining inequalities for $P(\mathscr{I}_1)$ and $P(\mathscr{I}_2)$, respectively, as in cases (i) and (ii). These latter inequalities must be of type (3.4) and (3.10) (or (3.5) and (3.9)). But then the circuit inequality $x(C)\leqslant |C|-1$ can be obtained from a mixed constraint by setting its coefficients for $e\notin C$ to zero. ∎

LEMMA (3.20). *Let* $k \in \{1, 2\}$. *If* $\sum_{e \in E_3} a_e x_e \leqslant a_0$ *is a facet-defining inequality for* $P(\mathscr{I}_3)$ *such that* $\{e \in E_3 : a_e > 0\}$ *is contained in* E_k, *then* $\sum_{e \in E_k} a_e x_e \leqslant a_0$ *is facet-defining for* $P(\mathscr{I}_k)$.

Proof. Obvious.

LEMMA (3.21). *Let* x^* *be a feasible solution of the linear system defining* P_1^* *and let at least one of the inequalities* (3.4), (3.9), *or* $x(S) \leqslant |S|$ *be satisfied with equality by* x^*. *Moreover, let* $\delta := |S| - x^*(S)$ $(\geqslant 0)$, e_0 *be a distinguished element from* T *resp.* V, *and*

$$x_e^1 := \begin{cases} x_e^* & \text{for } e \in E_1 \backslash T, \\ 1 & \text{for } e \in T \backslash \{e_0\}, \\ \delta & \text{for } e = e_0 \end{cases}$$

$$x_e^2 := \begin{cases} x_e^* & \text{for } e \in E_2 \backslash V, \\ 1 & \text{for } e \in V \backslash \{e_0\}, \\ \delta & \text{for } e = e_0. \end{cases}$$

Then x^1, x^2 *are both feasible for* $P(\mathscr{I}_1)$, $P(\mathscr{I}_2)$. *Furthermore, a mixed constraint* (3.14) *is satisfied by* x^* *with equality iff the associated inequality of type* (3.4) *is satisfied with equality by* x^1 *and that of type* (3.10) *is satisfied with equality by* x^2. *A similar statement holds for the case that two inequalities of type* (3.5) *and* (3.9) *define the mixed inequality.*

Proof. Let us first show that $x^1 \in P(\mathscr{I}_1)$ (the proof for $x^2 \in P(\mathscr{I}_2)$ is similar). We claim that x^* has to satisfy the inequality $x(S) \geqslant |S| - 1$. Otherwise, the constraint assumed to be satisfied with equality by x^* would be of type (3.4) or (3.9). But then

$$\sum_{e \in E_1'} a_e^i x_e^* > a_0^i - (|S| - 1) \quad \text{or} \quad \sum_{e \in E_2'} b_e^j x_e^* > b_0^j - (|U| - 1), \text{ respectively.}$$

By Lemma (3.17) this contradicts the validity of these inequalities with ">" replaced by "$\leqslant$." Consequently, $0 \leqslant \delta \leqslant 1$ and thus the trivial inequalities (3.7) are satisfied by x^1. Clearly, the constraints (3.3), (3.4) are satisfied by x^1. To show that x^1 also satisfies the constraints of type (3.5), we consider two cases:

Case 1. $x^*(S) = |S|$. Then $\delta = 0$. Take a constraint of type (3.5), i.e.,

(iii) $\sum_{e \in E_1'} a_e^i x_e + x(T) \leqslant a_0^i$, $i \in K_3$.

By hypothesis, the constraint

(iv) $\sum_{e \in E_1'} a_e^i x_e \leqslant a_0^i - (|T| - 1)$

is satisfied by x^* and, by definition, also by x^1. Since $\delta = 0$, x^1 also satisfies (iii).

Case 2. $x^*(S) < |S|$. Now there is an inequality of type (3.4) or (3.9) which is satisfied by x^* with equality, e.g.,

(v) $\sum_{e \in E_1'} a_e^{i'} x_e^* + x^*(S) = a_0^{i'}$ for some $i' \in K_2$.

Since the inequality

$$\sum_{e \in E_1'} (a_e^{i'} + a_e^i) x_e \leqslant a_0^{i'} + a_0^i - (|C_1| - 1)$$

obtained by mixing inequality (iii) with the one corresponding to (v) is satisfied by x^*, we get

(vi) $\sum_{e \in E_1'} (a_e^{i'} + a_e^i) x_e^* \leqslant a_0^{i'} + a_0^i - (|C_1| - 1)$.

Subtracting (v) from (vi) we obtain

$$\sum_{e \in E_1'} a_e^i x_e^* - x^*(S) + |C_1| - 1 \leqslant a_0^i,$$

which is equivalent to

$$\sum_{e \in E_1'} a_e^i x_e^* + \delta + |T| - 1 \leqslant a_0^i.$$

This shows that x^1 satisfies (iii).

If, on the other hand, a constraint of type (3.9) is satisfied by x^* with equality, we can show by a similar reasoning that x^1 satisfies (iii).

Finally, suppose that a mixed constraint (3.14) is satisfied with equality by x^*, i.e.,

$$\sum_{e \in E_1'} a_e^i x_e^* + \sum_{e \in E_2'} b_e^j x_e^* = a_0^i + b_0^j - (|C_1| - 1).$$

Suppose w.l.o.g. that

$$\sum_{e \in E_1'} a_e^i x_e^1 + x^1(S) < a_0^i \qquad \text{or} \qquad \sum_{e \in E_2'} b_e^j x_e^2 + x^2(V) < b_0^j.$$

By definition of x^1 and x^2 this yields

$$\sum_{e \in E_1'} a_e^i x_e^* + x^*(S) + \sum_{e \in E_2'} b_e^j x_e^* + |S| + |T| - 1 - x^*(S) < a_0^i + b_0^j.$$

a contradiction. ∎

We are now able to state the key result of this section:

THEOREM (3.22). $P(\mathscr{I}_1^*) = P_1^*$.

Proof. By Lemma (3.19) a point $x \in P_1^*$ is integral iff it is the incidence vector of a member of $\mathscr{I}_1^*$. Since $P(\mathscr{I}_1^*)$ is contained in P_1^*, we only have to show that every extreme point of P_1^* is integral. For this suppose there is an extreme point x^* of P_1^* having at least one nonintegral component.

Case 1. There is an element $e' \in S$ such that $0 < x_{e'}^* < 1$; in this case x^* must satisfy at least one of the constraints (3.4), (3.9) with equality, for otherwise x^* would not define an extreme point of P_1^*. Let x^1, x^2 be defined as in Lemma (3.21), by which we have seen that $x^1 \in P(\mathscr{I}_1)$ and $x^2 \in P(\mathscr{I}_2)$. Consequently, there is a representation of the form

$$x^1 = \sum_{i=1}^{z} v_i x^{I_i^1}, \qquad x^2 = \sum_{j=1}^{s} \mu_j x^{I_j^2}$$

with

$$\sum_{i=1}^{z} v_i = 1,$$

$$\sum_{j=1}^{s} \mu_j = 1, \qquad v_i, \mu_j \geqslant 0, \text{ and } I_i^1 \in \mathscr{I}_1,\ I_j^2 \in \mathscr{I}_2 \text{ for all } i, j.$$

We note at this place that any constraint of $P(\mathscr{I}_1)$ resp. $P(\mathscr{I}_2)$ satisfied by x^1 resp. x^2 with equality is at the same time satisfied with equality by $x^{I_i^1}$, $i = 1, ..., z$ and $x^{I_j^2}$, $j = 1, ..., s$, respectively. In particular, since $x^1(C_1) = x^2(C_2) = |C_1| - 1$, it follows that every set I_i^1, $i = 1, ..., z$, and every set I_j^2, $j = 1, ..., s$, contains $|C_1| - 1$ many elements of C_1 and C_2, respectively.

Now since $x_{e'}^* < 1$, there must exist indices i_0 and j_0 such that $e' \notin I_{i_0}^1$ and $e' \notin I_{j_0}^2$ (note that we identify S with U and T with V). By the previous remark it follows that $T \subseteq (I_{i_0}^1 \cap I_{j_0}^2)$. Let $I := (I_{i_0}^1 \cup I_{j_0}^2) \backslash T$.

CLAIM 1. *I is a member of $\mathscr{I}_1^*$.*

Proof. Suppose this were not the case. Then there must be a circuit $C' \in \mathscr{C}_1^*$ such that $C' \subseteq I$. Since $C' \cap I_{i_0}^1 \neq \varnothing \neq C' \cap I_{j_0}^2$, it follows by definition of $\mathscr{C}_1^*$ that C' is obtained by mixing two circuits $C_1' \in \mathscr{C}_1$ and $C_2' \in \mathscr{C}_2'$, where w.l.o.g. $S \subseteq C_1'$ and $T \subseteq C_2'$. But then $I_{j_0}^2$ contains C_2', a contradiction.

CLAIM 2. *Any constraint satisfied by x^* with equality is at the same time satisfied with equality by x^I.*

Proof. This is immediately clear for constraints of type (3.3), (3.4), (3.8), (3.9), (3.18) as well as the trivial inequalities. So let us consider a mixed constraint (3.14). By Lemma (3.21) the two constraints used for getting the mixed one are satisfied with equality by x^1 and x^2. In addition, the incidence vectors of $I^1_{i_0}$ and $I^2_{j_0}$ satisfy the inequality $x(C_1) \leqslant |C_1| - 1$ with equality, since this is the case for x^1 and x^2. Therefore, x^I satisfies the mixed constraint with equality, too. Since $x^I \neq x^*$, x^* cannot define an extreme point, a contradiction.

Case 2. For all $e \in S$ we have $x^*_e = 1$; in this case there are $I^1_{i_0}$ and $I^2_{j_0}$ such that $S \subseteq I^1_{i_0} \cap I^2_{j_0}$. Putting $I := (I^1_{i_0} \cup I^2_{j_0}) \backslash S$, we again obtain a member of $\mathcal{I}^*_1$, whose incidence vector satisfies the same constraints with equality as x^*, a contradiction.

Case 3. There is an element $e' \in S$ with $x^*_{e'} = 0$; if there is at least one inequality of type (3.4) or (3.9), which is satisfied with equality by x^*, Case 1 applies. Otherwise, we can augment $x^*_{e'}$ properly to obtain another extreme point of P^*_1, which satisfies one of the constraints (3.4) or (3.9) or $x(S) \leqslant |S|$ with equality. But then again Case 1 applies.

In a similar way we can derive

COROLLARY (3.23). *If $\mathscr{C}^*_2$ is obtained by removing from $\mathscr{C}_3$ all those circuits which contain S, the linear system given by the inequalities* (3.3), (3.5), (3.7), (3.8), (3.10) (*with V replaced by T*), (3.12) (*for $e \in E'_2$*), (3.18) *and all mixed inequalities* (3.14) *fully describe the polytope $P(\mathcal{I}^*_2)$*.

We are now able to fully describe $P(\mathcal{I}_3)$:

COROLLARY (3.24). *The polytope $P(\mathcal{I}_3)$ is fully described by the constraints* (3.3), (3.4), (3.5), (3.8), (3.9), (3.10), (3.14), *the inequality $x(C_1) \leqslant |C_1| - 1$, as well as the trivial inequalities $0 \leqslant x_e \leqslant 1$ for all $e \in E_3$*.

Proof. Since $(E_3, \mathcal{I}_3)$ satisfies conditions (1) and (2) with respect to C_1, S, and T, by Lemma (3.2) the constraint $x(C_1) \leqslant |C_1| - 1$ is the only facet-defining one for $P(\mathcal{I}_3)$, whose support contains C_1. Moreover, the support of any other such inequality coincides with C_1 in S, T, or the empty set. So if the constraints proposed do not suffice to describe $P(\mathcal{I}_3)$ an additional facet-defining inequality must exist. But such an inequality defines at the same time a facet for $P(\mathcal{I}^*_1)$ or $P(\mathcal{I}^*_2)$ (because $\mathcal{I}^*_1$ and $\mathcal{I}^*_2$ both contain $\mathcal{I}_3$). Consequently, such an inequality must be one of (3.18). But by Lemma (3.20) the inequalities (3.18) are all redundant with respect to the trivial ones and those given by (3.3), (3.8). Therefore, they cannot be facet-defining for $P(\mathcal{I}_3)$, a contradiction. ∎

The conclusion with particular reference to the class of linearly relaxable independence systems is the following:

COROLLARY (3.25). *If* $(E_1, \mathcal{I}_1)$ *and* $(E_2, \mathcal{I}_2)$ *as given above are both linearly relaxable, then* $(E_3, \mathcal{I}_3)$ *obtained by composition is also linearly relaxable.*

4. SOLVING THE INDEPENDENT SET PROBLEM FOR $(E_3, \mathcal{I}_3)$

Having discussed how to compose an independence system $(E_3, \mathcal{I}_3)$ out of $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ and, in particular, how to obtain a linear system sufficient to describe $P(\mathcal{I}_3)$ once we have such systems for $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$ we are now going to elaborate on algorithmic aspects of our composition. As above and throughout this section let $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ satisfy conditions (1), (2), and (3) with respect to distinguished circuits $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$, and partitions (S, T), (U, V), respectively, and let $(E_3, \mathcal{I}_3)$ be the composition of these two as defined above. For notational convenience we will write $\mathcal{I}_3 = \mathcal{I}_1 \Delta \mathcal{I}_2$. Moreover, let $w: E_3 \to \mathbb{R}$ be a weight-function on E_3, which can be assumed to be nonnegative without loss of generality. In the sequel we will show how a maximum-weight independent set (with respect to w) within $\mathcal{I}_3$ can be determined from maximum-weight independent sets in $\mathcal{I}_1$ and in $\mathcal{I}_2$ with respect to two appropriate weight-functions associated with $\mathcal{I}_1$ and $\mathcal{I}_2$. The method proposed is polynomial in $|E_3|$ provided polynomial algorithms for solving problem (ISP) over $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ are available. Of specific importance is the fact that this method can be generalized to a dynamic programming polynomial time algorithm for the solution of (ISP) over an independence system $(E, \mathcal{I})$, which is obtained by sequentially composing $(E_1, \mathcal{I}_1), ..., (E_m, \mathcal{I}_m)$, where m is polynomial in $|E|$.

First observe that by Lemma (2.3) any base of $(E_3, \mathcal{I}_3)$ contains $|C_1| - 1$ many elements from C_1, identified with C_2. For the given partition (S, T) of C_1 let $e^* \in S$ and $e^{**} \in T$ be chosen such that

$$w(e^*) = \min\{w(e) : e \in S\} \qquad \text{and} \qquad w(e^{**}) = \min\{w(e) : e \in T\}.$$

As a consequence, any maximum-weight base of $(E_3, \mathcal{I}_3)$ can be transformed into one containing the whole set $C_1 \setminus \{e^*, e^{**}\}$ and either e^* or e^{**}, but not both. This can be achieved by appropriate element exchanges within S or T and due to the special structure of the members of $\mathcal{C}_3$.

Let us now consider the restrictions w^1 and w^2 of w to E_1 and E_2, respectively, i.e., the weight-functions $w^i: E_i \to \mathbb{R}_+$ given by $w^i(e) = w(e)$ for all $e \in E_i$, $i = 1, 2$. Then let B_1^* and B_1^{**} (B_2^* and B_2^{**}) be bases within $\mathcal{I}_1$ ($\mathcal{I}_2$) having maximum total weight ω_1^* and ω_1^{**} (ω_2^* and ω_2^{**}) with respect to w^1 (w^2) and containing e^* and e^{**}, but not both, respectively. If then ω_3

is the maximum total weight of a member I_3 from $\mathscr{I}_3$ (with respect to w) we have

$$\omega_3 = \max\{\omega_1^* + \omega_2^* - w(C_1\backslash\{e^{**}\}), \omega_1^{**} + \omega_2^{**} - w(C_1\backslash\{e^*\})\},$$

since any such I_3 decomposes into optimal bases $B_1 \in \mathscr{I}_1$ and $B_2 \in \mathscr{I}_2$. We should remark at this point that a straightforward way to determine such an I_3 would consist just in determining optimal bases B_1^*, B_2^* and B_1^{**}, B_2^{**} and comparing the total weights of $B_1^* \cup B_2^*$ and $B_1^{**} \cup B_2^{**}$, respectively. This procedure, however, would provide us with an exponential algorithm when generalized to the composition of more than 2, say m, independence systems, even when m is polynomial in $|E|$.

As already mentioned the method presented here will require a maximum-weight base within $\mathscr{I}_2$ with respect to w'^2, an appropriate modification of w^2. We will now develop the details of this modification. First of all by putting $\gamma := w(C_1\backslash\{e^*, e^{**}\})$ and considering the system of 2 equations

$$\begin{aligned} x + \gamma &= \omega_1^* - \sigma \\ y + \gamma &= \omega_1^{**} - \sigma, \end{aligned} \tag{4.1}$$

in which x, y, σ are parameters, we are able to give a slightly different expression for ω_3 in terms of these parameters. Namely, if (4.1) admits a solution with x, y both nonnegative, we get

$$\omega_3 = \sigma + \max\{x + \omega_2^* - w(e^*), y + \omega_2^{**} - w(e^{**})\}. \tag{4.2}$$

So if we define

$$\alpha_0 := \omega_1^* - \omega_1^{**}, \qquad x := \max\{0, \alpha_0\}, \qquad y := x - \alpha_0,$$

one easily verifies that together with an arbitrary σ, x and y satisfy the system (4.1). As a modification of w^2 we now assign the weights x and y to e^* and e^{**}, respectively, and a weight $w(e) + M$, M chosen to be sufficiently large (for instance $M = 2(x + y)$) to the remaining elements of C_1. Together with $w^2(e)$ as weights for all $e \in E_2\backslash C_2$ we hereby obtain a new weight-function w'^2 over E_2, assuring by the choice of M that any maximum weight base of $(E_2, \mathscr{I}_2)$ with respect to w'^2 will not contain either e^* or e^{**}. The total weight of such a base is thus given by

$$\begin{aligned} \omega_2' &= \max\{\omega_2^* - w(e^*) + x + M(|C_1| - 2), \omega_2^{**} - w(e^{**}) + y + M(|C_1| - 2)\} \\ &= M(|C_1| - 2) + \max\{x + \omega_2^* - w(e^*), y + \omega_2^{**} - w(e^{**})\} \end{aligned}$$

implying by (4.2)

$$\omega_3 = \sigma + \omega_2' - M(|C_1| - 2). \tag{4.3}$$

This allows us to derive the following result:

THEOREM (4.4). *Let w'^2 be given as just introduced and B'_2 be a maximum-weight base within $(E_2, \mathscr{I}_2)$ with respect to w'^2. Moreover, let B_3 be a subset of E_3 defined as follows:*

(i) *if $e^* \in B'_2$ then $B_3 := B'_2 \cup B_1^*$,*

(ii) *if $e^{**} \in B'_2$ then $B_3 := B'_2 \cup B_1^{**}$.*

Then B_3 is a maximum-weight independent set within $(E_3, \mathscr{I}_3)$ with respect to w.

Proof. Clearly, B_3 is a member of $\mathscr{I}_3$ by definition of $\mathscr{C}_3$. Moreover, for $w(B_3)$ we have in case (i)

$$\omega'_2 + \omega_1^* - \gamma - x - M(|C_1| - 2) = \omega'_2 + \sigma - M(|C_1| - 2),$$

and in case (ii)

$$\omega'_2 + \omega_1^{**} - \gamma - y - M(|C_1| - 2) = \omega'_2 + \sigma - M(|C_1| - 2),$$

which in both cases equals ω_3 by (4.3). ∎

Let us now reflect on the situation that an independence system $(E, \mathscr{I})$ is the composition of more than two appropriate $(E_i, \mathscr{I}_i)$. For convenience let

$$H_n := \mathscr{I}_n, \qquad H_{n-i} := H_{n-i+1} \Delta \mathscr{I}_{n-i}, \qquad i = 1, ..., n-1, \text{ and } \mathscr{I} := H_1.$$

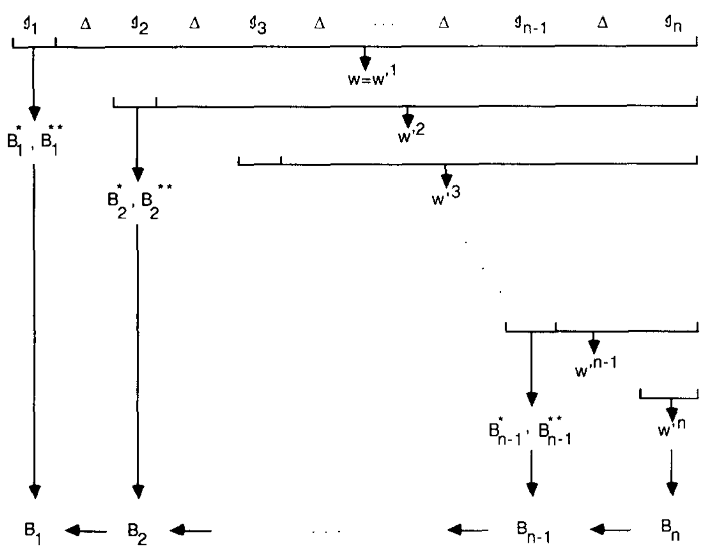

FIGURE (4.5)

By Theorem (4.4), a maximum-weight independent set with respect to w and $(E, \mathscr{I})$ can be found by determining B_1^*, B_1^{**} as already considered above and by solving (ISP) over H_2 with respect to a modified weight-function w'^2. This latter problem can now be "decomposed" identically and so on until we reach the point to determine an optimal independent set within $\mathscr{I}_n$ with respect to a weight-function w'^n. Together with B_{n-1}^*, B_{n-1}^{**} this yields by application of Theorem (4.4) an optimal solution for (ISP) over H_n and so on until we obtain such a solution for H_1, which equals $\mathscr{I}$. Figure (4.5) illustrates this procedure schematically, the B_i representing maximum-weight independent sets within H_i and with respect to weight-functions w'^i for $i = n, n-1, ..., 1$.

5. On the K_3-Cover Problem in Graphs Not Contractible to $K_5 - e$

The *K_3-cover problem* in a graph $G = [V, E]$ is the task to find an edge set $F \subseteq E$ not containing a triangle and having maximum total weight with respect to a weight-function $w: E \to \mathbb{R}_+$. The induced independence system $(E, \Delta(G))$ has already been introduced in Section 2, and what we are now going to study is the polytope $P(\Delta(G))$, i.e., the convex hull of all incidence vectors of triangle-free edge sets in G. For this we need some further definitions. If $G = [V, E]$ is a graph (without loops and multiple edges), and if e is an edge of G, let $G - e$ be the graph obtained from G by *deleting* edge e. *Contracting* edge e in G means identification of the endnodes of e, deletion of one of the two parallel edges, which may arise by this operation and deletion of e from the resulting graph. G is said to be *contractible* to a graph H if H can be obtained from G by repeated applications of deletion and contractions. In this section we will consider the class of all those graphs which are not contractible to $K_5 - e$, the complete graph on 5 vertices with one edge deleted (as depicted in Fig. (5.1)). This class has also been studied by Barahona and Grötschel [2] with respect to the travelling salesman problem.

Some examples of such graphs are illustrated in Fig. (5.2).

A constructive characterization of graphs not contractible to $K_5 - e$ has been given by Wagner [11]:

THEOREM (5.3). *Each maximal (w.r.t. its edge set) graph G not contractible to $K_5 - e$ can be obtained by taking repeated* 1- *or* 2-*sums of the graphs shown in Fig.* (5.2).

Let us study the implications of Wagner's theorem and of the previous section for the composition of associated polyhedra. If G_1 and G_2 are two graphs, their associated polyhedra $P(\Delta(G_1))$ and $P(\Delta(G_2))$ are given by the systems of linear inequalities $A^1x^1 \leqslant b^1$ and $A^2x^2 \leqslant b^2$, and G_3 is the 1-sum

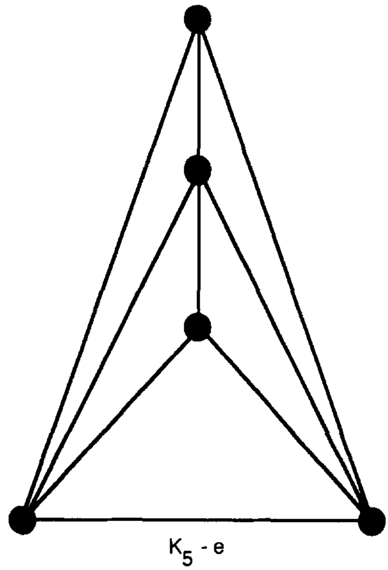

FIGURE (5.1)

of these two graphs, then $P(\Delta(G_3))$ is fully described by the system $A^1x^1 \leq b^1$, $A^2x^2 \leq b^2$.

Now assume that G_3 is the 2-sum of G_1 and G_2 with respect to $e_1 \in E_1$ and $e_4 \in E_2$ (cf. Section 2). Recall that the independence system $(E'_3, \Delta(G'_3))$ is the composition of $(E'_1, \Delta(G'_1))$ and $(E'_2, \Delta(G'_2))$ with respect to $C_1 = \{e_1, e_2, e_3\}$, $S = \{e_1\}$, $T = \{e_2, e_3\}$, and $C_2 = \{e_4, e_5, e_6\}$, $U = \{e_4\}$, $V = \{e_5, e_6\}$, the G'_i denoting the modifications of the G_i by adding e_2, e_3

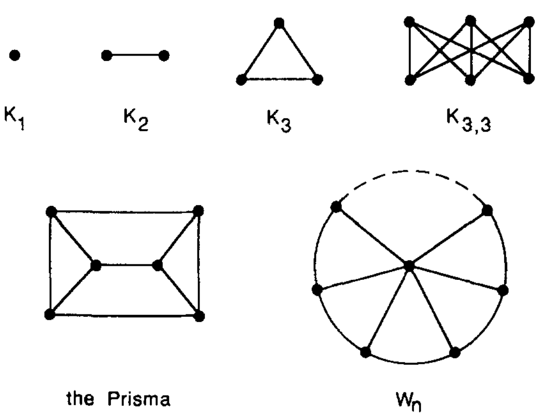

FIGURE (5.2)

and e_5, e_6 as shown in Fig. (2.2). We claim that apart from the distinguished inequality $x(C_1) \leqslant 2$, $[x(C_2) \leqslant 2]$ there is no facet-defining inequality for $P(\Delta(G_1'))$ $[P(\Delta(G_2'))]$, whose support contains $\{e_2, e_3\}$ $[\{e_5, e_6\}]$. If on the contrary there would be one for, say, $P(\Delta(G_1'))$, each incidence vector of a triangle-free edge set in G_1', which satisfies this inequality with equality, would have to satisfy $x(C_1) = 2$, a contradiction. Consequently, there is no mixed inequality necessary to define $P(\Delta(G_3'))$, and by Corollary (3.22) a full description of this polytope is given by $A^1x^1 \leqslant b^1$, $A^2x'^2 \leqslant b^2$, and $x(C_1) \leqslant 2$, where $A^2x'^2 \leqslant b^2$ is the linear system obtained from $A^2x^2 \leqslant b^2$ by identifying e_1, e_2, e_3 with e_4, e_5, e_6, respectively. This also implies that $P(\Delta(G_3))$ is fully described by $A^1x^1 \leqslant b^1$, $A^2x^2 \leqslant b^2$.

What about the polyhedra associated with the basic graphs as shown in Fig. (5.2)? First of all let us notice that for any graph G, $P(\Delta(G))$ is full-dimensional, which implies that (up to multiplication by a constant) the defining linear system is unique. Moreover, Conforti *et al.* [4] have shown that for a graph $G = [V, E]$ the following inequalities define facets of $P(\Delta(G))$:

$$x(C) \leqslant 2 \qquad \forall \text{ triangles } C \text{ in } G; \tag{5.4}$$

$$x(W_n) \leqslant 3k + 1 \qquad \forall\ n\text{-wheels } W_n,\ n = 2k + 1, \text{ and } k \in \mathbb{N}; \tag{5.5}$$

$$0 \leqslant x_e \leqslant 1 \qquad \forall \text{ edges } e \text{ in } G. \tag{5.6}$$

It is not difficult to verify that for the graphs K_1, K_2, K_3, $K_{3,3}$ the prisma, and W_n with n even, the polyhedron $P(\Delta(G))$ is fully described by the linear inequalities of type (5.4) and (5.6). It remains to establish a system of linear inequalities sufficient to describe the polyhedron associated with an n-wheel W_n, where $n \geqslant 3$ and odd.

THEOREM (5.7). *Given an n-wheel W_n, $n = 2k + 1$, $k \in \mathbb{N}$, the polyhedron $P(\Delta(W_n))$ is fully described by the inequalities* (5.4), (5.5), (5.6).

Proof. Let $a^Tx \leqslant a_0$ be a nontrivial facet-defining inequality of $P(\Delta(W_n))$. Then by Lemma (3.1) we know that $a \geqslant 0$. We will show that this inequality is necessarily of type (5.4) or (5.5). Let us denote by

— E_n the edge set of W_n;

— H that subgraph of W_n, which is induced by the set $\{e \in E_n : a_e > 0\}$;

— $\mathscr{T}$ the collection of those triangle-free edge sets of W_n, for which $a^Tx = a_0$.

Then the only equations satisfied by all members of $\mathscr{T}$ are positive multiples of $a^Tx = a_0$.

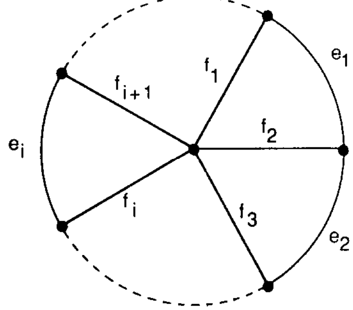

FIGURE (5.8)

Case 1. There is an edge $e \in E_n$ such that $a_e = 0$. Since $a^T x \leqslant a_0$ is at the same time facet-defining for $P(\Delta(H))$, which in this case is fully described by inequalities of type (5.4), (5.6), our inequality $a^T x \leqslant a_0$ is of type (5.4).

Case 2. $a_e > 0$ for all $e \in E_n$. Then $a^T x \leqslant a_0$ is not a triangle inequality. Let $e_1, e_2, ..., e_{2k+1}, f_1, f_2, ..., f_{2k+1}$ denote the edges of W_n, such that e_i and e_{i+1} are incident at a common node and $\{e_i, f_i, f_{i+1}\}$ form a triangle for $i = 1, ..., 2k+1$ (the indices are taken modulo $2k+1$; see Fig. (5.8) for an illustration).

Remark. By validity of $a^T x \leqslant a_0$, if $I \in \mathscr{T}$ is triangle-free such that either f_1 or f_2 is not in I, then necessarily $e_1 \in I$.

CLAIM 1. *There is a $T_1 \in \mathscr{T}$ containing both f_1, f_2.*

Proof. If this is not the case then, by our previous remark, every triangle-free set $T \in \mathscr{T}$ contains e_1. But then we have $x^T_{e_1} = 1$ for all $T \in \mathscr{T}$, a contradiction.

CLAIM 2. *There is a $T_2 \in \mathscr{T}$ containing neither f_1 nor f_2.*

Proof. If this is not the case then for every $T \in \mathscr{T}$ the following holds:

$$f_1 \notin T \Rightarrow e_1, f_2 \in T,$$
$$f_2 \notin T \Rightarrow e_1, f_1 \in T.$$

Thus $x^T(\{e_1, f_1, f_2\}) = 2$ for all $T \in \mathscr{T}$, a contradiction.

Now consider the edge sets $E^1 := (T_1 \backslash \{f_1\}) \cup \{e_1\}$, $E^2 := (T_1 \backslash \{f_2\}) \cup \{e_1\}$, where T_1 is the edge set exhibited by Claim 1. Clearly,

E^1, E^2 are triangle-free and their incidence vectors satisfy $a^T x \leqslant a_0$. This implies that

(i) $a_{e_1} \leqslant a_{f_1}, a_{f_2}$.

Considering e_2, we obtain

(ii) $a_{e_2} \leqslant a_{f_2}, a_{f_3}$.

Now we consider the edge set T_2 exhibited in Claim 2. We must have e_2, $f_3 \in T_2$. Otherwise, $T_2' := T_2 \cup \{f_2\}$ is triangle-free but violates $a^T x \leqslant a_0$. Let $E^3 := (T_2 \backslash \{e_2\}) \cup \{f_2\}$, $E^4 := (T_2 \backslash \{f_3\}) \cup \{f_2\}$. Again E^3, E^4 are triangle-free and we obtain

(iii) $a_{f_2} \leqslant a_{e_2}$,

(iv) $a_{f_2} \leqslant a_{f_3}$.

From (ii) and (iii) we get $a_{f_2} = a_{e_2}$, and by symmetry

(v) $a_{e_i} = a_{f_i}$ for $i = 1, ..., 2k+1$.

From (iv) and again by symmetry we get

(vi) $a_{f_1} \leqslant a_{f_2} \leqslant \cdots \leqslant a_{f_{2k+1}} \leqslant a_{f_1}$.

Altogether, this yields $a_e = \lambda$ for all $e \in E_n$ and some $\lambda \in \mathbb{R}_+$. Therefore, $a^T x \leqslant a_0$ is of the form (5.5). ∎

From Theorems (5.3), (5.7), and the above remarks concerning the polytopes associated with the graphs shown in Fig. (5.2) we obtain the following

THEOREM (5.9). *For each graph G not contractible to $K_5 - e$ the polytope $P(\Delta(G))$ is fully described by the constraints* (5.4), (5.5), (5.6).

In [4] Conforti *et al.* have presented a polynomial algorithm for the solution of the separation problem for $2k+1$-wheel constraints. This and Theorem (5.9) allow us to state the following result:

THEOREM (5.10). *The K_3-cover problem is polynomially solvable in graphs not contractible to $K_5 - e$.*

6. FINAL REMARKS

The following question has also been studied intensively: If we are given minimal linear systems sufficient to describe $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$, will the linear system sufficient to describe $P(\mathcal{I}_3)$ as derived in Section 3 also be a minimal one? We are able to show that the answer to this question is in the affirmative.

Another interesting property of linear systems has been studied within this context. Call a linear system $Ax \leqslant b$ *totally dual integral* (*TDI*) if the dual to the linear program

$$\text{Maximize } c^T x \text{ subject to } Ax \leqslant b$$

has an integral optimum solution for each integral c for which an optimum exists. This important concept goes back to Hoffman [9] and Edmonds and Giles [5]. Our main result in this area can be stated as follows.

If the linear systems (3.3)–(3.7) and (3.8)–(3.12) are TDI, then the linear system given by the inequalities (3.3)–(3.5), (3.8)–(3.10), (3.14), the circuit inequality $x(C_1) \leqslant |C_1|$, and all trivial inequalities is also a TDI-linear system.

What are the consequences of this when looking at the linear system (5.4), (5.5), (5.6) associated with K_3-covers in graphs not contractible to $K_5 - e$? It can easily be seen that for the basic graphs K_1, K_2, K_3, $K_{3,3}$ and the prisma the set of inequalities (5.4), (5.6) constitute TDI-linear systems. If we could show that for an n-wheel W_n, $3 \leqslant n$ odd, the linear system (5.4), (5.5), (5.6) is TDI, then we could conclude that this linear system for any graph not contractible to $K_5 - e$ would also be TDI. Similar results certainly hold for linear systems associated with bipartite subgraph systems or acyclic subdigraph systems.

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