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Critical extreme points of the 2-edge connected spanning subgraph polytope*

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Abstract. In this paper we study the extreme points of the polytope $P(G)$, the linear relaxation of the 2-edge connected spanning subgraph polytope of a graph G. We introduce a partial ordering on the extreme points of $P(G)$ and give necessary conditions for a non-integer extreme point of $P(G)$ to be minimal with respect to that ordering. We show that, if $\bar{x}$ is a non-integer minimal extreme point of $P(G)$, then G and $\bar{x}$ can be reduced, by means of some reduction operations, to a graph G' and an extreme point $\bar{x}'$ of $P(G')$ where G' and $\bar{x}'$ satisfy some simple properties. As a consequence we obtain a characterization of the perfectly 2-edge connected graphs, the graphs for which the polytope $P(G)$ is integral.

Key words. Polytope – Cut – 2-edge connected graph – Critical extreme point

1. Introduction

A graph $G = (V, E)$ is called *2-edge connected* if for every pair of nodes (u, v) there are at least two edge-disjoint paths between u and v. Given a graph $G = (V, E)$ and a weight function w which associates to each edge e a weight $w(e)$, the *2-edge connected subgraph problem* (TECSP) consits in finding a 2-edge connected subgraph $H = (V, F)$ of G, spanning all the nodes of G and such that $\sum_{e \in F} w(e)$ is minimum. This problem arises in the design of reliable transportation and communication networks [8], [34], [26] [30]. It is NP-hard in general. It has been shown to be polynomial in series-parallel graphs [36] and Halin graphs [35]. (A graph is called *series-parallel* if it can be obtained from an edge by subdivisions and duplications of edges. A graph is said to be a *Halin graph* if it consists of a cycle and a tree without nodes of degree 2 whose leaves are precisely the nodes of the cycle.)

Given a graph $G = (V, E)$ and an edge subset $F \subseteq E$, the $0-1$ vector x^F of $\mathbb{R}^E$ such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ if $e \in E \setminus F$ is called the *incidence vector*

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of F. The convex hull of the incidence vectors of the edge sets of the connected spanning subgraphs of G, denoted by TECP(G), is called the *2-edge connected spanning subgraph polytope* of G.

Let $G = (V, E)$ be a graph. Given $b : E \to \mathbb{R}$ and F a subset of E, $b(F)$ will denote $\sum_{e\in F} b(e)$. For $W \subseteq V$ we let $\overline{W} = V \setminus W$. If $W \subset V$ is a node subset of G, then the set of edges that have only one node in W is called a *cut* and denoted by $\delta(W)$. (Note that $\delta(W) = \delta(\overline{W})$). For $v \in V$, we will write $\delta(v)$ for $\delta(\{v\})$. An *edge cutset* $F \subseteq E$ of G is a set of edges such that $F = \delta(S)$ for some non-empty set $S \subset V$.

If x^F is the incidence vector of the edge set F of a 2-edge connected spanning subgraph of G, then x^F satisfies the inequalities

$$x(e) \geq 0 \quad \forall\, e \in E, \tag{1.1}$$

$$x(e) \leq 1 \quad \forall\, e \in E, \tag{1.2}$$

$$x(\delta(S)) \geq 2 \quad \forall\, S \subset V,\ S \neq \emptyset. \tag{1.3}$$

Inequalities (1.1) and (1.2) are called *trivial inequalities* and inequalities (1.3) are called *cut inequalities*.

Given a graph $G = (V, E)$ we will denote by $P(G)$ the polytope given by inequalities (1.1), (1.2), (1.3). The TECSP is then equivalent to the integer program

$$\min\{wx,\ x \in P(G),\ x \text{ integer}\}.$$

The TECSP is closely related to the widely studied traveling salesman problem. In fact, as pointed out in [16], the problem of determining whether a graph contains a hamiltonian cycle can be reduced to the TECSP.

The *subtour polytope* of the traveling salesman problem is the set of all solutions of the system given by inequalities (1.1)–(1.3) together with the constraints $x(\delta(v)) = 2$ for all $v \in V$. Clearly, the polytope $P(G)$ is a relaxation of both the TECP(G) and the subtour polytope. Thus minimizing wx over $P(G)$ may provide a good lower bound for both the TECSP and the traveling salesman problem.

Using network flows [14] [15], one can compute in polynomial time a minimum cut in a weighted undirected graph. Hence the separation problem for inequalities (1.3) (i.e. the problem that consists in finding whether a given solution $\bar{y} \in \mathbb{R}^{|E|}$ satisfies inequalities (1.3) and if not to find an inequality which is violated by $\bar{y}$) can be solved in polynomial time. This implies by the ellipsoid method [23] that the TECSP can be solved in polynomial time on graphs G for which TECP(G) $= P(G)$. Mahjoub [32] called these graphs *perfectly 2-edge connected graphs* (perfectly-TEC). Thus an interesting question would be to characterize these graphs. In [31], Mahjoub showed that series-parallel graphs are perfectly-TEC. In [32] he described sufficient conditions for a graph to be perfectly-TEC.

In [18], Fonlupt and Naddef studied the graphs for which the polyhedron given by the inequalities (1.1) and (1.3) is the convex hull of the tours of G. (Here a tour is a cycle going through each node at least once). This is in connection with the graphical traveling salesman problem. They gave a characterization of these graphs in terms of excluded minors. (A minor of a graph G is a graph obtained from G by deletions and contractions of edges). A natural question that may arise here is whether or not one can obtain a similar characterization for perfectly-TEC graphs. The answer to this question is, unfortunately, in the negative. If we add the constraints $x(e) \leq 1$ for all $e \in E$, the approach developed by Fonlupt and Naddef [18] would not be appropriate. In fact, consider a non perfectly-TEC graph G (for instance a complete graph on four nodes or more). Subdivide G by inserting a new node of degree 2 on each edge and let $G' = (V', E')$ be the resulting graph. Clearly, each edge e of G' belongs to a 2-edge cutset. So $x(e) = 1$ for all $e \in E'$ is the unique solution of $P(G')$ and hence G' is perfectly-TEC. However, the graph G, which is a minor of G', is not.

In this paper we study the fractional extreme points of the polytope $P(G)$. We introduce an ordering on the extreme points of $P(G)$ and give necessary conditions for a non-integer extreme point of $P(G)$ to be minimal with respect to that ordering. We show that if $\bar{x}$ is a minimal non-integer extreme point of $P(G)$, then G and $\bar{x}$ can be reduced, by means of some reduction operations, to a graph G' and an extreme point $\bar{x}'$ of $P(G')$, where G' and $\bar{x}'$ satisfy some simple properties . As a consequence, we obtain a characterization of perfectly-TEC graphs.

The polytope TECP(G) has seen much attention in the last few years. Grötschel and Monma [24] and Grötschel, Monma and Stoer [25], [27], [28] consider a more general model related to the design of minimum survivable networks. They discuss polyhedral aspects of this model. In particular, they study the polytope, the extreme points of which are the incidence vectors of the edge-sets of the k-edge connected subgraphs of a graph G, where k is a fixed positive integer. In [27], they devise cutting plane algorithms along with computational results are presented. A complete survey of that model can be found in Stoer [34] (see also [26], [30]). In [5], [6] Chopra studies the minimum k-edge connected spanning subgraph problem when multiple copies of an edge may be used. In [5] he considers the problem in directed graphs, for $k = 2$. He shows how facets of the associated polyhedron on undirected graphs can be obtained by projection and he develops a cutting plane algorithm for that problem. In [6], he studies the problem for k odd. In particular, he characterizes the associated polyhedron in this case, for the class of outerplanar graphs. (A graph is called *outerplanar* if it can be drawn in the plane as one cycle with noncrossing chords. Outerplanar graphs are also series-parallel.) In [12] Didi Biha and Mahjoub give a complete description of the k-edge connected subgraph polytope, for all k, when the graph is series-parallel. As a consequence, they show that Chopra's result remains true if the graph is series-parallel, which has also been independently proved by Chopra and Stoer [7]. In [2], Barahona and Mahjoub characterize the polytope TECP(G) for the class of Halin graphs. Baïou and Mahjoub [1] characterize the Steiner 2-edge connected subgraph polytope for series-parallel graphs. Boyd and Hao [4] discuss a class of facets for the TECP(G) related to the the traveling salesman polytope. Extensions of the TECSP are addressed in [19], [21] (see also [20]), [22], [29], [13].

Related work can be found in [9], [10], [11]. In [9] Cornuéjols, Fonlupt and Naddef study the TECSP when multiple copies of an edge may be used. They showed that when the graph is series-parallel, the associated polyhedron is completly described by inequalities (1.1) and (1.3). In [10] Coullard et al. characterize the dominant of the 2-node connected subgraph polytope in the graphs which do not have W_4 (the wheel on 5 nodes) as a minor. In [11] they devise a linear algorithm for the Steiner 2-node connected subgraph problem on Halin graphs and the graphs with no W_4 as a minor.

This paper is organized as follows. In the next section we give more notation and definitions and present some preliminary results. In Section 3 we study an ordering relation on the extreme points of $P(G)$, and introduce the concept of critical extreme points of $P(G)$. In Section 4 we prove our main result. In Section 5 we discuss some applications and give some concluding remarks.

2. Notation, definitions and preliminary results

2.1. Notation and definitions

We assume familiarity with graphs and polyhedra. For specific details, the reader is referred to [3] and [33], respectively. We consider finite, undirected and loopless 2-edge connected graphs, which may have multiple edges. We denote a graph by $G = (V, E)$ where V is the *node set* and E the *edge set*. If e is an edge with endnodes u and v, then we write $e = uv$. If $W \subseteq V$, we denote by $G(W)$ the subgraph of G induced by W and by $E(W)$ the set of edges having both nodes in W. Given W, W' two disjoint subsets of V, $\delta(W, W')$ will denote the set of edges of G having one endnode in W and the other in W'. If $W = \{v\}$, we will write $\delta(v, W')$ instead of $\delta(\{v\}, W')$. If G is a graph and e is an edge of E, then $G - e$ will denote the graph obtained from G by removing e. A *path* P of G is an alternate sequence of nodes and edges $(u_1, e_1, u_2, e_2, \dots, u_{q-1}, e_{q-1}, u_q)$ where $e_i \in \delta(u_i, u_{i+1})$ for $i = 1, \dots, q-1$. We will denote a path P by either its node sequence $(u_1, \dots, u_q)$ or its edge sequence $(e_1, \dots, e_{q-1})$. The *length* of a path is the number of its edges. A path is said to be *even* (resp. *odd*) if its length is even (resp. odd).

A graph $G = (V, E)$ is said to be *contractible* to a graph G' if G' can be obtained from G by deletions and contractions of edges. The contraction of an edge $e = uv$ consists in deleting e, identifying u and v and in preserving all the adjacencies. If $W \subset V$, then the graph obtained by contracting W is the graph obtained by contracting all the edges of $E(W)$.

Throughout the paper, we will consider a graph $G = (V, E)$ and an extreme point $\bar{x}$ of $P(G)$. We suppose that $P(G)$ is not integral and $\bar{x}$ is fractional. If $\bar{x}(f) = 0$ for some edge $f \in E$, then the solution $\bar{x}'$ such that $\bar{x}'(e) = \bar{x}(e)$ for all $e \in E \setminus \{f\}$ is an extreme point of $P(G - f)$ if and only if $\bar{x}$ is an extreme point of $P(G)$. Therefore studying the structure of $\bar{x}$ on G is equivalent to studying that of $\bar{x}'$ on $G - f$. Hence we will suppose, w.l.o.g. that $\bar{x}(e) > 0$ for all $e \in E$. We will denote by E_0 (resp. E_1) the subset of edges $e \in E$ such that $0 < \bar{x}(e) < 1$ (resp. $x(e) = 1$). The edges of E_0 (resp. E_1) will be called *fractional edges* (resp. *integer edges*). We will denote by $G_0 = (V_0, E_0)$ (resp. $G_1 = (V_1, E_1)$) the subgraph induced by E_0 (resp. E_1). The graph G_0 (resp. G_1) will be called the *fractional graph* (resp. *integer graph*). We will use the subscript 0 (resp. 1) for the notation which refer to G_0 (resp. G_1). For instance, if $W \subset V$, then $\delta_0(W)$ (resp. $\delta_1(W)$) will denote $\delta(W) \cap E_0$ (resp. $\delta(W) \cap E_1$), and if x is a solution of $P(G)$, then x_0 will denote the restriction of x on E_0. Hence $\bar{x}_0$ will denote the restriction of $\bar{x}$ on E_0.

2.2. Cuts and polytopes

Let $\delta(W)$ be a cut of G. If $G(W)$ is not 2-edge connected, then the constraint $x(\delta(W)) \geq 2$ is redundant in the decription of $P(G)$. In fact suppose $G(W)$ is not 2-edge connected, and let W_1, W_2 be a partition of W such that $|\delta(W_1, W_2)| \leq 1$. From the cut constraints corresponding to W_1 and W_2, we have $x(\delta(W_1, \overline{W})) \geq 1$ and $x(\delta(W_2, \overline{W})) \geq 1$. By summing these inequalities, we get $x(\delta(W)) \geq 2$. Let $\mathcal{S}(G)$ be the set of subsets S of V such that $G(S)$ and $G(V \setminus S)$ are both 2-edge connected. Thus $P(G)$ can be written as

$$P(G) \begin{cases} 0 \leq x(e) \leq 1 & \forall\, e \in E, \\ x(\delta_0(S)) + x(\delta_1(S)) \geq 2 & \forall\, S \in \mathcal{S}(G). \end{cases}$$

For $S \subseteq V$, let $b_S = 2 - |\delta_1(S)|$. Let $P_0(G)$ be the polytope given by

$$P_0(G) = \begin{cases} 0 \leq x(e) \leq 1 & \forall e \in E_0, \\ x(\delta_0(S)) \geq b_S & \forall\, S \in \mathcal{S}(G). \end{cases}$$

Note that $P_0(G) \subset \mathbb{R}^{E_0}$ and $\dim(P_0(G)) = |E_0|$. Since $\bar{x}$ is an extreme point of $P(G)$, we have that $\bar{x}_0$ is an extreme point of $P_0(G)$. Moreover, if $\delta(S)$ contains no (resp. contains one) integer edge, then $b_S = 2$ (resp. $b_S = 1$). If $\delta(S)$ contains more than one integer edge, then $b_S \leq 0$, and hence $x(\delta_0(S)) \geq b_S$ is redundant in $P_0(G)$.

Given a solution $x \in \mathbb{R}^E$, a constraint $ax \geq \alpha$ is said to be *tight* for x if $ax = \alpha$.

If $\delta(S)$ induces a cut inequality tight for $\bar{x}$, then $\delta(S)$ will be called a *tight cut*. If $\delta(S)$ is tight and $\delta_0(S) \neq \emptyset$, we will say that $\delta_0(S)$ is tight for $\bar{x}_0$. We will denote by $\mathcal{S}_0$ the set of elements $S \in \mathcal{S}(G)$ such that $\delta_0(S)$ is tight for $\bar{x}_0$. A cut $\delta(S)$ tight for $\bar{x}$ will be called a *degree-cut* if $\min(|S|, |\overline{S}|) = 1$. A node $v \in V$ is said to be *tight* for $\bar{x}$ (or just *tight*) if it induces a degree-cut. A cut $\delta(W)$ is said to be *proper* if it is tight for $\bar{x}$ and $E_0(W) \neq \emptyset \neq E_0(\overline{W})$. We will denote by Π the set of proper cuts.

The following result, due to Cornuéjols, Fonlupt and Naddef [9], will be useful in the sequel.

Lemma 1. [9] *Let $\delta(W)$ be a cut tight for $\bar{x}$. Then the polytope $P(G) \cap \{x \in \mathbb{R}^E;\ x(\delta(W)) = 2\}$ can be described by the following system of linear inequalities*

$$\begin{cases} 0 \leq x(e) \leq 1 & \forall\, e \in E, \\ x(\delta(S)) \geq 2 & \forall\, S \subset W,\ S \in \mathcal{S}(G), \\ x(\delta(S)) \geq 2 & \forall\, S \subset V \setminus W,\ S \in \mathcal{S}(G), \\ x(\delta(W)) = 2. \end{cases}$$

Given a node subset $W \subseteq V$ we let $\Gamma(W) = \delta_0(W) \cup E_0(W)$.

Let $\delta(W)$ be a tight cut for $\bar{x}_0$. As $\bar{x}_0$ is an extreme point of $P_0(G)$, it follows that $\bar{x}_0$ is an extreme point of the polytope $P_0(G) \cap \{x \in \mathbb{R}^{E_0};\ x(\delta_0(W)) = 2\}$. From Lemma 1, it follows that this polytope is given by the inequalities

$$\begin{cases} 0 \leq x(e) \leq 1 & \forall\, e \in E_0, \\ x(\delta_0(S)) \geq b_S & \forall\, S \subset W,\ S \in \mathcal{S}(G), \\ x(\delta_0(S)) \geq b_S & \forall\, S \subset V \setminus W,\ S \in \mathcal{S}(G), \\ x(\delta_0(W)) = b_W. \end{cases}$$

As $0 < \bar{x}_0(e) < 1$ for all $e \in E_0$, we then have that $\bar{x}_0$ is the unique solution of the system

$$(\mathcal{L}) \begin{cases} x(\delta_0(S)) = b_S & \forall\, S \subset W,\ S \in \mathcal{S}_0, \\ x(\delta_0(S)) = b_S & \forall\, S \subset V \setminus W,\ S \in \mathcal{S}_0, \\ x(\delta_0(W)) = b_W. \end{cases}$$

Let $Ax = b$ (resp. $\bar{A}x = \bar{b}$) be the equality system given by the first (resp. second) set of equations of system $(\mathcal{L})$. System $(\mathcal{L})$ can then be written as

$$(\mathcal{L}) = \{x \in \mathbb{R}^{E_0};\ Ax = b,\ \bar{A}x = \bar{b},\ x(\delta_0(W)) = b_W\}.$$

In the sequel we will denote by P_W the polytope in $\mathbb{R}^{\Gamma(W)}$ given by

$$P_W \begin{cases} 0 \leq x(e) \leq 1 & \forall\, e \in \Gamma(W), \\ x(\delta_0(S)) \geq b_S & \forall\, S \subseteq W,\ S \in \mathcal{S}_0. \end{cases}$$

Given an edge subset $F \subseteq E$ and $x \in \mathbb{R}^F$, we will call *extension* of x the solution $y \in \mathbb{R}^E$ given by $y(e) = x(e)$ if $e \in F$ and $y(e) = 1$ if $e \in E \setminus F$.

2.3. Redundant relations

Let $\delta(W)$ be a tight cut. Let $A_i x = b_i,\ i = 1, \ldots, k$ be the rows of the system $Ax = b$. Let c be a row-vector of $\mathbb{R}^{\Gamma(W)}$ such that $c(e) = 0$ for all $e \in E_0(W)$. Suppose there is a row-vector $\mu = (\mu_1, \ldots, \mu_k)$ such that $c = \mu A$. Then the equation

$$cx = \delta,$$

where $\delta = \mu b$, will be called a *redundant relation* produced by $Ax = b$. If the scalar μ_i is uniquely defined, μ_i will be called the *weight* of $A_i x = b_i$ in the redundant relation $cx = \delta$; note that in this case, the equation $A_i x = b_i$ is linearly independent from the other rows of the linear system $Ax = b$.

Let $Cx = d$ be a linear system of $\mathbb{R}^{\delta_0(W)}$ that generates all the redundant relations of $Ax = b$, that is any redundant relation of $Ax = b$ can be written as a linear combination of equations of $Cx = d$. Such a system will be called *a redundant system* for $Ax = b$. Note that the affine subspace $\{x \in \mathbb{R}^{\delta_0(W)};\ Cx = d\}$ is nothing but the projection of the affine subspace $\{x \in \mathbb{R}^{\delta_0(W)};\ Ax = b\}$ onto $\mathbb{R}^{\delta_0(W)}$. We have the following lemma.

Lemma 2. *i) If $y \in \mathbb{R}^{\delta_0(W)}$ is a solution of $Cx = d$, then there exists a unique solution $\tilde{x} \in \mathbb{R}^{\Gamma(W)}$ of $Ax = b$ (resp. $\tilde{x}' \in \mathbb{R}^{\Gamma(\overline{W})}$ of $\bar{A}x = \bar{b}$) such that $\tilde{x}(e) = y(e)$ for all $e \in \delta_0(W)$ (resp. $\tilde{x}'(e) = y(e)$ for all $e \in \delta_0(W)$).*

ii) Let y_0 be the restriction of $\bar{x}_0$ on $\Gamma(\overline{W})$. Then y_0 is the unique solution of the system

$$(\mathcal{L}') \begin{cases} \bar{A}x = \bar{b}, \\ Cx = d, \\ x(\delta_0(W)) = b_W. \end{cases}$$

iii) If $P_{\overline{W}}$ is integral, then there exists at least one redundant relation of $Ax = b$ different from $x(\delta_0(W)) = b_W$.

Proof. *i)* We prove the statement for $Ax = b$, the proof for $\bar{A}x = \bar{b}$ is similar. W.l.o.g., we may suppose that $Ax = b$ is nonsingular. Hence, by elementary linear algebra operations, $Ax = b$ can be written as a system of the form

$$\begin{aligned} Ix_1 + A''x_2 &= b', \\ Cx_2 &= d, \end{aligned}$$

where x_1 and x_2 are the restrictions of x on $E_0(W)$ and $\delta_0(W)$, respectively, and I is the identity matrix. As y is a solution of $Cx_2 = d$, clearly, there is a unique solution $y_1 \in \mathbb{R}^{E_0(W)}$ such that $Iy_1 + A''y = b'$. Hence $\tilde{x}$ such that $\tilde{x}(e) = y_1(e)$ if $e \in E_0(W)$ and $\tilde{x}(e) = y(e)$ if $e \in \delta_0(W)$ is the desired solution.

ii) Suppose there are two solutions y_1 and y_2 of system $(\mathcal{L}')$. Let y_1' and y_2' be the restrictions of y_1 and y_2 on $\delta_0(W)$, respectively. As y_1' (resp. y_2') is a solution of $Cx = d$, by *i)* there is a solution $x_1' \in \mathbb{R}^{E_0(W)}$ (resp. $x_2' \in \mathbb{R}^{E_0(W)}$) such that the solution $\tilde{x}_1$ (resp. $\tilde{x}_2$) given by $\tilde{x}_1(e) = x_1'(e)$ if $e \in E_0(W)$ and $\tilde{x}_1(e) = y_1'(e)$ if $e \in \delta_0(W)$ (resp. $\tilde{x}_2(e) = x_2'(e)$ if $e \in E_0(W)$ and $\tilde{x}_2(e) = y_2'(e)$ if $e \in \delta_0(W)$) is a solution of $Ax = b$. Let $\bar{x}_1$ (resp. $\bar{x}_2$) be the solution such that $\bar{x}_1(e) = x_1'(e)$ if $e \in E_0(W)$ and $\bar{x}_1(e) = y_1(e)$ if $e \in \Gamma(\overline{W})$ (resp. $\bar{x}_2(e) = x_2'(e)$ if $e \in E_0(W)$ and $\bar{x}_2(e) = y_2(e)$ if $e \in \Gamma(\overline{W})$). We then have that $\bar{x}_1$ and $\bar{x}_2$ are both solutions of system $(\mathcal{L})$. As $y_1 \neq y_2$ and hence $\bar{x}_1 \neq \bar{x}_2$, this contradicts the fact that $\bar{x}_0$ is the unique solution of $(\mathcal{L})$.

iii) Suppose that $P_{\overline{W}}$ is integral. If system $Ax = b$ does not produce any redundant relation different from $\delta_0(W) = b_W$, then by *ii)*, the solution y_0 is the unique solution of the system

$$\begin{cases} \bar{A}x = \bar{b}, \\ x(\delta_0(W)) = b_W. \end{cases}$$

But this implies that y_0 is an extreme point of $P_{\overline{W}}$. As $\delta_0(W)$ is tight for $\bar{x}_0$, and therefore y_0 is fractional, this is a contradiction. □

2.4. Basic linear systems

Definition 2.1. *Let $\delta(W)$ be a tight cut. Consider a partition of $Ax = b$ into two subsystems $A_1x = b_1$ and $\tilde{A}_1x = \tilde{b}_1$. We say that $A_1x = b_1$ is a basic system for W if the following conditions hold:*

i) *$x \in \mathbb{R}^{\Gamma(W)}$ is a solution of $Ax = b$ if and only if x is a solution of $\{A_1x = b_1;\ x(\delta_0(W)) = b_W\}$.*

ii) *The solutions set of the system $\{A_1x = b_1;\ x(\delta_0(W)) > b_W\}$ is non-empty, and any solution x of this system also satisfies $\tilde{A}_1x > \tilde{b}_1$.*

If the equation $x(\delta_0(W)) = b_W$ is not a redundant relation of $Ax = b$, then $Ax = b$ is itself a basic linear system. If this is not the case, then there exists a proper subsystem $A_1x = b_1$ of $Ax = b$ in which $x(\delta_0(W)) = b_W$ is not redundant. Also a solution x satisfies $Ax = b$ if and only if x satisfies $\{A_1x = b_1,\ x(\delta_0(W)) = b_W\}$.

Let $A_1x = b_1$ be a basic system for W. Let $C_1x = d_1$ be a redundant system for $A_1x = b_1$. We will now establish a lemma which will allow us to deal with $A_1x = b_1$ instead of $Ax = b$.

Lemma 3. *i) If $x(\delta_0(W)) = b_W$ is not a redundant equation of $Ax = b$, then $C_1x = d_1$ is also a redundant system for $Ax = b$,*

ii) If $x(\delta_0(W)) = b_W$ is a redundant equation of $Ax = b$, then the linear system $\{C_1x = d_1;\ x(\delta_0(W)) = b_W\}$ is a redundant system for $Ax = b$.

Proof. *i)* This easily follows from the fact that $A_1x = b_1$ is a basic system for W.
ii) Since $\{A_1x = b_1;\ x(\delta_0(W)) = b_W\}$ generates the whole system $Ax = b$, any redundant equation produced by $Ax = b$ can also be obtained from $\{A_1x = b_1;\ x(\delta_0(W)) = b_W\}$. The result then follows.

□

3. Critical extreme points of $P(G)$

In this section we introduce the concept of critical extreme points and give our main result.

Let x be a fractional extreme point of $P(G)$. Let x' be a solution obtained by replacing some (but at least one) non-integer components of x by 0 or 1 (and keeping all the other components of x unchanged). If x' is a point of $P(G)$, then x' can be written as a convex combination of extreme points of $P(G)$. If y is such an extreme point, then y is said to be *dominated* by x and we write $x \succ y$. Note that an extreme point of $P(G)$ may dominate more than one extreme point of $P(G)$. Also note that if x dominates y, then $\{e \in E;\ 0 < y(e) < 1\} \subset \{e \in E;\ 0 < x(e) < 1\}$, and if $x(e)$ is integer, then $y(e)$ is so.

The relation $\succ$ defines a partial ordering on the extreme points of $P(G)$. The minimal elements of this ordering (*i.e.* the extreme points x of $P(G)$ for which there is no extreme point $y \neq x$ such that $x \succ y$) correspond to the integer extreme points of $P(G)$. In what follows we are going to define, in a recursive way, a rank function on the extreme points of $P(G)$.

The minimal extreme points of $P(G)$ will be said extreme points of *rank* 0. An extreme point x of $P(G)$ will be said of *rank* k, for fixed k, if x dominates only extreme points of rank $\leq k-1$ and if x dominates at least one extreme point of rank $k-1$.

The maximum of the ranks of the extreme points of $P(G)$ will be called the *rank* of $P(G)$.

In this paper we will be mainly interested in the study of the extreme points of $P(G)$ of rank 1. Note that if x is an extreme point of $P(G)$ of rank 1 and if we replace one fractional component of x by 1, keeping unchanged all the other components, we obtain a feasible point x' of $P(G)$ which can be written as a convex combination of integer extreme points of $P(G)$.

In [32], Mahjoub introduced the following operations:

θ_1: Delete an edge.
θ_2: Contract an edge whose both endnodes are of degree two.
θ_3: Contract a node subset W such that $G(W)$ is 2-edge connected.

And he proved the following.

Theorem 1. [32] *If $G = (V, E)$ is perfectly-TEC and G' is a graph obtained by repeated applications of the operations $\theta_1, \theta_2, \theta_3$, then G' is perfectly-TEC.*

Now we are going to consider somewhat similar operations but defined with respect to a given solution x of $P(G)$.

θ_1': Delete an edge e with $x(e) = 0$.
θ_2': Contract an edge e whose one of its endnodes is of degree 2.
θ_3': Contract a node subset W such that $G(W)$ is 2-edge connected and $x(e) = 1$ for all l $e \in E(W)$.

In contrast with operations θ_1 and θ_1' (θ_3 and θ_3') which are similar, operations θ_2 and θ_2' are quite different. Suppose for instance that G contains a path P of length $k \geq 3$ whose internal nodes are of degree two. Using θ_2 one can replace P by a path P' of length ≥ 2. However, by θ_2', P can be replaced by only one edge. This is because with θ_2' we do not only deal with the graph G (which is the case of θ_2) but we are also dealing with a feasible solution x of $P(G)$. If $P = (e_1, \dots e_k)$, then we have $x(e_i) = 1$, for $i = 1, \dots, k$. And by replacing P by only one edge, say e_k, and assigning again the value 1 to $x(e_k)$, we keep a record of the values of the variables $x(e)$ on the edges of P.

Starting from a graph G and a solution x of $P(G)$, and applying operations $\theta_1', \theta_2', \theta_3'$ one obtains a reduced graph G' and a solution $x' \in P(G')$. The following lemmas establish the relation between x and x'. The proof of the first lemma is omitted because it is similar to that of Theorem 1.

Lemma 4. *The solution x is an extreme point of $P(G)$ if and only if x' is an extreme point of $P(G')$.*

Lemma 5. *The solution x is an extreme point of $P(G)$ of rank* 1 *if and only if x' is an extreme point of $P(G')$ of rank* 1.

Proof. Suppose that x is an extreme point of $P(G)$ of rank 1. By Lemma 4, x' is an extreme point of $P(G')$. If x' is not of rank 1, then there must exist a fractional extreme point x'' of $P(G')$ that is dominated by x'. Let $y \in \mathbb{R}^E$ be the solution such that

$$y(e) = \begin{cases} x''(e) & \text{if } e \in E', \\ 1 & \text{if } x(e) = 1 \text{ and } e \in E \setminus E', \\ 0 & \text{if } x(e) = 0 \text{ and } e \in E \setminus E'. \end{cases}$$

Note that x'' can be obtained from y using the same operations as x' from x. By Lemma 4, y is then an extreme point of $P(G)$. Moreover, we have that $x \succ y$. As y is fractional, this contradicts the fact that x is of rank 1.
If x' is of rank 1, we can prove in a similar way that x is also of rank 1. □

An extreme point of $P(G)$ will be called *critical* if it is of rank 1 and if none of the operations $\theta_1', \theta_2', \theta_3'$ can be applied for it. By Lemma 5, the characterization of the extreme points of rank 1 reduces then to that of the critical extreme points of $P(G)$. In the rest of the paper we restrict ourselves to that class of extreme points. Our aim is to give necessary conditions for an extreme point of $P(G)$ to be critical.

Definition 3.1. *Given a graph $G = (V, E)$ and a solution $x \in P(G)$, the pair (G, x) is called a* basic pair *if the following hold.*

i) $V = V^1 \cup V^2$ with $V^1 \cap V^2 = \emptyset$,
$E = E^1 \cup E^2$ with $E^1 \cap E^2 = \emptyset$,

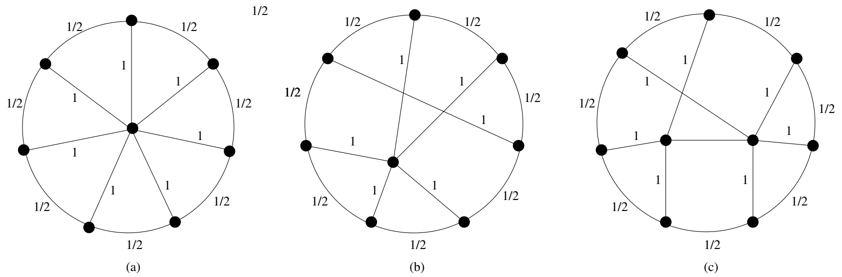

Fig. 1. Basic pairs

(V^1, E^1) is an odd cycle,
(V, E^2) is a forest whose set of pending nodes is V^1, and such that all the nodes of V^2 have degree at least 3,

ii) $x(e) = \frac{1}{2}$ for $e \in E^1$,
$x(e) = 1$ for $e \in E^2$, and

iii) $x(\delta(W)) > 2$ for any cut $\delta(W)$ such that $|W| \geq 2$ and $|\overline{W}| \geq 2$.

Note that by the last condition of Definition 3.1, if (G, x) is a basic pair, then G is 3-edge connected. Figure 1 shows some examples of basic pairs. In Figure 1 (a) the forest is a star, Figure 1 (b) shows an example where one component of the forest is an edge. In Figure 1 (c) the forest is a tree but not a star.

Lemma 6. *If (G, x) is a basic pair, then x is an extreme point of $P(G)$.*

Proof. First note that conditions ii) and iii) of Definition 3.1 ensure that x is a solution of $P(G)$. Moreover, since (V^1, E^1) is an odd cycle, x is the unique solution of the system

$$\begin{cases} x(e) = 1 & \forall\, e \in E^2, \\ x(\delta(v)) = 2 & \forall\, v \in V^1. \end{cases}$$

This implies that x is an extreme point of $P(G)$. □

We can now state the main result of the paper.

Theorem 2. *If $\bar{x}$ is a critical extreme point of $P(G)$, then $(G, \bar{x})$ is a basic pair.*

The proof of Theorem 2 will be given in the next section. In what follows we give, as a consequence of Theorem 2, a characterisation of the perfectly-TEC graphs.

Let Ω be the class of graphs $G = (V, E)$ such that G can be obtained from a graph satisfying condition i) of Definition 3.1 by replacing some edges of E^2 by paths of length two. Note that the graphs of Ω are not perfectly-TEC. In fact, let $\tilde{G} = (\tilde{V}, \tilde{E})$ and $\tilde{x} \in \mathbb{R}^{\tilde{E}}$ form a basic pair and $G = (V, E)$ be a graph of Ω obtained from $\tilde{G}$ by inserting nodes of degree two on some edges of the forest of $\tilde{G}$. Let $x \in \mathbb{R}^E$ be the solution of $P(G)$ such that $x(e) = \tilde{x}(e)$ if $e \in \tilde{E}$ and $x(e) = 1$ if $e \in E \setminus \tilde{E}$. As by Lemma 6 $\tilde{x}$ is an extreme point of $P(\tilde{G})$, x is an extreme point of $P(G)$. Since x is fractional, G is not perfectly-TEC.

A consequence of Theorem 2 is the following.

Corollary 1. *A graph G is perfectly-TEC if and only if G is not reducible to a graph of* Ω *by means of the operations* $\theta_1, \theta_2, \theta_3$.

Proof. Assume that $G = (V, E)$ reduces to a graph $G' = (V', E')$ of Ω by means of the operations $\theta_1, \theta_2, \theta_3$. By Lemma 6, G' is not perfectly-TEC. Hence from Theorem 1, it follows that G is not perfectly-TEC.

Conversely, suppose G is not perfectly-TEC. Then $P(G)$ contains fractional extreme points, and, in consequence, there existo an extreme point y of $P(G)$ of rank 1. By Lemma 5 together with Theorem 2, G and y can be reduced by operations $\theta_1', \theta_2', \theta_3'$ to a basic pair (G', y'). If instead of applying θ_2' we apply θ_2, we obtain a graph of Ω. □

4. Proof of Theorem 2

We first present the main steps of the proof of Theorem 2. We will suppose that $\bar{x}$ is a critical extreme point of $P(G)$. We first establish some useful properties of $\bar{x}$ and the tight cuts for $\bar{x}$ in G. These are given in Lemmas 7, 8 and 9. Then the proof is divided into two parts. In the first part, we suppose that the set of proper cuts Π is empty. We will show that $\bar{x}$ and G satisfy in this case the conditions of Definition 3.1, and hence induce a basic pair. This is given by Proposition 1. In the second part, which is the crucial part of the proof, we will show that $\Pi = \emptyset$, and thus, by the first part, the theorem follows.

So suppose that $\bar{x}$ is critical. Then $\bar{x}$ is of rank 1, and $\bar{x}$ cannot be reduced by any of the operations θ_1', θ_2', θ_3'. (Recall that a cut $\delta(W)$ is proper if it is tight for $\bar{x}$ and $E_0(W) \neq \emptyset \neq E_0(\overline{W})$.)

4.1. Structural properties

We have the following lemmas. The first one, given without proof, is a direct consequence of the fact that $\bar{x}$ is critical.

Lemma 7. *i)* $\bar{x}(e) > 0$ *for all* $e \in E$.
ii) G contains no nodes of degree 2.
iii) If for some $W \subseteq V$, $G(W)$ *is 2-edge connected, then* $E_0(W) \neq \emptyset$.

Lemma 8. *Let* $\delta(W)$ *be a proper cut tight for* $\bar{x}$. *Then*

i) $\overline{x}(f) + \overline{x}(g) \leq 1$ *for every two edges* f, g *of* $\delta(W)$,
ii) $|\delta(W)| \geq 4$.

Proof. *i)* Suppose that $\bar{x}(f) + \bar{x}(g) > 1$. Let $y \in \mathbb{R}^{\Gamma(W)}$ be the restriction of $\bar{x}$ on $\Gamma(W)$. Clearly y is a solution of $Ax = b$. Let $\bar{y}$ be the extension of y. Note that $\bar{y} \in P(G)$ and $\delta(W)$ is tight for $\bar{y}$. Also note that, as $\delta(W)$ is proper, and hence, $E_0(\overline{W}) \neq \emptyset$, $\{e \in E;\ 0 < \bar{y}(e) < 1\} \subset \{e \in E;\ 0 < \bar{x} < 1\}$. As $\bar{x}$ is critical, $\bar{y}$ can be written as a convex combination of integer extreme points of $P(G)$. Let $\bar{y}^1$ be one of these extreme points, and let $\bar{y}_0^1$ be its restriction on $\Gamma(W)$. Notice that any constraint of $P(G)$ that is tight for $\bar{y}$ is also tight for $\bar{y}^1$. Hence $\bar{y}_0^1$ is a solution of $Ax = b$, and as $\delta(W)$ is tight for $\bar{y}$, we have that $\bar{y}^1(\delta(W)) = 2$. Moreover, since $\bar{x}(f) + \bar{x}(g) > 1$, $\bar{y}^1$ can be chosen so that $\bar{y}^1(f) = \bar{y}^1(g) = 1$ and hence $\bar{y}^1(e) = 0$ for all $e \in \delta(W) \setminus \{f, g\}$. By considering the restriction of $\bar{x}$ on $\Gamma(\overline{W})$ we can similarly show that there exists an integer solution $\bar{y}^2$ of $P(G)$ such that its restriction on $\Gamma(\overline{W})$, $\bar{y}_0^2$ is a solution of $\bar{A}x = \bar{b}$, $\bar{y}^2(f) = \bar{y}^2(g) = 1$ and $\bar{y}^2(e) = 0$ for all $e \in \delta(W)\setminus\{f, g\}$. Observe that $\bar{y}_0^1(e) = \bar{y}_0^2(e)$ for all $e \in \delta_0(W)$. Let $\tilde{y} \in \mathbb{R}^{E_0}$ be the solution given by

$$\tilde{y}(e) = \begin{cases} \bar{y}_0^1(e) & \text{if } e \in \Gamma(W), \\ \bar{y}_0^2(e) & \text{if } e \in E_0(\overline{W}). \end{cases}$$

We have that $\tilde{y}$ is a solution of $(\mathcal{L})$. Since $\tilde{y} \neq \bar{x}_0$, this is impossible.

ii) If $|\delta(W)| \leq 3$, then there exists two edges f, g in $\delta(W)$ with $\bar{x}(f) + \bar{x}(g) > 1$. But this contradicts *i)*. □

Lemma 9. *If $\delta(S)$ is a cut of $\mathcal{S}(G)$ tight for $\bar{x}$ with $|S| \geq 2$, then $\delta(S)$ is proper.*

Proof. If $\delta(S)$ is not proper, then at leat one of the sets $E_0(S)$ and $E_0(\overline{S})$ is empty. Suppose, for instance, that $E_0(S) = \emptyset$. Then $E(S) = E_1(S)$. Since $G(S)$ is 2-edge connected, $\bar{x}$ can be reduced by operation θ_3', a contradiction. □

4.2. Case $\Pi = \emptyset$

Proposition 1. *If $\Pi = \emptyset$, then ($G, \overline{x}$) is a basic pair.*

Proof. Assume that $\Pi = \emptyset$. By Lemma 9, $\bar{x}_0$ is the unique solution of the linear system

$$x(\delta_0(v)) = b_v, \quad \forall\, v \in V^*,$$

where V^* is the set of tight nodes of V. Let $G_0' = (V', E_0')$ be a connected component of the fractional graph G_0. Let $V^{*\prime} = V^* \cap V'$. Then the linear system

$$x(\delta_0(v)) = b_v, \qquad \forall\, v \in V^{*\prime}, \tag{4.1}$$

has also a unique solution, namely the restriction of $\bar{x}$ to E_0'. Thus $|V^{*\prime}| \geq |E_0'|$. On the other hand, since G_0' is connected, $|V^{*\prime}| \leq |V'| \leq |E_0'| + 1$. Hence

$$|V^{*\prime}| \geq |V'| - 1. \tag{4.2}$$

In addition, if $v \in V^*$, then v is adjacent to at least two fractional edges. If G_0' is a tree, then G_0' has at least two (pending) nodes which do not induce tight cuts and therefore $|V^{*\prime}| \leq |V'| - 2$, contradicting (4.2). Thus G_0' is not a tree, and hence $|V'| \leq |E_0'|$. We then have that $|V^{*\prime}| \leq |V'| \leq |E_0'| \leq |V^{*'}|$. Consequently, $|V^{*\prime}| = |V'|$. This implies that $V^{*\prime} = V'$, and the matrix of system (4.1) is the node-edge incidence matrix of a cycle. Moreover, as this matrix is non-singular, this cycle must be odd, and therefore the unique solution of (4.1) is $\bar{x}(e) = \frac{1}{2}$ for all $e \in E_0'$. We claim that $E_0' = E_0$. In fact, if $E_0' \subset E_0$, let $\bar{y} \in E$ be the solution such that

$$\bar{y}(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E_0', \\ 1 & \text{if } e \in E_0 \setminus E_0'. \end{cases}$$

Obviously, $\bar{y} \in P(G)$. Also as $\bar{y}$ is a unique solution of system 4.1 and $\bar{y}(e) = 1$ for all $e \in E \setminus E_0'$, $\bar{y}$ is an extreme point of $P(G)$. Since $\bar{x} \succ \bar{y}$, this implies that $\bar{x}$ is not critical, a contradiction.

Consequently, G_0 is an odd cycle such that $\bar{x}(e) = \frac{1}{2}$ for all $e \in E_0$. Moreover, as $\bar{x}$ is critical, by Lemma 7 *i)* $\bar{x}(e) > 0$ for all $e \in E$. Hence $\bar{x} = 1$ for all $e \in E \setminus E_0$. Also, as every node of V_0 is tight, it follows by Lemma 7 *iii)* that the integer graph, G_1 is a forest whose pending nodes are precisely the nodes of V_0. Thus conditions i) and iii) of Definition 3.1 are satisfied with respect to G and $\bar{x}$.

In addition, if $|\delta(W)| = 2$ for some cut $\delta(W)$ and, say $W = \{v\}$, then v is a node of degree two. But this contradicts Lemma 7 *ii)*. Therefore condition iii) of Definition 3.1 is also satisfied with respect to G and $\bar{x}$. Consequently, G and $\bar{x}$ form a basic pair. □

4.3. Case $\Pi \neq \emptyset$

4.3.1. Further structural properties We first prove a basic lemma, which establishes further structural properties of $\bar{x}$ and the tight cuts of G when $\Pi \neq \emptyset$. In particular we will show in this case that the components of $\bar{x}$ are 0, 1 and $\frac{1}{2}$, and all the proper cuts have exactly four edges.

Lemma 10. *If* $\Pi \neq \emptyset$*, then*

i) $\bar{x}(e) = \frac{1}{2}$ *for all* $e \in E_0$*, and*
ii) $|\delta(S)| = 4$ *for any proper cut* $\delta(S)$ *of* Π.

Proof. Let $\delta(S)$ be a proper cut. Let $\delta(W)$ be a proper cut such that $W \subseteq S$ and $|W|$ is minimum, that is $|W| \leq |S'|$ for any proper cut $\delta(S')$ with $S' \subseteq S$. Thus, if $\delta(T)$ is a tight cut with $T \subset W$, then either $x(\delta_0(T)) = b_T$ is redundant in $(\mathcal{L})$ and hence in $Ax = b$ or $\delta_0(T)$ is a degree-cut. Thus we may suppose that the equations of $Ax = b$ all correspond to degree-cuts.

Let $f = uv \in E_0(W)$ and let $x^* \in \mathbb{R}^E$ be defined as

$$x^*(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E \setminus \{f\}, \\ 1 & \text{if } e = f. \end{cases}$$

Clearly, $x^* \in P(G)$. If none of the nodes u and v induces a tight cut for $\bar{x}$, then clearly x_0^* is a solution of $(\mathcal{L})$. As $x^* \neq \bar{x}_0$, this is a contradiction.

If only one of these nodes, say u is tight, then we have $x^*(\delta(u)) = 3 - \bar{x}(f)$. As $x^* \in P(G)$ and $\bar{x}$ is critical, x^* can be written as a convex combination of integer extreme points of $P(G)$. Let y^1 be one of these extreme points. Since $\bar{x}(f) > 0$ and thus $x^*(\delta(u)) < 3$, y^1 can be chosen so that $y^1(\delta(u)) = 2$. Moreover, every constraint of $P(G)$ which is tight for x^* is also tight for y^1. Thus y_0^1 is a solution of $(\mathcal{L})$. As $y_0^1 \neq \bar{x}_0$, we have again a contradiction. (Recall that y_0^1 is the restrictions of y^1 on E_0.)

So assume that both nodes u and v are tight for $\bar{x}$. Hence $x^*(\delta(u)) < 3$ and $x^*(\delta(v)) < 3$. As we did in the previous case, we can show that there exists an integer solution y^2 of $P(G)$ such that $y^2(\delta(v)) = 2$ and y^2 satisfies with equality every constraint of $P(G)$ which is tight for x^*. Thus $y^1(W) = y^2(W) = 2$. Moreover, we have

$y^1(\delta(v)) > 2$ and $y^2(\delta(u)) > 2$. For otherwise, either y^1_0 or y^2_0 would satisfy system $(\mathcal{L})$, contradicting the fact that $\bar{x}_0$ is the unique solution of $(\mathcal{L})$.

Now it is not hard to see that there exists $0 \leq \alpha \leq 1$ such that the solution

$$\bar{y} = \alpha y^1 + (1-\alpha) y^2 \tag{5.1}$$

satisfies

$$\bar{y}(\delta(u)) = \bar{y}(\delta(v)) = \rho > 2.$$

Note that $\bar{y}$ satisfies with equality all the constraints tight for $\bar{x}$ other than $x(\delta(u)) \geq 2$ and $x(\delta(v)) \geq 2$. Also note that $\bar{y}(f) = 1$. Let $z \in \mathbb{R}^E$ such that

$$z(e) = \begin{cases} \bar{y}(e) & \text{if } e \in E \setminus \{f\}, \\ 3 - \rho & \text{if } e = f. \end{cases}$$

As $z(\delta(u)) = z(\delta(v)) = 2$, we have that z is a solution of $(\mathcal{L})$. Since $\bar{x}_0$ is the unique solution of $(\mathcal{L})$, we then have $\bar{x} = z$.

Consider now our initial cut $\delta(S)$. Since $y^1(\delta(S)) = 2$ and $y^2(\delta(S)) = 2$, there are four edges $e_1, e_2, e_3, e_4 \in \delta(S)$ such that

$$\begin{cases} y^1(e_1) = y^1(e_2) = 1, \text{ and } y^1(e) = 0 & \text{for all } e \in \delta(S) \setminus \{e_1, e_2\}, \\ y^2(e_3) = y^2(e_4) = 1, \text{ and } y^1(e) = 0 & \text{for all } e \in \delta(S) \setminus \{e_3, e_4\}. \end{cases}$$

Since $\bar{x}(e) > 0$ for all $e \in \delta(S)$, it follows that $|\delta(S)| \leq 4$. For otherwise, there would exist an edge $e \in \delta(S)$ such that $y^1(e) = y^2(e) = 0$. But this implies that $\bar{x}(e) = z(e) = 0$, a contradiction.

If $e_i = e_j$ for some i, j where $i \in \{1, 2\}$ and $j \in \{3, 4\}$, then $\bar{y}(e_i) = z(e_i) = 1$ and hence $\bar{x}(e_i) = 1$. But this implies that $\bar{x}(e_i) + \bar{x}(e_j) > 1$, contradicting Lemma 8 *i)*. Consequently, e_1, e_2, e_3, e_4 are all distinct, and therefore $\delta(S) = \{e_1, e_2, e_3, e_4\}$. Hence $|\delta(S)| = 4$, and *ii)* is proved.

Moreover we have

$$\begin{aligned} \bar{x}(e_1) + \bar{x}(e_2) &= 2\alpha, \\ \bar{x}(e_3) + \bar{x}(e_4) &= 2(1-\alpha). \end{aligned}$$

By Lemma 8 *i)*, we should also have $\overline{y}(e_1) + \overline{y}(e_2) = 2\alpha \leq 1$ and $\overline{y}(e_3) + \overline{y}(e_4) = 2 - 2\alpha \leq 1$. Thus $\alpha = \frac{1}{2}$. This implies that $\overline{x}(e) = \frac{1}{2}$ for all $e \in E_0 \setminus \{f\}$. Since $\bar{x}(\delta(u)) = 2$ and $\bar{x}(f) > 0$, we should also have $\bar{x}(f) = \frac{1}{2}$, which proves *i)*, and ends the proof of the lemma. □

4.3.2. No proper cuts exist In what follows we are going to show that no proper cuts exist. Once this fact is established, Theorem 2 follows from Proposition 1.

Suppose, on the contrary, that there exists a proper cut $\delta(W)$. By Lemma 10 *ii)* there are four edges $e_1, e_2, e_3, e_4 \in E_0$ such that $\delta(W) = \{e_1, e_2, e_3, e_4\}$. The set W is called a *nice set* if there exists a basic system $A_1 x = b_1$ for W such that, after eventual permutation of the four edges of $\delta(W)$, the relation $x(e_1) + x(e_2) = 1$ is the unique redundant relation produced by $A_1 x = b_1$.

Now the proof will be organized as follows. First we will prove the following crucial result (Proposition 2): any proper cut is induced by a nice set. The proof of this proposition uses two preliminary results. The first, established by Lemma 11, states that the family of nice sets contained in W is a nested family. (The definition of a nested family is given below.) And the second, given in Lemma 12, shows that the polytopes P_W and $P_{\overline{W}}$ are integral. As a consequence, we will obtain that the sets W and $\overline{W}$ are both nice. Using this, we will finally show that there is an integer solution that satisfies system ($\mathcal{L}$). As this solution is different from $\bar{x}$, we will thus get a contradiction.

By Lemma 3, if $x(\delta(W)) = 2$ is linearly independent in the system $Ax = b$, then $x(e_1)+x(e_2) = 1$ is also the unique redundant equation produced by $Ax = b$. However, if $x(\delta(W)) = 2$ is a redundant equation of $Ax = b$, then $\{x(e_1)+x(e_2) = 1,\ x(\delta(W)) = 2\}$ is a redundant system for $Ax = b$. Thus, if W is a nice set and x is a solution of

$$P_W \cap \{x \in \mathbb{R}^{\Gamma(W)};\ Ax = b,\ x(\delta(W)) = 2\},$$

then $x(e_1) + x(e_2) = 1$ and $x(e_3) + x(e_4) = 1$.

Two sets S and T are said to be *crossing* if $S \cap T \neq \emptyset$, $S \cap \overline{T} \neq \emptyset$, $\overline{S} \cap \overline{T} \neq \emptyset$ and $\overline{S} \cap T \neq \emptyset$. A family of sets $S_1, \dots, S_k$ is said to be *nested* if $S_1, \dots, S_k$ are pairwise noncrossing.

Lemma 11. *Let $W \subset V$ be a nice set. Then for every proper cut $\delta(T)$, W and T are noncrossing.*

Proof. Assume, on the contrary, that T and W are crossing. Let $W_1 = W \cap T$ and $W_2 = W \cap \overline{T}$. As $\delta(W)$ and $\delta(T)$ are tight for $\bar{x}$, it is easy to see that $\delta(W_1)$ and $\delta(W_2)$ are also tight for $\bar{x}$, and $\bar{x}(\delta(W_1, W_2)) = \bar{x}(\delta(W_1, \overline{W})) = \bar{x}(\delta(W_2, \overline{W})) = 1$. As $W \in \mathcal{S}_0$, and by Lemma 10 *i)*, $\bar{x}(e) = \frac{1}{2}$ for all $e \in E_0$, we have that $|\delta(W_1, \overline{W})| = |\delta(W_2, \overline{W})| = |\delta(W_1, W_2)| = 2$. So we may suppose that $\delta(W_1, \overline{W}) = \{e_1, e_2\}$ and $\delta(W_2, \overline{W}) = \{e_3, e_4\}$. Let $\delta(W_1, W_2) = \{f_1, f_2\}$. Then the following equations are among those of $Ax = b$,

$$x(\delta_0(W_1)) = x(e_1) + x(e_2) + x(f_1) + x(f_2) = 2,$$
$$x(\delta_0(W_2)) = x(e_3) + x(e_4) + x(f_1) + x(f_2) = 2.$$

As W is a nice set, it then follows that the unique redundant relation of $Ax = b$ is either $x(e_1) + x(e_2) = 1$ or $x(e_3) + x(e_4) = 1$. Consider the following solution $y \in \mathbb{R}^E$ such that

$$y(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E(\overline{W}) \cup \delta(W), \\ 1 & \text{if } e \in E(W) \setminus \{f_2\}, \\ 0 & \text{if } e = f_2. \end{cases}$$

As $\delta(W)$ is proper, and hence $G(W)$ is 2-edge connected, by Lemma 1, $y \in P(G)$. Also observe that both cuts $\delta(W_1)$ and $\delta(W_2)$ are tight for y. Hence y is a solution of the system

$$(\mathcal{L}_1)\begin{cases} x(e) = 1 & \text{if } e \in (E_1 \cup E(W)) \setminus f_2, \\ x(e) = 0 & \text{if } e = f_2, \\ x(e_1) + x(e_2) = 1, \\ x(e_3) + x(e_4) = 1, \\ \bar{A}x = \bar{b}. \end{cases}$$

By Lemma 2 *ii)*, it follows that y is the unique solution of $(\mathcal{L}_1)$, and in consequence y is an extreme point of $P(G)$. Moreover we have that $\bar{x} \succ y$. Since y is fractional, this contradicts the fact that $\bar{x}$ is critical. □

Lemma 12. *P_W and $P_{\overline{W}}$ are integral.*

Proof. We show the statement for P_W, the proof for $P_{\overline{W}}$ is similar. Assume the contrary, and let $y \in \mathbb{R}^{\Gamma(W)}$ be a fractional extreme point of P_W. Let $\bar{y} \in \mathbb{R}^E$ be the extension of y. As $G(\overline{W})$ is 2-edge connected, from Lemma 1 it follows that $\bar{y} \in P(G)$. Furthermore since y is an extreme point of P_W, we have that $\bar{y}$ is also an extreme point of $P(G)$. Since $\delta(W)$ is proper, and hence $E_0(\overline{W}) \neq \emptyset$, we have that $\bar{x} \succ \bar{y}$. As $\bar{y}$ is fractional, this is a contradiction. □

Now suppose that the cut $\delta(W)$ is such that if $S \subset W$, then S is a nice set. As $\delta(W) \in \Pi$. By Lemma 11 there exists a partition $S_1, \ldots, S_r, \{w_1\}, \ldots, \{w_s\}$ such that $S_1, \ldots, S_r$ are maximal noncrossing nice sets and $w_1, \ldots, w_s$ are tight nodes. Note that if $T \subset W$ induces a proper cut, then $T \subseteq S_i$ for some $i = 1, \ldots, r$. Also notice that, as by Lemma 10 *i)* $\bar{x}(e) = \frac{1}{2}$ for all $e \in E_0$, every $w_i,\ i = 1, \ldots, s$, is either of degree 2 or of degree 4 in G_0. If w_i, for $i = 1, \ldots, s$, is of degree 2 (resp. 4), then w_i will be called a *2-singleton* (resp. 4-*singleton*). Observe that if w_i is a 2-singleton (resp. 4-singleton), then $\bar{x}(\delta_0(w_i)) = 1$ (resp. $\bar{x}(\delta_0(w_i)) = 2$), and hence $x(\delta_0(w_i)) = 1$ (resp. $x(\delta_0(w_i)) = 2$) is one of the row of $Ax = b$.

We now establish the following which is the crucial point of the proof. It shows that any proper cut is induced by a nice set.

Proposition 2. *Any proper cut is induced by a nice set.*

Proof. We will show that W is a nice set. To this end we will consider two cases.

Case 1. W contains no nice sets.

Let W^* be the set of tight nodes in W (*i.e.* $W^* = \{w_1, \ldots, w_s\}$). As W does not contain nice sets, it follows that $Ax = b$ is given by

$$x(\delta_0(v)) = b_v, \qquad \forall\, v \in W^*. \tag{4.3}$$

Let $v_i,\ i = 1, \ldots, 4$ be the nodes (not necessarily distinct) of W incident to e_i; $i = 1, \ldots, 4$, respectively.

Claim 1. Let $W_1 \subseteq W$ be a node subset that induces a connected component of $G_0(W)$. Let $A_1x = b_1$ be the linear system given by the inequalities (4.3) corresponding to the nodes of $W^* \cap W_1$. Then

i) $|\delta(W_1) \cap \delta(W)| \geq 1$,

ii) Assume that $A_1x = b_1$ produces a redundant constraint distinct from $x(\delta(W)) = 2$. Then

ii.1) $G_0(W_1)$ is a tree $\mathcal{T}$ and all the nodes of W_1 belong to W^*,

ii.2) $\mathcal{T}$ has at most one 4-singleton,

ii.3) if $\mathcal{T}$ has one 4-singleton, then $\mathcal{T}$ has at least one 2-singleton, and (after eventual permutation of edges) $A_1x = b_1$ produces a unique redundant relation given by

$$x(e_1) + x(e_2) - x(e_3) - x(e_4) = 0, \tag{4.4}$$

ii.4) if all the nodes of $\mathcal{T}$ are 2-singletons, then $\mathcal{T}$ is a path. Moreover, if $\mathcal{T}$ is an even (resp. odd) path, then $A_1x = b_1$ produces a unique redundant relation given by

$$x(e_1) + x(e_2) = 1, \tag{4.5}$$

$$(\text{resp. } x(e_1) - x(e_2) = 0.) \tag{4.6}$$

ii.5) The rows of $A_1x = b_1$ are linearly independent.

Proof. *i)* If $\delta(W_1) \cap \delta(W) = \emptyset$, then $\delta_0(W_1, \overline{W}) = \emptyset$, and therefore $A_1x = b_1$ has a unique solution, namely the restriction of $\bar{x}$ to $E_0(W_1)$. Note that this solution is fractional. As any extreme point of the polytope $P'_W = P_W \cap \{x \in \mathbb{R}^{\Gamma(W)};\ Ax = b,\ x(\delta(W)) = 2\}$ is also a solution of the system $A_1x = b_1$, it then follows that all the extreme points of P'_W are fractional. Since by Lemma 12 P_W is integral, and hence P'_W is so, this is impossible.

ii.1) We first prove that $G_0(W)$ does not contain even cycles. Indeed, assume, on the contrary, that $G_0(W)$ contains an even cycle $\mathcal{C} = \{f_1, \ldots, f_{2p}\}$, $(k > 1)$. Consider the solution $y \in \mathbb{R}^E$ defined as

$$y(e) = \begin{cases} \bar{x}(e) & \forall\, e \in E \setminus \mathcal{C}, \\ \bar{x}(f_{2i+1}) + \epsilon & \forall\, i = 0, \ldots, p-1, \\ \bar{x}(f_{2i}) - \epsilon & \forall\, i = 1, \ldots, p, \end{cases}$$

where $\epsilon > 0$. Since all the equations of system $Ax = b$ correspond to degree-cuts, we have that y is a solution of system $(\mathcal{L})$. As $y \neq \bar{x}$, this is a contradiction.

Now by *i)*, we may assume that $v_1 \in W_1$. Let $cx = d$ be a redundant relation produced by $A_1x = b_1$ and distinct from $x(\delta(W)) = 2$. If $v \in W^* \cap W_1$, let $\mu(v)$ be the weight of the equation $x(\delta(v)) = b_v$ in $cx = d$. W.l.o.g., we may assume that $c(e_1) = +1$, and hence $\mu(v_1) = +1$. Let $e = vw$ be an edge of $E_0(W_1)$. As $c(e) = 0$, if $v \in W^*$ and $\mu(v) \neq 0$, one should have $w \in W^*$ and $\mu(w) = -\mu(v)$. As $v_1 \in W_1 \cap W^*$, and $\mu(v_1) = +1$, this implies that all the nodes of W_1 belong to W^*, and $\mu(u) = +1$ or -1 for all $u \in W_1$. Let ${W_1}^+$ (resp. ${W_1}^-$) be the set of nodes $u \in W_1$ with $\mu(u) = +1$ (resp. $\mu(u) = -1$). Note that ${W_1}^+$ and ${W_1}^-$ are determined in a unique way. Hence ${W_1}^+$ and ${W_1}^-$ is a partition of W_1, and therefore the graph $G_0(W_1)$ is bipartite. In consequence, $G_0(W_1)$ does not contain odd cycle. Since $G_0(W_1)$ does not also contain even cycles, it follows that $G_0(W_1)$ is a tree $\mathcal{T}$.

ii.2) By *ii.1)* every pending node of $\mathcal{T}$ belongs to W^* and hence is either a 2 or a 4-singleton. Also at least one of the edges incident to it belongs to $\delta(W)$. Hence every pending node of $\mathcal{T}$ is one of the nodes $v_1, \dots, v_4$. In consequence, $\mathcal{T}$ cannot contain more than one 4-singleton.

ii.3) If $\mathcal{T}$ has a 4-singleton, then the four nodes $v_1, \dots, v_4$ belong to $\mathcal{T}$. Now recall that $\mu(v_1) = +1$ and that the weights of the equations of $A_1x = b_1$, involved in the generation of the redundant relation $cx = d$ are $+1$ and -1. Also these weights are determined in a unique way. In consequence, there is a unique redundant relation. If the weights of the constraints induced by the four nodes is $+1$, then the redundant relation is $x(e_1) + x(e_2) + x(e_3) + x(e_4) = 2$. But this contradics the fact that $cx = d$ is different from $x(\delta_0(W)) = 2$. If only one node among $v_1, \dots, v_4$, say v_4, induces a constraint with weight -1, then the redundant relation would be $x(e_1) + x(e_2) + x(e_3) - x(e_4) = 1$. As we also have $x(e_1) + x(e_2) + x(e_3) + x(e_4) = 2$, from both equalities we get $x(e_4) = \frac{1}{2}$. So this implies that the polytope $P_W \cap \{x \in \mathbb{R}^{\Gamma(W)},\ Ax = b,\ x(\delta(W)) = 2\}$ is not integral, contradicting Lemma 12. Consequently, exactly two nodes among $v_1, \dots, v_4$ induce constraints with weight -1, and therefore the redundant relation is of the form (4.4). Also note that $\mathcal{T}$ contains at least one 2-singleton.

ii.4) Suppose that $\mathcal{T}$ is an even path, and, w.l.o.g., v_1 and v_2 are the extremities of $\mathcal{T}$. As $\mu(v_1) = +1$, it follows that $|W^+| = |W^-| + 1$, and $\mu(v_2) = +1$. This yields the redundant relation (4.5).
If $\mathcal{T}$ is an odd path, and v_1 and v_2 are its extremities, then $|W^+| = |W^-|$ and so $\mu(v_2) = -1$. Hence the redundant relation is of type (4.6).

ii.5) As $\mathcal{T}$ is a tree and the weights $\mu(v)$ for all $v \in W_1$ are uniquely defined, the statement follows.

□

As $|\delta(W)| = 4$, from Claim 1 it follows that, if W contains no nice sets, $G_0(W)$ has at most two connected components each being a tree and producing a redundant relation. If $Ax = b$ produces two redundant relations, then each relation is of type either (4.6) or (4.5), and $G_0(W)$ has no node of degree 4. If $Ax = b$ produces one redundant relation, this is of type either (4.4), (4.6) or (4.5).

Case 2. W contains a nice set.

Thus $r \geq 1$. As $\delta(S_1) \in \Pi$, by Lemma 10 *ii)*, there are four edges $f_1, f_2, f_3, f_4 \in E_0$ such that $\delta(S_1) = \{f_1, f_2, f_3, f_4\}$. Let $A_1x = b_1$ be a basic system for S_1. Note that $A_1x = b_1$ can be considered as a subsystem of $Ax = b$. As S_1 is a nice set, we may suppose that $x(f_1) + x(f_2) = 1$ is the unique redundant relation of $A_1x = b_1$. Let $A_2x = b_2$ be the system given by

$$\begin{cases} x(f_1) + x(f_2) = 1, \\ x(f_3) + x(f_4) = 1, \\ x(\delta_0(T)) = b_T, \qquad \forall\, T \subseteq W,\ T \cap S_1 = \emptyset,\ T \in \mathcal{S}_0. \end{cases}$$

As by Lemma 11, nice sets and node sets that induce proper cuts are noncrossing, it follows that a solution of $\mathbb{R}^{\Gamma(W)}$ satisfies $Ax = b$ if and only if it satisfies both systems $A_1x = b_1$ and $A_2x = b_2$.

Claim 2. Systems $Ax = b$ and $A_2x = b_2$ have the same sets of redundant relations.

Proof. Since $x(f_1) + x(f_2) = 1$ is a redundant relation of $A_1x = b_1$, it is clear that a redundant relation of $A_2x = b_2$ is also a redundant relation of $Ax = b$. Now consider a redundant relation $cx = d$ of $Ax = b$. If this equation is not redundant in $A_2x = b_2$, then there must exist a solution $y_2 \in \mathbb{R}^{\Gamma(W)\setminus E_0(S)}$ of $A_2x = b_2$ such that $cy_2 \neq d$. By Lemma 2 *i)* there exists a solution $y_1 \in \mathbb{R}^{\Gamma(S_1)}$ of $A_1x = b_1$ such that $y_1(f) = y_2(f)$ for all $f \in \delta(S_1)$. Let $\tilde{y} \in \mathbb{R}^{\Gamma(W)}$ such that

$$\tilde{y}(e) = \begin{cases} y_1(e) & \text{if } e \in E_0(S_1), \\ y_2(e) & \text{if } e \in \Gamma(W) \setminus E_0(S_1). \end{cases}$$

Clearly $\tilde{y}$ is a solution of both systems $A_1x = b_1$ and $A_2x = b_2$, and hence is a solution of $Ax = b$. As $cx = d$ is redundant in $Ax = b$, it then follows that $c\tilde{y} = d$. However, $c\tilde{y} = cy_2 \neq d$, a contradiction. □

Claim 2 allows a reduction on the graph G_0. Indeed, consider the graph $\tilde{G}_0 = (\tilde{V}_0, \tilde{E}_0)$ obtained from G_0 as follows:

- remove the nodes of S_1 and the edges of $E_0(S_1)$,
- add two new nodes s_1, s_2, and
- link s_1 to the edges f_1, f_2 and s_2 to the edges f_3, f_4.

Let $\tilde{W} = (W \setminus S_1) \cup \{s_1, s_2\}$. We have that $A_2x = b_2$ is the linear system corresponding to $\tilde{W}$ in $\tilde{G}_0$.

By Claims 1 and 2 we can assume, by induction on the number of maximally nice sets of W, that $A_2x = b_2$ (and therefore $Ax = b$) has either one redundant relation of type either (4.4), (4.5) or (4.6) or two redundant relations each being of type either (4.5) or (4.6). Now the rest of the proof consists in studying all the possible cases of the redundant relations produced by $A_2x_2 = b_2$ and showing for each one either W is nice or the considered case does not hold. As it is quite technical, this part of the proof is omitted, the complete proof can be found in [17]. □

By Proposition 2, it follows that every proper cut in $Ax = b$ is induced by a nice set. In consequence, we have that both W and $\overline{W}$ are nice sets. W.l.o.g., we may suppose that $x(e_1)+x(e_2) = 1$ is the unique redundant relation of $Ax = b$, and that the unique redundant relation of $\bar{A}x = \bar{b}$ is one of the relations $x(e_i) + x(e_j) = 1,\ i, j \in \{1, \dots, 4\}$. Suppose for instance that the redundant relation of $\bar{A}x = \bar{b}$ is $x(e_1) + x(e_3) = 1$. (The other cases can be treated in a similar way.) Consider the solution $y \in \mathbb{R}^4$ such that $y(e_1) = 1,\ y(e_2) = 0,\ y(e_3) = 1,\ y(e_4) = 0$. Clearly, y satisfies both redundant relations. By Lemma 2 *i)* there exists a solution $z_1 \in \mathbb{R}^{\Gamma(W)}$ of $Ax = b$ with $z_1(e_i) = y(e_i)$ for $i = 1, \dots, 4$. Similarly there is a solution $z_2 \in \mathbb{R}^{\Gamma(\overline{W})}$ of $\bar{A}x = \bar{b}$ such that $z_2(e_i) = y(e_i)$ for $i = 1, \dots, 4$. Let $z \in \mathbb{R}^{E_0}$ be given by

$$z(e) = \begin{cases} z_1(e) & \text{if } e \in \Gamma(W), \\ z_2(e) & \text{if } e \in E_0(\overline{W}). \end{cases}$$

In consequence, we have that z is a solution of system $(\mathcal{L})$. Since $z \neq \bar{x}_0$, this is a contradiction.

Consequently, the set of proper cuts Π is empty. By Proposition 1 $(G, \bar{x})$ is then a basic pair, and the proof of our theorem is complete.

5. Concluding remarks

In this paper we have introduced the concept of critical extreme points of the polytope $P(G)$, and we have given necessary conditions for a non-integer extreme point of $P(G)$ to be critical. As a consequence, we have obtained a characterisation of perfectly 2-edge connected graphs.

Theorem 2 has interesting polyhedral and algorithmic consequences. First note that operations $\theta_1', \theta_2', \theta_3'$ can be performed in polynomial time and in any order. Now consider a graph $G = (V, E)$ and a critical extreme point $\bar{x}$. By Theorem 2, there exists an odd cycle $\mathcal{C}$ of G such that $\bar{x}(e) = \frac{1}{2}$ and $\bar{x}(e) = 1$ for $e \in E \setminus \mathcal{C}$. Morover $E \setminus \mathcal{C}$ induces a forest whose pending nodes are precisely the nodes of $\mathcal{C}$. The following inequality, which is valid for the TECP, is then violated by $\bar{x}$.

$$\sum_{e \in \mathcal{C}} x(e) \geq \frac{|\mathcal{C}| + 1}{2}. \tag{5.1}$$

Thus we have the following.

Theorem 3. *Critical extreme points can be separated from the 2-edge connected subgraph polytope in polynomial time.*

Also, if G is for instance a wheel, that is the forest induced by $E \setminus \mathcal{C}$ is a star, constraint (5.1) defines a facet of the TECP [31]. Thus by Theorem 3 critical extreme point may be separated by facet defining inequalities of the TECP in polynomial time. So a natural question that arises here is to characterise the facets that can be produced by critical extreme points.

Moreover, given an extreme point x of $P(G)$ of rank k, one can obtain from x extreme points of rank $\leq k$. Thus a further question that may be asked here is to find out whether it is possible to lift facets induced by extreme points of rank k to facets induced by extreme points of rank $\leq k$. It may be however possible that when all the non-integer extreme points of $P(G)$ have rank 1, the problem of the descrption of the facets of $P(G)$ is tractable. So we shall give the following conjecture.

Conjecture 6.2. *For k fixed, if all the non-integer extreme points of $P(G)$ have rank $\leq k$, then TECSP can be solved in polynomial time.*

As a final remark, let us note that the rank concept introduced in Section 3 for the extreme points of $P(G)$ can be extended to the faces of a polytope.

Let $H \subset \mathcal{R}^n$ be the hypercube, *i.e.,*

$$H = \{x \in \mathcal{R}^n;\ 0 \leq x_i \leq 1,\ i = 1, \ldots, n\},$$

and $\mathcal{S}$ a subset of the extreme points of H. (Each element of $\mathcal{S}$ is the incidence vector of a subset of the set $\{1, \ldots, n\}$).

Let P be a polytope included in H and containing $\mathcal{S}$. A non-empty face F of P is said to be *fundamental* if there exists a face L of H such that $F = P \cap L$. The polytope P will be considered itself as a fundamental face. Let us denote by $\mathcal{F}(P)$ the set of fundamental faces of P, and for a point x of P by $F(x)$ the smallest fundamental face containing x.

Assume now that P satisfies the following property:

Any fundamental face of P contains at least one element of $\mathcal{S}$.

Note that if x is an integer extreme point of P, then the set $\{x\}$ is a face of P and therefore x is an element of $\mathcal{S}$. (Thus if P is not an integer polytope, P is a relaxed polytope of the convex hull of the elements of $\mathcal{S}$). We will now define the rank of a face (extreme point) F of P, which will be denoted by *rank* (F), in a recursive way.

1) The rank of an extreme point x of P is equal to $rank(F(x))$.
2) The rank of a fundamental face is 0 if and only if it is integral. Thus all the integer extreme points of P have rank 0.
3) If F is a fundamental face which is not integral, then

$$rank(F) = 1 + \max\{rank(F');\ F' \in \mathcal{F}(P),\ F' \subset F\}.$$

A fundamental face F (resp. an extreme point x) of P is called *critical* if $rank(F) = 1$ (resp. $rank(x) = 1$).

Note that if $\mathcal{S}$ is the set of the incidence vectors of the 2-edge connected subgraphs of a 2-connected graph G and if P is the polytope $P(G)$, then the definitions related to the rank of an extreme point and to the critical extreme points, given in Section 3, fit in this more general concept.

A further illustration of this new general concept comes from perfect graphs. Here $\mathcal{S}$ is the set of incidence vectors of the stable sets of a graph G and P is the polytope described by the non-negativity and the so-called clique constraints. We have that P is critical if and only if the graph G is critically imperfect.

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