# Polyhedral results for the bipartite induced subgraph problem 

Pierre Fouilhoux, A. Ridha Mahjoub<br>Laboratoire LIMOS, CNRS UMR 6158, Université Blaise Pascal Clermont II, Complexe Scientifique des Cézeaux, 63177 Aubière Cedex, France

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#### Abstract

Given a graph $G=(V, E)$ with node weights, the Bipartite Induced Subgraph Problem (BISP) is to find a maximum weight subset of nodes $V^{\prime}$ of $G$ such that the subgraph induced by $V^{\prime}$ is bipartite. In this paper we study the facial structure of the polytope associated with that problem. We describe two classes of valid inequalities for this polytope and give necessary and sufficient conditions for these inequalities to be facet defining. For one of these classes, induced by the so-called wheels of order $q$, we give a polynomial time separation algorithm. We also describe some lifting procedures and discuss separation heuristics. We finally describe a Branch-and-Cut algorithm based on these results and present some computational results.


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## 1. Introduction

Let $G=(V, E)$ be a graph. If $W \subset V$, then $E(W)$ denotes the set of all edges of $G$ with both endnodes in $W$. The graph $H=(W, E(W))$ is the subgraph of $G$ induced by $W$. A graph is called bipartite if its node set can be partitioned into two nonempty disjoint sets $V_{1}$ and $V_{2}$ such that no two nodes in $V_{1}$ and no two nodes in $V_{2}$ are linked by an edge. Given a weight function $c: V \rightarrow \mathbb{R}$ that associates with every node $v$ a weight $c(v)$, the bipartite induced subgraph problem (BISP for short) is to find a bipartite induced subgraph ( $W, E(W)$ ) of $G$ such that $c(W)=\sum_{v \in W} c(v)$ is as large as possible.

A stable set of a graph is a set of pairwise nonadjacent nodes. The stable set problem consists of finding a stable of maximum weight. The BISP is a generalization of the maximum stable set problem. In fact, if $H=(W, F)$ is a graph, then the maximum stable set problem in $H$ can be reduced to the BISP in the graph $G=(V, E)$ obtained from $H$ by adding for every edge $u v$ of $H$, a node $w$ with weight $M$, where $M$ is a big positive value, and the edges $w u$ and $w v$. This implies that the BISP is NP-hard. The BISP has been shown to be NP-hard even in graphs with maximum degree three and in planar graphs when the maximum degree is $\geqslant 4$ [8]. The BISP is solvable in polynomial time in series-parallel graphs [4] and in planar graphs when the maximum node degree is limited to three [8]. (A graph is called series-parallel if it can be obtained from a graph constituted of one edge by the operations of subdividing and doubling edges.)

The BISP has applications to the via minimization problem which arises in the design of integrated circuits and printed circuit boards [ $8,17,6,14,5,9]$. The design of printed circuit board and integrated circuits is usually broken into

[^0]three phases, placement of the components, determination of the physical routing and layer assignment. A net is a collection of wires (straight lines) that electrically connects a specified set of components. The latter phase consists in assigning the wire segments of each net to the layers of the circuit so that no two wire segments of different nets cross in the same layer. Physically a change of layers is achieved by placing a via, a contact cut or a hole to be drilled, to electrically connect wire segments. Since vias degrade the circuit performance and cause additional cost, it is desirable to minimize the number of vias without affecting routability. The via minimization problem is to find an assignment with a minimum number of vias. A junction is a point on which two or more wire segments of the same net meet. The number of wire segments which are incident to the same junction is called junction degree. The via minimization problem has been shown to be polynomial when the maximum junction degree is $\leqslant 3$ [17,6]. In [8] Choi et al. showed that the via minimization problem, where the maximum junction degree is limited to 4 , is NP-complete by showing that the BISP can be transformed to the via minimization problem in that case. In [9] it is shown that the via minimization problem with any maximum junction degree can be reduced to the BISP. So studying the BISP may permit to develop efficient approaches for the via minimization problem. This was our motivation for investigating the polyhedral structure of the BISP in this paper.

If $W \subset V$, let $x^{W} \in \mathbb{R}^{V}$ such that $x^{W}(u)=1$ if $u \in W$ and $x^{W}(u)=0$ if not, $x^{W}$ is called the incidence vector of $W$. The convex hull $P(G)$ of the incidence vectors of the node sets of all the induced bipartite subgraphs of $G$, i.e.

$$
P(G)=\operatorname{conv}\left\{x^{W} \in \mathbb{R}^{V} \mid W \subset V,(W, E(W)) \text { is bipartite }\right\},
$$

is called the bipartite induced subgraph polytope (BIS polytope) of $G$.

$$
\beta(G)=\{W \subset V,(W, E(W)) \text { is bipartite }\}
$$

denotes the family of node sets of bipartite induced subgraph.
In [3] Barahona and Mahjoub studied the polytope $P(G)$. They exhibited some basic classes of facet defining inequalities and described several lifting methods. In [4] they studied a composition technique for the polytope $P(G)$ in graphs which are decomposable by two-node cutsets. If $G$ decomposes into $G_{1}$ and $G_{2}$, they showed that $P(G)$ can be obtained from two linear systems related to $G_{1}$ and $G_{2}$. Using this, they gave a polynomial time algorithm to solve the BISP in series-parallel graphs.
The closely related bipartite subgraph problem has been extensively investigated. In [2] Grötschel et al. describe several classes of facet defining inequalities of the bipartite subgraph polytope. A graph is said to be weakly bipartite if its bipartite subgraph polytope coincides with the polyhedron given by the trivial inequalities and the so-called odd cycle inequalities. Grötschel and Pulleyblank [11] showed that the bipartite subgraph problem can be solved in polynomial time in that class of graphs. Barahona [1] showed that planar graphs belong to that class of graphs. In [13] Mahjoub extended this result by showing that the graphs noncontractible to $K_{5}$ are weakly bipartite. Recently Guenin [12] gave a characterization for this class of graphs.

The paper is organized as follows. In the following section we describe two classes of valid inequality for $P(G)$ and give necessary and sufficient conditions for these inequalities to be facet defining. We also give a polynomial time separation algorithm for the class produced by the so-called wheels of order $q$. In Section 3 we describe some lifting procedures of facets. In Section 4 we present our computational study. We discuss separation heuristics, give our Branch-and-Cut algorithm and present some computational results. In the rest of this section, we give more definitions and notations. The reader is supposed to be familiar with polyhedral theory, for more details see [16].
The graphs we consider are finite, undirected and without loops and multiple edges. We denote a graph by $G=(V, E)$ where $V$ is the node set and $E$ the edge set of $G$. If $e \in E$ is an edge with endnodes $u$ and $v$, we also write $u v$ to denote the edge. If $G=(V, E)$ is a graph and $F \subset E$, then $V(F)$ denotes the set of nodes of $V$ which occur at least once as an endnode of an edge in $F$. If $W \subset V$, then $\delta(W)$ is the set of edges with one endnode in $W$ and the other in $V \backslash W$. The set $\delta(W)$ is called cut. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$ and call $\delta(v)$ the star of $v$. For $v \in V$, we denote by $N(v)$ the set of nodes adjacent to $v$. If $W \subset V$, we let $N(W)=\left(\bigcup_{v \in W} N(v)\right) \backslash W$, and we shall call $N(W)$ the neighbour set of $W$.

A walk $P$ in $G=(V, E)$ is a sequence of edges $e_{1}, e_{2}, \ldots, e_{k}$ such that $e_{1}=v_{0} v_{1}, e_{2}=v_{1} v_{2}, \ldots, e_{k}=v_{k-1} v_{k}$. A path is a walk that does not go through the same node more than once. The nodes $v_{0}$ and $v_{k}$ are the endnodes of $P$ and we say that $P$ links $v_{0}$ and $v_{k}$ or goes from $v_{0}$ to $v_{k}$. The number $k$ of edges of $P$ is called the length of $P$. If $P=e_{1}, e_{2}, \ldots, e_{k}$ is a path linking $v_{0}$ and $v_{k}$ and $e_{k+1}=v_{0} v_{k} \in E$, then the sequence $e_{1}, e_{2}, \ldots, e_{k}, e_{k+1}$ is
called cycle of length $k+1$. A cycle (walk) is called odd if its length is odd, otherwise it is called even. If $P$ is a cycle or a path and $u v$ an edge of $E \backslash P$ with $u, v \in V(P)$, then $u v$ is called a chord of $P$. A hole of $G$ is a cycle without chords.

## 2. Facets of $P(G)$

In this section we introduce two new classes of facets of $P(G)$. For one of the classes we present a polynomial time separation algorithm.

Let $G=(V, E)$ be a graph. If $W \subset V$ is a node subset and $(W, E(W))$ is a bipartite subgraph of $G$, then the incidence vector of $W, x^{W}$ satisfies the inequalities

$$
\begin{align*}
& 0 \leqslant x(v) \leqslant 1  \tag{1}\\
& x(W) \leqslant|W|-1 \quad \text { for all } W \subset V \text { such that }(W, E(W)) \text { is an odd cycle. } \tag{2}
\end{align*}
$$

Moreover, any $0-1$ solution of inequalities (1), (2) is the incidence vector of a bipartite induced subgraph of $G$. The constraints of type (1) are called trivial inequalities and the constraints of type (2) are called odd cycle inequalities. In, [3], it is shown that inequalities (1) define facets for $P(G)$. Moreover, necessary and sufficient conditions for inequality (2) to define facets are provided. In particular an inequality (2) is facet defining for $P(G)$ only if ( $W, E(W)$ ) is an odd hole.
A node subset $W \subset V$ is called a clique if each pair of nodes in $W$ is joined by an edge. A clique is called maximal if it is not strictly contained in a clique. If $W \subset V$ is a clique then any bipartite induced subgraph of $G$ cannot contain more than two nodes. This implies that the inequality

$$
\begin{equation*}
x(W) \leqslant 2 \tag{3}
\end{equation*}
$$

is valid for $P(G)$. In [3], it is shown that an inequality of type (3) defines a facet of $P(G)$ if and only if $W$ is a maximal clique with $|W| \geqslant 3$.

Our first class of facets is defined by the wheels of order $q$ of $G$.
A graph $H=(W, F)$ is called a wheel of order $q$ where $q$ is a positive integer, if $W=W_{\mathrm{h}} \cup W_{\mathrm{c}}$ where $W_{\mathrm{h}}$ induces a hole and $W_{\mathrm{c}}$ is a clique of $q$ universal nodes, which are adjacent to every node in the hole (see Fig. 1 for a wheel of order 3). We will also write $H=\left(W_{\mathrm{h}} \cup W_{\mathrm{c}}, F\right)$.

Theorem 2.1. Let $G=(V, E)$ be a graph and $\left(W_{\mathrm{h}} \cup W_{\mathrm{c}}, F\right)$ an induced wheel of order $q$ of $G$. Let $p=\left|W_{\mathrm{h}}\right|$. Then the inequality

$$
\begin{equation*}
x\left(W_{\mathrm{h}}\right)+\left\lfloor\frac{p}{2}\right\rfloor x\left(W_{\mathrm{c}}\right) \leqslant 2\left\lfloor\frac{p}{2}\right\rfloor \tag{4}
\end{equation*}
$$

is valid for $P(G)$. Moreover, (4) is facet defining for $P(G)$ if and only if
(i) $p$ is odd,


Fig. 1. A wheel of order 3.
(ii) for every node $u$ of $V \backslash\left(W_{\mathrm{h}} \cup W_{\mathrm{c}}\right)$ where $\{u\} \cup W_{\mathrm{c}}$ is a clique, at least one of the following statements holds:
(a) There is a set $S \subset W_{\mathrm{h}}$ of $\lfloor p / 2\rfloor$ nodes such that $S \cup\{u\}$ is a stable set in $G$.
(b) There is a node $v$ of $W_{\mathrm{h}}$ such that the graph induced by $\left(W_{\mathrm{h}} \backslash\{v\}\right) \cup\{u\}$ is bipartite.

Proof. First we show that inequality (4) is valid for $P(G)$. Indeed, let $B \in \beta(G)$. As $W_{\mathrm{c}}$ is a clique, $B$ contains at most two nodes of $W_{\mathrm{c}}$.

- If $\left|B \cap W_{\mathrm{c}}\right|=2$, then $B \cap W_{\mathrm{h}}=\emptyset$ and thus $x^{B}$ satisfies (4).
- If $\left|B \cap W_{\mathrm{c}}\right|=1$, then $B \cap W_{\mathrm{h}}$ is a stable set of $W_{\mathrm{h}}$, and then $\left|B \cap W_{\mathrm{h}}\right| \leqslant\lfloor p / 2\rfloor$. Therefore $x^{B}$ satisfies (4).
- If $B \cap W_{\mathrm{c}}=\emptyset$, then $B \cap W \subset W_{\mathrm{h}}$. If $p$ is odd then by inequality (2), we have $\left|B \cap W_{\mathrm{h}}\right| \leqslant p-1$. And if $p$ is even, then trivially $\left|B \cap W_{\mathrm{h}}\right| \leqslant p$. In both cases we have that $\left|B \cap W_{\mathrm{h}}\right| \leqslant 2\lfloor p / 2\rfloor$.

Consequently, (4) is valid for $P(G)$.
Suppose that $p=2 k$ for some $k \geqslant 2$. Let $W_{\mathrm{h}}=\left\{u_{1}, \ldots, u_{2 k}\right\}$ where $u_{i} u_{i+1} \in E$ for $i=1, \ldots, p$ (the indices are taken modulo $p$ ). Consider the inequalities

$$
x\left(W_{\mathrm{c}}\right)+x_{u_{i}}+x_{u_{i+1}} \leqslant 2 \text { for } i=1,3, \ldots, 2 k-1
$$

which are of type (3), and hence valid for $P(G)$. By summing these inequalities, we obtain constraint (4). This implies that (4) cannot define a facet.

From now on, we consider that $p$ is odd. Note that in this case inequality (4) can be written as

$$
x\left(W_{\mathrm{h}}\right)+\frac{p-1}{2} x\left(W_{\mathrm{c}}\right) \leqslant p-1 .
$$

Let $u \in V \backslash\left(W_{\mathrm{h}} \cup W_{\mathrm{c}}\right)$ such that $W_{\mathrm{c}} \cup\{u\}$ is a clique. Suppose that neither (a) nor (b) of (ii) hold. Let $B \in \beta(G)$ such that $x^{B}$ satisfies (4) with equality. We claim that $u \notin B$. In fact, if this is not the case, then $\left|B \cap W_{\mathrm{c}}\right| \leqslant 1$, for otherwise the graph induced by $B$ would contain an odd cycle.

- If $B \cap W_{\mathrm{c}}=\emptyset$, then $B$ would contain $p-1$ nodes from $W_{\mathrm{h}}$. However, by our hypothesis, this implies that the graph induced by $B$ is not bipartite, a contradiction.
- If $\left|B \cap W_{\mathrm{c}}\right|=1$, then $B \cap W_{\mathrm{h}}$ is a stable set of $(p-1) / 2$ nodes. As (ii) (a) does not hold, there must exist a node $v \in B \cap W_{\mathrm{h}}$, which is adjacent to $u$. Then $\{u, v, w\}$ induces an odd cycle, where $w$ is the node of $B \cap W_{\mathrm{c}}$. As $u, v, w \in B$, this is again a contradiction.

Thus $u$ does not belong to any set $B \in \beta(G)$, whose incidence vector satisfies (4) with equality. But this implies that (4) is equivalent to the constraint $x(u) \geqslant 0$, a contradiction.

Now, suppose that (i) and (ii) hold and let $p=2 k+1, k \geqslant 1$. Let $S \subset W_{\mathrm{h}}$ be a stable set of $k$ nodes and let $v_{0} \in W_{\mathrm{c}}$. Let $V_{1} \subset V \backslash\left(W_{\mathrm{h}} \cup W_{\mathrm{c}}\right)$ be the set of nodes $u$ such that $\{u\} \cup W_{\mathrm{c}}$ is a clique. Let $V_{2}=V \backslash\left(W_{\mathrm{h}} \cup W_{\mathrm{c}} \cup V_{1}\right)$. Let $V_{1}^{\prime} \subset V_{1}$ be a node subset of $V_{1}^{\prime}$ such that for every $u \in V_{1}^{\prime}$ there is a node subset $S_{u}$ of $W_{\mathrm{h}}$ of size $k$ where $S_{u} \cup\{u\}$ is a stable set of $G$. By (ii) (b), for every $u \in V_{1} \backslash V_{1}^{\prime}$, there is a node, say $w_{u}$, of $W_{\mathrm{h}}$ such that the graph induced by $\left(W_{\mathrm{h}} \backslash\left\{w_{u}\right\}\right) \cup\{u\}$ is bipartite. Also if $u \in V_{2}$ then there exists a node, say $v_{u}$, of $W_{\mathrm{c}}$ that is not adjacent to $u$. Set

$$
B_{u}= \begin{cases}W_{\mathrm{h}} \backslash\{u\} & \text { if } u \in W_{\mathrm{h}}, \\ S \cup\{u\} & \text { if } u \in W_{\mathrm{c}}, \\ S_{u} \cup\left\{u, v_{0}\right\} & \text { if } u \in V_{1}^{\prime}, \\ \left(W_{\mathrm{h}} \backslash\left\{w_{u}\right\}\right) \cup\{u\} & \text { if } u \in V_{1} \backslash V_{1}^{\prime}, \\ S \cup\left\{u, v_{u}\right\} & \text { if } u \in V_{2} .\end{cases}
$$

For all $u \in V$, clearly $B_{u} \in \beta(G)$ and $x^{B_{u}}$ satisfies (4) with equality, for all $u \in V$. Moreover it is not hard to see that the vectors $x^{B_{u}}, u \in V$, are linearly independent.

Inequalities (4) will be called wheel inequalities.
The separation problem for a class of inequalities consists to decide whether a given vector $\bar{x} \in \mathbb{R}^{V}$ satisfies the inequalities, and, if not, to find an inequality that is violated by $\bar{x}$. An algorithm which solves this problem is called


Fig. 2. A path-cycle of order 4.
a separation algorithm. A fundamental result in combinatorial optimization is the well known equivalence between optimization and separation. That is there exists a polynomial time algorithm for optimizing over a class of inequalities if and only if the separation problem for this class can then be solved in polynomial time. Thus if for a class of inequalities there exists a polynomial time separation algorithm, it can then be used efficiently in the framework of a cutting plane algorithm for solving the corresponding optimization problem. The following theorem shows that the wheel inequalities associated with wheels of order $q$ can be separated in polynomial time, when $q$ is fixed.

Theorem 2.2. For $q$ fixed, the wheel inequalities associated with wheels of order $q$, with $\left|W_{\mathrm{h}}\right|$ odd, can be separated in polynomial time.

Proof. Let $\bar{x}$ be a given solution of $\mathbb{R}^{V}$. If $q$ is fixed, then we can test in polynomial time the inequalities of the form

$$
\begin{equation*}
x(W)+x_{u}+x_{v} \leqslant 2, \tag{5}
\end{equation*}
$$

where $W$ is a clique of $q$ nodes, and $u, v \in V$ are adjacent to each other and universal to $W$. So one may assume that inequalities (5) are satisfied by $\bar{x}$. Now, let $\bar{G}=(\bar{V}, \bar{E})$ be the graph where $\bar{V}=\{u \in V \mid u$ is universal to $W\}$ and $\bar{E}=\{u v \in E \mid u, v \in \bar{V}\}$. Let $\lambda=\bar{x}(W)$. With every edge $u v \in \bar{E}$, let us associate the weight $x_{u v}^{*}=(2-\lambda)-\left(\bar{x}_{u}+\bar{x}_{v}\right)$. Note that the weights in $\bar{G}$ are all nonnegative. Thus, there is an inequality of type (4), corresponding to $W$, which is violated by $\bar{x}$ if and only if the minimum weight of an odd cycle in $\bar{G}$, with respect to the weight vector $x^{*}$, is less than (2- $\lambda$ ). Indeed, an odd cycle $C$ in $\bar{G}$, where $|C|=2 k+1$, has a weight

$$
\begin{aligned}
x^{*}(C) & =(2 k+1)(2-\lambda)-2 \bar{x}(V(C)) \\
& =2(2 k-k \lambda-\bar{x}(V(C)))+(2-\lambda) .
\end{aligned}
$$

Thus, $x^{*}(C)<2-\lambda$ if and only if $\bar{x}(V(C))+k \lambda>2 k$, that is (4) is violated. Since finding a minimum weight odd cycle in a graph with nonnegative weights can be done in polynomial time [10], the statement follows.

The second class of facets is induced by the so-called path-cycles of $G$.
A graph $H=(W, F)$ is called a path-cycle of order $p$ where $p$ is a positive integer, if $H$ consists of $p$ odd holes $C_{1}, \ldots, C_{p}$ and a chordless path $P^{*}$ such that
(i) $C_{i} \cap C_{i+1}=P_{i}$ where $P_{i}$ is a chordless path consisting of at least one edge, for $i=1, \ldots, p-1$,
(ii) the paths $P^{*}, P_{1}, \ldots, P_{p-1}$ are pairwise node disjoint,
(iii) for all $i, j \in\{1, \ldots, p\}$ such that $i+1<j, C_{i}$ and $C_{j}$ are node disjoint,
(iv) the path $P^{*}$ joins a node of $C_{1}$, say $w_{0}$, to a node of $C_{p}$, say $w_{p}$, such that $w_{0} \in V\left(C_{1}\right) \backslash V\left(P_{1}\right), w_{p} \in$ $V\left(C_{p}\right) \backslash V\left(P_{p-1}\right)$ and $V\left(P^{*}\right) \backslash\left\{w_{0}, w_{p}\right\} \cap\left(\bigcup_{i=1, \ldots, p} V\left(C_{i}\right)\right)=\emptyset$.

Note that a path-cycle of order $p$ is a planar graph with $p+2$ faces (including the exterior face) and a maximum degree equal to three. A path-cycle of order 4 is shown in Fig. 2.

Given a path-cycle $H=(W, F)$ of order $p$, let $H^{\prime}=\left(W^{\prime}, F^{\prime}\right)$ be the subgraph of $H$ obtained by deleting all the internal nodes of the paths $P_{1}, \ldots, P_{p-1}$. Note that the graph $H^{\prime}$ consists of three edge-disjoint paths $P^{*}, P^{\prime}, P^{\prime \prime}$ between $w_{0}$ and $w_{p}$. We shall denote by $C^{\prime}$ and $C^{\prime \prime}$ the holes of $H^{\prime}$ formed by the paths $P^{\prime}$ and $P^{*}$, and $P^{\prime \prime}$ and $P^{*}$, respectively.


Fig. 3.

Let $w_{1}, \ldots, w_{p-1}$ and $z_{1}, \ldots, z_{p-1}$ be the nodes of $P^{\prime}$ and $P^{\prime \prime}$ which belong to the paths $P_{1}, \ldots, P_{p-1}$, respectively, so that each path $P_{i}$ joins the node $w_{i}$ to the node $z_{i}$. For convenience of notations, we let $z_{0}=w_{0}$ and $z_{p}=w_{p}$ and we denote by $P_{0}$ and $P_{p}$, respectively, the paths from $w_{0}$ to $z_{0}$ and from $w_{p}$ to $z_{p}$, respectively. (Note that $P_{0}$ and $P_{p}$ are empty.)

If $u$ and $v$ are two nodes of $P^{\prime}\left(P^{\prime \prime}\right)$ then $u \sim v$ denotes the subpath of $P^{\prime}\left(P^{\prime \prime}\right)$ between $u$ and $v$. Given two integers $j, k, 1 \leqslant j \leqslant k \leqslant p$, we let $H_{j, k}$ denote the subgraph of $H$ induced by $\bigcup_{i=j, \ldots, k} V\left(C_{i}\right)$.

Before showing that the path-cycles are facet inducing, we give a technical lemma which will be frequently used in the sequel.

Lemma 2.3. Let $H=(W, T)$ be a path-cycle of order $p$. Let $j, k, 1 \leqslant j<k \leqslant p$, be two integers such that $k-j+1$ is even. Let $u_{i} \in\left\{w_{i}, z_{i}\right\}$ be a fixed node of $P_{i}$ for $i=j, j+2, \ldots, k-1$. Let $v_{1}$ and $v_{2}$ be respectively two nodes of $V\left(C_{j}\right) \backslash\left\{u_{j}\right\}$ and $V\left(C_{k}\right) \backslash\left\{u_{k-1}\right\}$, which both belong either to $V\left(P^{\prime}\right)$ or to $V\left(P^{\prime \prime}\right)$. Then the subgraph $\bar{H}_{j, k}$ of $H_{j, k}$ induced by $\bigcup_{i=j, \ldots, k} V\left(C_{i}\right) \backslash\left\{u_{j}, u_{j+2}, \ldots, u_{k-1}\right\}$ is a tree. Moreover, the path of $\bar{H}_{j, k}$ between $v_{1}$ and $v_{2}$ has the same parity as $v_{1} \sim v_{2}$.

Proof. Let $U=\left\{u_{j}, u_{j+2}, \ldots, u_{k-1}\right\}$. It is easily seen that $\bar{H}_{j, k}$ is a tree and hence contains exactly one path joining $v_{1}$ and $v_{2}$. Let us denote by $R$ this path (see Fig. 3).

Note that $\left\{w_{j-1}, w_{j+1}, w_{j+3}, \ldots, w_{k}\right\} \in W \backslash U$. Now, let $R_{i}$ be the unique path of $\bar{H}_{j, k}$ between $w_{i}$ and $w_{i+2}$ for $i=j-1, \ldots, k-2$. Pick an arbitrary node $w_{i}$ in $\left\{w_{j-1}, w_{j+1}, w_{j+3}, \ldots, w_{k-2}\right\}$. If $w_{i+1} \in V(R)$, then $R_{i}$ is precisely the subpath $w_{i} \sim w_{i+2}$. If this is not the case, then $R_{i}$ is the path $P_{i} \cup\left(z_{i} \sim z_{i+2}\right) \cup P_{i+2}$. And, in consequence, $R_{i} \cup\left(w_{i} \sim w_{i+2}\right)$ is the hole $H_{i, i+2} \backslash P_{i+1}$. Since $H_{i, i+2}$ consists of two odd cycles with a path in common, it follows that $H_{i, i+2} \backslash P_{i+1}$ is even and, therefore, $R_{i}$ has the same parity as ( $w_{i} \sim w_{i+2}$ ). This implies that the path, say $R^{\prime}$, in $\bar{H}_{j, k}$ between $w_{j-1}$ and $w_{k}$ has the same parity as $w_{j-1} \sim w_{k}$. Suppose that both $v_{1}$ and $v_{2}$ belong to $V\left(P^{\prime}\right)$. The case where $v_{1}, v_{2}$ belong to $V\left(P^{\prime \prime}\right)$ follows by symmetry. Note that $w_{j-1} \sim w_{k}=\left(v_{1} \sim\right.$ $\left.v_{2}\right) \cup\left(w_{j-1} \sim v_{1}\right) \cup\left(v_{2} \sim w_{k}\right)$. Hence, $R^{\prime}$ has the same parity as the set $\left(v_{1} \sim v_{2}\right) \cup\left(w_{j-1} \sim v_{1}\right) \cup\left(v_{2} \sim w_{k}\right)$. Suppose that $w_{j} \in U$ and $w_{k-1} \notin U$. (The other cases: $w_{j}, w_{k-1} \in U, w_{j}, w_{k-1} \notin U, w_{j} \notin U$ and $w_{k-1} \in U$, can be treated in a similar way.) Hence $R^{\prime}=\left(R \backslash\left(w_{j-1} \sim v_{1}\right)\right) \cup\left(v_{2} \sim w_{k}\right)$. And therefore the sets $\left(v_{1} \sim v_{2}\right) \cup\left(w_{j-1} \sim\right.$ $\left.v_{1}\right) \cup\left(v_{2} \sim w_{k}\right)$ and $\left(R \backslash\left(w_{j-1} \sim v_{1}\right)\right) \cup\left(v_{2} \sim w_{k}\right)$ have the same parity. It then follows that $R$ has the same parity as $v_{1} \sim v_{2}$.

Remark 2.4. $p$ is even if and only if $P^{\prime}$ and $P^{\prime \prime}$ have the same parity.
Proof. Let $q=\sum_{i=1}^{p}\left|E\left(C_{i}\right)\right|$. Clearly $p$ is even if and only if $q$ is even. Since each edge of $P_{i}, i=1, \ldots, p-1$, is counted twice in the sum defining $q$, we have that $q$ is even if and only if $P^{\prime}$ and $P^{\prime \prime}$ have the same parity.

Now let $G=(V, E)$ be a graph and let $H=(W, F)$ be an induced subgraph of $G$ which is a path-cycle of order $p$ with $p \geqslant 1$ such that $C^{\prime}$ and $C^{\prime \prime}$ are both odd. By the remark above, it follows that $p$ is even. Let us associate with $H=(W, F)$ the inequality

$$
\begin{equation*}
x(W) \leqslant|W|-\left(\frac{p}{2}+1\right) . \tag{6}
\end{equation*}
$$

Inequalities of type (6) will be called path-cycle inequalities. We have the following.
Theorem 2.5. Inequality (6) is valid for $P(G)$.

Proof. Let $B \in \beta(G)$, and suppose that $|B \cap W|$ is maximum. (Obviously, if inequality (6) is satisfied by $x^{B}$ then it is satisfied by the incidence vector of any node set inducing a bipartite subgraph of $G$.) Let $\bar{B}=W \backslash B$ and $\bar{W}=$ $W \backslash \bigcup_{i=1, \ldots, p-1} V\left(P_{i}\right)$. We distinguish two cases.

Case $1: \bar{W} \cap \bar{B} \neq \emptyset$. Then let $u \in \bar{W} \cap \bar{B}$. If $u \in V\left(P^{*}\right) \backslash\left\{w_{0}, w_{p}\right\}$, then since $V\left(C_{i}\right) \cap \bar{B} \neq \emptyset$ for $i=1,3, \ldots, p-1$, we have that $|\bar{B}| \geqslant(p / 2)+1$, and thus (6) is satisfied by $x^{B}$. If not, then $u \in V\left(C_{j}\right)$ for some $j \in\{1,2, \ldots, p\}$. We may assume that $j$ is even. (The case where $j$ is odd follows by symmetry.) Hence, since $V\left(C_{i}\right) \cap \bar{B} \neq \emptyset$ for $i=1,3, \ldots, j-1, j+1, j+3, \ldots, p-1$, it follows that $|\bar{B}| \geqslant p / 2+1$, implying that (6) is satisfied by $x^{B}$.

Case 2: $\bar{W} \cap \bar{B}=\emptyset$. Since $V\left(C_{i}\right) \cap \bar{B} \neq \emptyset$, for $i=1,3, \ldots, p-1$, we have that $|\bar{B}| \geqslant p / 2$. Now, let us suppose that $|\bar{B}|=p / 2$. It then follows that $\left|\bar{B} \cap V\left(P_{i}\right)\right|=1$ for $i=1,3, \ldots, p-1$. Let $u_{i}$ be the node of $\bar{B} \cap V\left(P_{i}\right)$ for $i=1,3, \ldots, p-1$. Since $|B \cap W|$ is maximum, we may assume that $u_{1}, u_{3}, \ldots, u_{p-1}$ all are in $\left\{w_{1}, \ldots, w_{p-1}, z_{1}, \ldots, z_{p-1}\right\}$. From Lemma 2.3 it follows that $\bar{H}=H \backslash\left\{u_{1}, u_{3}, \ldots, u_{p-1}\right\}$ has a path, say $R$, between $w_{0}$ and $w_{p}$ which is different from $P^{*}$ and has the same parity as $P_{p}=w_{0} \sim w_{p}$. Since $C^{\prime}$ is odd, we have that the hole which consists of the paths $R$ and $P^{*}$ is also odd. Therefore, $\bar{H}$ is not bipartite, a contradiction. Consequently, $|\bar{B}| \geqslant p / 2+1$ and thus $x^{B}$ satisfies (6).

Theorem 2.6. Inequality (6) defines a facet of $P(H)$.
Proof. Denote the inequality (6) by $a x \leqslant \alpha$ and suppose that there exists an inequality $b x \leqslant \beta$ that defines a facet of $P(H)$ such that $\{x \in P(H) \mid a x=\alpha\} \subset\{x \in P(H) \mid b x=\beta\}$. Since, by Theorem 2.5, inequality (6) is valid for $P(G)$, to prove that it is also facet defining it suffices to show that there is $\rho>0$ such that $b=\rho a$.

For this, we first show that there is $\rho \in \mathbb{R}$ such that $b(u)=\rho$ for every node $u \in V\left(C^{\prime}\right) \cup V\left(C^{\prime \prime}\right)$. To this end, let $j$ be an odd integer in $\{1,3, \ldots, p-1\}$. Pick an arbitrary node $u$ of $V\left(C^{\prime}\right)$. Consider the node sets

$$
\begin{aligned}
& B_{1}=W \backslash\left\{z_{1}, z_{3}, \ldots, z_{j}, w_{j}, z_{j+2}, z_{j+4}, \ldots, z_{p-1}\right\}, \\
& B_{2}=\left(B_{1} \backslash\{u\}\right) \cup\left\{w_{j}\right\} .
\end{aligned}
$$

From Lemma 2.3 w.r.t. $v_{1}=w_{0}$ and $v_{2}=w_{p}$, it follows that the graph obtained from $H_{0, p}$ by removing the nodes $z_{1}, z_{3}, \ldots, z_{p-1}$ is a tree. Hence the graph induced by $W \backslash\left\{z_{1}, z_{3}, \ldots, z_{p-1}\right\}$ contains a unique (odd) cycle. Therefore the graphs induced by $B_{1}$ and $B_{2}$ are bipartite. Moreover, $a x^{B_{1}}=a x^{B_{2}}=|W|-((p / 2)+1)=\alpha$. Thus $0=b x^{B_{1}}-$ $b x^{B_{2}}=b(u)-b\left(w_{j}\right)$. By letting $\rho_{1}=b\left(w_{j}\right)$, we then get $b(u)=\rho_{1}$ for every node $u \in V\left(C^{\prime}\right)$. Similarly, we obtain that there exists $\rho_{2} \in \mathbb{R}$ such that $b(v)=\rho_{2}$ for every node $v \in V\left(C^{\prime \prime}\right)$. Since $w_{0} \in V\left(C^{\prime}\right) \cap V\left(C^{\prime \prime}\right)$, it follows that $\rho_{1}=\rho_{2}=\rho$ and thus we have proved that

$$
\begin{equation*}
b(u)=\rho \quad \text { for all } u \in V\left(C^{\prime}\right) \cup V\left(C^{\prime \prime}\right) \tag{7}
\end{equation*}
$$

Next we show that $b(w)=\rho$, for all $w \in V\left(P_{i}\right) \backslash\left\{w_{i}, z_{i}\right\}$, for $i=1,2, \ldots, p-1$. For this, consider a path $P_{i}$, for $i=1,2, \ldots, p-1$, whose set of internal nodes is nonempty and let $w \in V\left(P_{i}\right) \backslash\left\{w_{i}, z_{i}\right\}$. If $i$ is even (resp. odd), consider the following node sets

$$
\begin{aligned}
& B_{3}=W \backslash\left\{z_{1}, z_{3}, \ldots, z_{i-1}, z_{i+1}, w_{i+2}, z_{i+3}, z_{i+5}, \ldots, z_{p-1}\right\}, \\
& B_{4}=\left(B_{3} \backslash\{w\}\right) \cup\left\{z_{i+1}\right\} \\
& \text { (resp. } B_{3}^{\prime}=W \backslash\left\{z_{1}, z_{3}, \ldots, z_{i}, w_{i+1}, z_{i+2}, z_{i+4}, \ldots, z_{p-1\}},\right. \\
& \left.B_{4}^{\prime}=\left(B_{3}^{\prime} \backslash\{w\}\right) \cup\left\{z_{i}\right\}\right) .
\end{aligned}
$$

As before, using Lemma 2.3 we can show that both $B_{3}$ and $B_{4}$ (resp. $B_{3}^{\prime}$ and $B_{4}^{\prime}$ ) induce bipartite subgraphs of $H$. Moreover we have that $a x^{B_{3}}=a x^{B_{4}}=\alpha$ (resp. $a x^{B_{3}^{\prime}}=a x^{B_{4}^{\prime}}=\alpha$ ). Thus, $0=b x^{B_{3}}-b x^{B_{4}}$ (resp. $0=b x^{B_{3}^{\prime}}-b x^{B_{4}^{\prime}}$ ). Which implies that $b(w)=b\left(z_{i+1}\right)\left(\right.$ resp. $\left.b(w)=b\left(z_{i}\right)\right)$. By (7), it follows that $b(w)=\rho$.

Thus, $b=\rho a$. Moreover, since $a x \leqslant \alpha$ is nontrivial, $b x \leqslant \gamma$ defines a nontrivial facet of $P(H)$, and thus $b \geqslant 0$. Consequently $\rho>0$, which finishes the proof of our theorem.

Theorem 2.7. Let $H^{*}$ be the subgraph of $G$ induced by $W \cup N(W)$. Suppose that $H^{*}$ is of maximum degree three and that the neighbours in $H^{*}$ of each node $u \in N(W)$ all belong to the same face of $H$. Let $H_{u}$, for $u \in N(W)$, be the subgraph induced by $W \cup\{u\}$. Then inequality (6) defines a facet of $P(G)$ if and only if for each node $u \in N(W)$ of degree three in $H^{*}$, at least one of the faces of $H_{u}$ is even.

Proof. First of all, note that, for $u \in N(W), H_{u}$ is planar and every node of $H_{u}$ is of degree $\leqslant 3$.
$(\Leftarrow)$ : Since inequality (6) defines a facet of $P(H)$, to show that it also defines a facet of $P(G)$, it suffices to show that for every node $u \in V \backslash W$, there is a node set $B$ which induces a bipartite subgraph of $G$, contains $u$ and whose incidence vector satisfies (6) with equality. For this, pick an arbitrary node $u$ of $V \backslash W$. Note that $u$ is contained in one of the faces of $H$. (This face may be the exterior face of $H$.) Suppose that $u$ is contained in $C^{\prime}$ and its neighbours in $H^{*}$ all belong to $V\left(P^{\prime}\right)$. (The other cases can be treated in a similar way). Since $C^{\prime}$ is odd, it follows that exactly one of the faces of $H_{u}$ containing $u$ is odd. Let $u_{1}$ be a neighbour of $u$ which belongs to this face such that $u_{1} \in V\left(C_{j}\right)$, where $j \in\{1, \ldots, p\}$. If $j$ is odd (resp. even), consider the node set

$$
\begin{aligned}
& B_{u}=\left(W \backslash\left\{z_{1}, z_{3}, \ldots, z_{j-2}, u_{1}, z_{j}, z_{j+2}, \ldots, z_{p-1}\right\}\right) \cup\{u\} \\
& \text { (resp. } B_{u}=\left(W \backslash\left\{z_{1}, z_{3}, \ldots, z_{j-1}, u_{1}, z_{j+1}, \ldots, z_{p-1}\right\} \cup\{u\}\right) .
\end{aligned}
$$

Clearly, $B_{u}$ induces a bipartite subgraph of $G$, and its incidence vector satisfies (6) with equality.
$(\Rightarrow)$ : Let us suppose that the claim does not hold and let $u \in V \backslash W$ be a node of $N(W)$ of degree three whose neighbours, say $u_{1}, u_{2}, u_{3}$, are in $W$ and such that the three faces of $H_{u}$, which contain $u$, are all odd. By hypothesis, we have that the nodes $u_{1}, u_{2}, u_{3}$ all belong to one of the faces $C^{\prime}, C^{\prime \prime}, C_{i}, i=1, \ldots, p$. Since $H^{*}$ is of maximum degree three, we have that $u_{1}, u_{2}, u_{3}$ are of degree two in $H$.

To show that inequality (6) does not define a facet in this case, we shall show that every node subset $B$ which induces a bipartite subgraph of $G$ and whose incidence vector $x^{B}$ satisfies (6) with equality, cannot contain $u$. In fact, let us assume to the contrary, that there is a node set $B$, containing $u$, which induces a bipartite subgraph of $G$ and whose incidence vector $x^{B}$ satisfies (6) with equality.

Let $\bar{B}=W \backslash B$. Given two integers $r, s, 1 \leqslant r \leqslant s \leqslant p$, we let $\bar{B}_{r, s}$ denote the set of nodes of $\bar{B}$ which are in $\bigcup_{i=r, r+1, \ldots, s} V\left(C_{i}\right)$ and $\bar{H}_{r, s}$ the graph obtained from $H_{r, s}$ by deleting the nodes of $\bar{B}$. Note that if $s-r+1$ is even (resp. odd), then $\left|\bar{B}_{r, s}\right| \geqslant(s-r+1) / 2$ (resp. $\left|\bar{B}_{r, s}\right| \geqslant(s-r+2) / 2$ ). Also note that in case $s-r+1$ is even, and $\left|\bar{B}_{r, s}\right|=(s-r+1) / 2$, the nodes of $\bar{B}_{r, s}$ all belong to the paths $P_{r}, P_{r+2}, \ldots, P_{s-1}$ and $\left|\bar{B}_{r, s} \cap V\left(P_{i}\right)\right|=1$ for $i=r, r+2, \ldots, s-1$.

Claim 1. Let $r, s \in\{1, \ldots, p\}, r \leqslant s$. Suppose that $s-r+1$ is even and $\left|\bar{B}_{r, s}\right|=(s-r+1) 2$. Let $v_{1}$ and $v_{2}$ be two nodes of $V\left(w_{r-1} \sim w_{r}\right) \backslash\left\{w_{r}\right\}$ and $V\left(w_{s-1} \sim w_{s}\right) \backslash\left\{w_{s-1}\right\}$, respectively. Then there exists a path $R$ in $\bar{H}_{r, s}$ between $v_{1}$ and $v_{2}$ with the same parity as $v_{1} \sim v_{2}$.

Proof. Since the cycles $C_{r}, \ldots, C_{s}$ are all odd and $\left|\bar{B}_{r, s}\right|=(s-r+1) / 2$, one should have $\left|V\left(P_{i}\right) \cap \bar{B}\right|=1$ for $i=r, r+2, \ldots, s-1$. W.l.o.g., we may assume that $V\left(P_{i}\right) \cap \bar{B} \subset\left\{z_{i}, w_{i}\right\}$ for $i=r, r+2, \ldots, s-1$. Now, from Lemma 2.3, it follows that there is a path $R^{\prime}$ in $\bar{H}_{r, s}$ between $w_{r-1}$ and $w_{s}$ which has the same parity as $w_{r-1} \sim w_{s}$.

If $w_{r} \in R^{\prime}$, then we let $R^{\prime \prime}$ be the path $R^{\prime} \backslash\left(w_{r-1} \sim v_{1}\right)$ and if not we let $R^{\prime \prime}=R^{\prime} \cup\left(w_{r-1} \sim v_{1}\right)$. Now, if $w_{s-1} \in R^{\prime \prime}$, then we let $R$ be the path $R^{\prime \prime} \backslash\left(v_{2} \sim w_{s}\right)$ and if not be the path $R^{\prime \prime} \cup\left(v_{2} \sim w_{s}\right)$. Clearly, $R$ joins $v_{1}$ to $v_{2}$, and has the same parity as $v_{1} \sim v_{2}$.

Claim 2. Let $r, s \in\{1, \ldots, p\}$ such that $r<s$ and $s-r+1$ is even. Let $v_{1}$ and $v_{2}$ be two nodes of $V\left(P^{\prime}\right) \cap V\left(C_{r^{\prime}}\right)$ and $V\left(P^{\prime}\right) \cap V\left(C_{s^{\prime}}\right)$, respectively, where $r^{\prime} \leqslant r+1$ and $s^{\prime} \geqslant s$. Suppose that there are two paths $R_{1}$ and $R_{2}$ of $\bar{H}_{r^{\prime}, s^{\prime}}$ from $v_{1}$ to $w_{r}$ and from $v_{2}$ to $w_{s}$, respectively. If $\left(V\left(P_{r}\right) \cup V\left(z_{r} \sim z_{s}\right) \cup V\left(P_{s}\right)\right) \subset B$, then there is a path $R$ of $\bar{H}_{r^{\prime}, s^{\prime}}$, between $v_{1}$ and $v_{2}$, having opposite parity to that of the path $\left(R_{1}, w_{r} \sim w_{s}, R_{2}\right)$.

Proof. We shall consider the case where $z_{r} \in V\left(R_{1}\right)$ and $z_{s} \notin V\left(R_{2}\right)$ (The other cases are similar). Let $R=\left(R_{1} \backslash P_{r}, z_{r} \sim\right.$ $\left.z_{s}, P_{s}, R_{2}\right)$. Since the cycles $C_{r}, \ldots, C_{s}$ are all odd and $s-r$ is odd, the paths $\left(P_{r}, z_{r} \sim z_{s}, P_{s}\right)$ and $w_{r} \sim w_{s}$ have opposite parities. Thus the claim follows.

Now we consider two cases.
Case 1: $u$ is contained in a face $C_{j}$ for some $j \in\{1, \ldots, p\}$. Then $u_{1}, u_{2}, u_{3} \in V\left(C_{j}\right)$. We may assume that $j$ is odd. The case where $j$ is even follows by symmetry (see Fig. 4).

We can easily show that for all $i \in\{1, \ldots, p\},\left|V\left(C_{i}\right) \cap \bar{B}\right| \leqslant 2$. Since the three faces of $H_{u}$ containing $u$ are all odd and, consequently, must be covered by at least two nodes of $\bar{B},\left|V\left(C_{j}\right) \cap \bar{B}\right|=2$ with $\left|\left\{u_{1}, u_{2}, u_{3}\right\} \cap \bar{B}\right| \geqslant 1$.


Fig. 4.


Fig. 5.

Let us suppose that $u_{1} \in V\left(z_{j-1} \sim z_{j}\right), u_{2} \in V\left(P_{j}\right)$ and $u_{3} \in V\left(w_{j-1} \sim w_{j}\right)$ with $u_{3} \in \bar{B}$ (the other cases can be treated similarly). Then the subpath of $C_{j}$ between $u_{1}$ and $u_{2}$ which does not go through $u_{3}$ contains exactly one node of $\bar{B}$.

- If $\left(V\left(u_{1} \sim z_{j}\right) \backslash\left\{z_{j}\right\}\right) \cap \bar{B} \neq \emptyset$, then $|\bar{B}| \geqslant\left|\bar{B}_{1, j-1}\right|+2+\left|\bar{B}_{j+1, p}\right| \geqslant p / 2+2$, a contradiction.
- If $V\left(u_{1} \sim z_{j}\right) \cap \bar{B}=\emptyset$, then $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2$ and $\left|\bar{B}_{j+2, p}\right|=(p-(j+2)+1) / 2$. Thus, from Claim 1, there are two paths $R_{1}$ and $R_{2}$ having same endnodes and parities as $w_{0} \sim w_{j-1}$ and $w_{j+1} \sim w_{p}$, respectively. Since $V\left(P_{j-1}\right) \cup V\left(z_{j-1} \sim z_{j+1}\right) \cup V\left(P_{j+1}\right) \subset B$, it is easy to prove that, from $R_{1}$ and $R_{2}$, one can construct a path $R$ between $w_{0}$ and $w_{p}$ with the same parity as $P^{\prime}$. Then $R^{\prime} \cup P^{*}$ is an odd cycle whose node set is in $B$, a contradiction.

Let us suppose that $z_{j} \in \bar{B}$. Using the paths $R_{1},\left(P_{j-1}, z_{j-1} \sim z_{j}, P_{j}\right)$ and ( $w_{j} \sim w_{j+1}, R_{2}$ ) together with arguments similar to those used in the proof of Claim 2, one can show that there is a path $R^{\prime}$ between $w_{0}$ and $w_{p}$ having opposite parity to that of $P^{\prime}$. Moreover $R^{\prime}$ goes through $u_{1}, z_{j}$ and $u_{2}$. Let $R$ be the path constructed from $R^{\prime}$ by deleting the subpath between $u_{1}$ and $u_{2}$ and adding the edges $u_{1} u, u u_{2}$. Since $V\left(w_{j} \sim w_{j+1}\right) \cap \bar{B}=\emptyset, R$ and $P^{*}$ form an odd cycle whose node set is in $B$, a contradiction.

Case 2: $u$ is contained in the face $C^{\prime}$. (Note that this case is the same as the one where $u$ is in the exterior face of $H$. We have just to move path $P^{*}$.)

Suppose that the nodes $u_{1}, u_{2}, u_{3}$, all belong to the path $P^{\prime}$. (The case where some of the nodes $u_{1}, u_{2}, u_{3}$ are in the path $P^{*}$ can be studied along the same way.) We may assume that $u_{1}, u_{2}, u_{3}$ do not belong to the same cycle $C_{i}$. If $u_{1}, u_{2}, u_{3}$ are in the same cycle, then case 1 applies. So let $j, k, l, 1 \leqslant j \leqslant k \leqslant l \leqslant p$ such that $u_{1}, u_{2}, u_{3}$ belong to $V\left(C_{j}\right), V\left(C_{k}\right), V\left(C_{l}\right)$, respectively (see Fig. 5).

Suppose first that $\left\{u_{1}, u_{2}, u_{3}\right\} \subset \bar{B}$. Since $\left|V\left(C_{i}\right) \cap \bar{B}\right| \geqslant 1$ for $i=1, \ldots, j-1, j+1, \ldots, k-1, k+2, \ldots, l-$ $1, l+1, \ldots, p,|\bar{B}| \geqslant(p / 2)+2$, a contradiction. Hence, $\left|\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}\right| \leqslant 2$.
We shall consider the case where $j$ is odd, $k$ is even and $l$ is odd (the other cases can be treated similarly). Moreover, w.l.o.g., we can assume that $j>1, j+1<k<l-1$ and $l<p-1$.

Case 2.1. $\left|\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}\right|=2$.
If $\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}=\left\{u_{1}, u_{3}\right\}$ or $\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}=\left\{u_{2}, u_{3}\right\}$, then $|\bar{B}| \geqslant p / 2+2$, a contradiction. Thus $\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}=$ $\left\{u_{1}, u_{2}\right\}$. In consequence, $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2,\left|\bar{B}_{j+1, k-1}\right|=((k-1)-(j+1)+1) / 2$ and $\left|\bar{B}_{k+1, p}\right|=(p-(k+1)+1) / 2$. From Claim 1, there are three paths $R_{1}, R_{2}$ and $R_{3}$ having the same endnodes and parities as $w_{0} \sim w_{j-1}, w_{j} \sim w_{k-1}$
and $w_{k} \sim w_{p}$, respectively. Thus, by Claim 2 w.r.t. $R_{1}$ and $R_{2}$, there is a path $R$ between $w_{0}$ and $w_{k-1}$ having opposite parity to that of $w_{0} \sim w_{k-1}$. Now, using again Claim 2 w.r.t. $R$ and $R_{3}$, one gets a path $R^{\prime}$ between $w_{0}$ and $w_{p}$ with the same parity as $P^{\prime}$. Therefore, $R^{\prime}$ together with $P^{*}$ form an odd cycle. Since the node set of this cycle is in $B$, we have a contradiction.

Case 2.2: $\left|\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}\right|=1$.
We may assume that $u_{1} \in \bar{B}$ (the cases where $u_{2} \in \bar{B}$ and $u_{3} \in \bar{B}$ are similar). Thus $\left|V\left(C_{k}\right) \cap \bar{B}\right|=1$. Let $V\left(C_{k}\right) \cap \bar{B}=\{z\}$.

If $z \notin V\left(\left(u_{2} \sim w_{k}\right) \cup P_{k}\right)$, then $\left|\bar{B}_{k+1, l-1}\right|=(l-k-1) / 2$. Therefore by Claim 1 , there is a path $R$ between $w_{k}$ and $w_{l-1}$ of the same parity as $w_{k} \sim w_{l-1}$. Thus, the cycle ( $\left.u u_{2}, u_{2} \sim w_{k}, R, w_{l-1} \sim u_{3}, u_{3} u\right)$ is odd and its node set is in $B$, a contradiction.

So suppose that $z \in V\left(\left(u_{2} \sim w_{k}\right) \cup P_{k}\right)$. Thus $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2$ and $\left|\bar{B}_{j+1, k-1}\right|=(k-j-1) / 2$. In consequence, by Claim 1, there are two paths $R_{1}$ and $R_{2}$ having the same endnodes and parities as $w_{0} \sim w_{j-1}$ and $w_{j} \sim w_{k-1}$, respectively. From Claim 2 w.r.t. $R_{1}$ and $R_{2}$, there is a path, say $R^{*}$, between $w_{0}$ and $w_{k-1}$ having opposite parity to that of $w_{0} \sim w_{k-1}$. Then, we have the following claims.

Claim 3. $\left|\bar{B}_{l, p}\right|=(p-l+1) / 2+1$.
Proof. Obviously, $\left|\bar{B}_{l, p}\right| \geqslant(p-l+1) / 2$. Suppose the statement does not hold. Then $\left|\bar{B}_{l, p}\right|=(p-l+1) / 2$, and by Claim 1 , there is a path $R$ having the same endnodes and parity as $u_{3} \sim w_{p}$. However, since the cycles ( $u u_{1}, u_{1} \sim u_{2}, u_{2} u$ ) and $\left(u u_{1}, u_{1} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u\right)$ are odd, it follows that the cycle $\left(u u_{2}, u_{2} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u\right)$ is even. Furthermore, since $R^{*}$ has opposite parity to that of $w_{0} \sim w_{k-1}$ and $R$ has the same parity as $u_{3} \sim w_{p}$, the cycle ( $u u_{2}, u_{2} \sim w_{k-1}, R^{*}, P^{*}, R, u_{3} u$ ) is odd, a contradiction.

Claim 4. $\left\{w_{l-1}, w_{l}\right\} \subset \bar{B}$.
Proof. If $w_{l-1} \notin \bar{B}$, then $V\left(u_{3} \sim w_{l-2}\right) \cap \bar{B}=\emptyset$. Moreover, since $\left|\bar{B}_{k, l-2}\right|=(l-k-1) / 2$, from Claim 1, there is a path $R$ between $u_{2}$ and $w_{l-2}$ with the same parity as $u_{2} \sim w_{l-2}$. Thus the cycle ( $u u_{2}, R, w_{l-2} \sim u_{3}, u_{3} u$ ) is odd, a contradiction. Hence $w_{l-1} \in \bar{B}$.

Now in order to show that $w_{l} \in \bar{B}$, we first show that $\left|V\left(P_{l}\right) \cap \bar{B}\right|=1$ and $\left|\bar{B}_{l+2, p}\right|=(p-l-1) / 2$. If $V\left(P_{l}\right) \cap \bar{B}=\emptyset$, then $\left|\bar{B}_{k, l}\right|=(l-k+1) / 2$, and hence from Claim 1 there exists a path $R$ between $u_{2}$ and $u_{3}$ with the same parity as $u_{2} \sim u_{3}$. Thus the cycle $\left(u u_{2}, R, u_{3} u\right)$ is odd, a contradiction.

Now suppose $w_{l} \notin \bar{B}$. Then $V\left(u_{3} \sim w_{l+1}\right) \cap \bar{B}=\emptyset$. By Claim 1, there is a path $R$ between $w_{l+1}$ and $w_{p}$ with the same parity as $w_{l+1} \sim w_{p}$. However, since the cycles ( $u u_{1}, u_{1} \sim u_{2}, u_{2} u$ ) and ( $u u_{1}, u_{1} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u$ ) are odd, it follows that the cycle $\left(u u_{2}, u_{2} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u\right)$ is even. Since $R^{*}$ has opposite parity to that of $w_{0} \sim w_{k-1}$, the cycle $\left(u u_{2}, u_{2} \sim w_{k-1}, R^{*}, P^{*}, R, w_{l+1} \sim u_{3}, u_{3} u\right)$ is odd, a contradiction.

By Claims 3 and $4,\left|\bar{B}_{k, l-2}\right|=(l-k-1) / 2$ and $\left|\bar{B}_{l+2, p}\right|=(p-l-1) / 2$. Then from Claim 1, there are two paths $R_{1}$ and $R_{2}$ having the same endnodes and parities as $w_{k-1} \sim w_{l-2}$ and $w_{l+1} \sim w_{p}$, respectively. Consequently, by Claim 2 w.r.t. $R_{1}$ and $R_{2}$, there is a path $R$ between $w_{k-1}$ and $w_{p}$ having opposite parity to that of $w_{k-1} \sim w_{p}$. Note that if we replace in $C^{\prime}$ the path $w_{0} \sim w_{k-1}$ by $R^{*}$, as $R^{*}$ has opposite parity to $w_{0} \sim w_{k-1}$, the resulting cycle, say $\widetilde{C}^{\prime}$, is even. If we also replace in $\widetilde{C}^{\prime}, w_{k-1} \sim w_{p}$ by $R$, as the two paths have opposite parities, one gets an odd cycle. However we have that all the nodes of this latter cycle are in $B$, which is impossible.

Case 2.3: $\bar{B} \cap\left\{u_{1}, u_{2}, u_{3}\right\}=\emptyset$.
Let

$$
\begin{aligned}
& U_{1}=V\left(\left(u_{1} \sim w_{j}\right) \cup P_{j}\right) \cup V\left(P_{k-1} \cup\left(w_{k-1} \sim u_{2}\right)\right), \\
& U_{2}=V\left(\left(u_{2} \sim w_{k}\right) \cup P_{k}\right) \cup V\left(P_{l-1} \cup\left(w_{l-1} \sim u_{3}\right)\right) .
\end{aligned}
$$

If $U_{1} \cap \bar{B}=\emptyset$ (the case where $U_{2} \cap \bar{B}=\emptyset$ follows by symmetry), then $\left|\bar{B}_{j+1, k-1}\right|=(k-j-1) / 2$. Thus from Claim 1 , it follows that there is a path $R$ having the same extremities and parity as $w_{j} \sim w_{k-1}$. Since $V\left(u_{1} \sim w_{j}\right) \cap \bar{B}=\emptyset$ and $V\left(w_{k-1} \sim u_{2}\right) \cap \bar{B}=\emptyset$, it follows that the cycle ( $\left.u u_{1}, u_{1} \sim w_{j}, R, w_{k-1} \sim u_{2}, u_{2} u\right)$ is odd and its node set is contained in $B$, a contradiction.
Now suppose that $U_{1} \cap \bar{B} \neq \emptyset \neq U_{2} \cap \bar{B}$.

## Claim 5.

(i) $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2$,
(ii) $\left|\bar{B}_{j, k}\right|=((k-j+1) / 2)+1$,
(iii) $\left|\bar{B}_{l, p}\right|=((p-l+1) / 2)+1$.

Proof. (i) First observe that $(j-1) / 2 \leqslant\left|\bar{B}_{1, j-1}\right| \leqslant(j-1) / 2+1$. Suppose that $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2+1$. If $V\left(\left(u_{1} \sim\right.\right.$ $\left.\left.w_{j}\right) \cup P_{j}\right) \cap \bar{B} \neq \emptyset$, then $\left|\bar{B}_{j+2, p}\right| \geqslant(p-j-1) / 2$. This implies that $\left|V\left(P_{i}\right) \cap \bar{B}\right|=1$ for $i=j+2, j+4, \ldots, p-1$ and thus $U_{2} \cap \bar{B}=\emptyset$, a contradiction. If this is not the case, then $V\left(P_{k-1} \cup\left(w_{k-1} \sim u_{2}\right)\right) \cap \bar{B} \neq \emptyset$, and hence $\left|\bar{B}_{j+1, k-2}\right| \geqslant(k-j-1) / 2+1$. Consequently, $\left|\bar{B}_{k+1, p}\right|=(p-k) / 2$ and therefore $\left|V\left(P_{i}\right) \cap \bar{B}\right|=1$ for $i=k+1, k+$ $3, \ldots, p-1$. We then have $U_{2} \cap \bar{B}=\emptyset$, a contradiction.
(ii) We have that $(k-j+1) / 2 \leqslant\left|\bar{B}_{j, k}\right| \leqslant((k-j+1) / 2)+1$. Let us suppose that $\left|\bar{B}_{j, k}\right|=(k-j+1) / 2$. By Claim 1, there exists a path $R$ having the same endnodes and parity as $u_{1} \sim u_{2}$. Thus the cycle ( $u u_{1}, R, u_{2} u$ ) is odd, a contradiction.
(iii) We have that $(p-l+1) / 2 \leqslant\left|\bar{B}_{l, p}\right| \leqslant((p-l+1) / 2)+1$. Suppose that $\left|\bar{B}_{l, p}\right|=(p-l+1) / 2$. Then, since $\left|\bar{B}_{1, j-1}\right| \leqslant(j-1) / 2$, from Claim 1, there are two paths $R_{1}$ and $R_{2}$ having the same endnodes and parities as $w_{0} \sim u_{1}$ and $u_{3} \sim w_{p}$, respectively. Since the cycle $\left(u u_{1}, u_{1} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u\right)$ is odd, the cycle ( $u u_{1}, R_{1}, P^{*}, R_{2}, u_{3} u$ ) is also odd. As the node set of the latter cycle is in $B$, we have a contradiction.

## Claim 6.

(i) $\left|V\left(P_{k}\right) \cap \bar{B}\right|=1$,
(ii) $\left|V\left(P_{l-1}\right) \cap \bar{B}\right|=1$.

Proof. We will prove (i), the proof of (ii) is similar. It suffices to show that $V\left(P_{k}\right) \cap \bar{B} \neq \emptyset$. If $\left|V\left(P_{k}\right) \cap \bar{B}\right| \geqslant 2$, as by Claim 5(ii), $\left|\bar{B}_{j, k}\right|=((k-j+1) / 2)+1$, it follows that $\bar{B} \cap V\left(C_{t}\right)=\emptyset$ for some $t \in\{j, \ldots, k-1\}$, a contradiction. Now assume, on the contrary, that $V\left(P_{k}\right) \cap \bar{B}=\emptyset$. As $k<l-1, k$ is even and $l$ is odd, we have $l-k \geqslant 3$. If $l-k=3$, then from Claim 5, we have $\left|\bar{B}_{1, j-1}\right|+\left|\bar{B}_{j, k}\right|+\left|\bar{B}_{l, p}\right|=(p / 2)+1$ (Note that by Claim 5(i), $\bar{B}_{1, j-1} \cap \bar{B}_{j, k}=\emptyset$ ). Therefore $V\left(C_{k+1}\right) \cap \bar{B}=\emptyset$, a contradiction. If $l-k>3$, then $((l-2)-(k+2)+1) / 2 \leqslant\left|\bar{B}_{k+2, l-2}\right| \leqslant(p / 2)+1-$ $\left|\bar{B}_{1, j-1}\right|-\left|\bar{B}_{j, k}\right|-\left|\bar{B}_{l, p}\right| \leqslant(((l-2)-(k-2)+1) / 2)-1$, and we have again a contradiction.

Claim 7. If $\left|V\left(P_{k-1}\right) \cap \bar{B}\right|=1$, then $w_{k-1} \in \bar{B}$.
If $\left|V\left(P_{l}\right) \cap \bar{B}\right|=1$, then $w_{l} \in \bar{B}$.
Proof. Let us suppose that $\left|V\left(P_{k-1}\right) \cap \bar{B}\right|=1$ with $w_{k-1} \notin \bar{B}$. Then $V\left(u_{2} \sim w_{k-2}\right) \cap \bar{B}=\emptyset$. By Claims 5(ii) and 6(i) it follows that $\left|\bar{B}_{j, k-2}\right|=((k-2)-j+1) / 2$. Hence from Claim 1, there is a path $R$ between $u_{1}$ and $w_{k-2}$ with the same parity as $u_{1} \sim w_{k-2}$. But this implies that the cycle ( $u u_{1}, R, w_{k-2} \sim u_{2}, u_{2} u$ ) is odd, a contradiction.

Now, let us suppose that $\left|V\left(P_{l}\right) \cap \bar{B}\right|=1$ with $w_{l} \notin \bar{B}$. Then $V\left(u_{3} \sim w_{l+1}\right) \cap \bar{B}=\emptyset$. Observe that by Claim 5(i), $\left|\bar{B}_{1, j-1}\right|=(j-1) / 2$, and by Claims 5 (iii) and 6(ii), $\left|\bar{B}_{l+2, p}\right|=(p-(l+2)+1) / 2$. Thus from Claim 1, there are two paths $R_{1}$ and $R_{2}$ with the same endnodes and parities as $w_{0} \sim w_{j-1}$ and $w_{l+1} \sim w_{p}$, respectively. Then the cycle ( $\left.u u_{1}, u_{1} \sim w_{j-1}, R_{1}, P^{*}, R_{2}, u_{3} \sim w_{l+1}, u_{3} u\right)$ is odd, a contradiction.

Now we distinguish two cases.
Case 2.3.1: Either $w_{k} \notin \bar{B}$ or $w_{k} \in \bar{B}$ and $V\left(P_{k-1}\right) \cap \bar{B}=\emptyset$.
First suppose that either $w_{l-1} \notin \bar{B}$ or $w_{l-1} \in \bar{B}$ and $V\left(P_{l}\right) \cap \bar{B}=\emptyset$. In both cases, it is easy to prove that there is a path $R$ between $u_{2}$ and $u_{3}$ with the same parity as $u_{2} \sim u_{3}$. Then $\left(u u_{2}, R, u_{3} u\right)$ is an odd cycle whose node set is in $B$, a contradiction.

Consequently, $w_{l-1} \in \bar{B}$ and $\left|V\left(P_{l}\right) \cap \bar{B}\right|=1$. From Claim 5 together with Claim 1, it is easy to prove that there are three paths $R_{1}, R_{2}$ and $R_{3}$ having the same endnodes and parities as $w_{0} \sim u_{1}, u_{2} \sim w_{l-2}$ and $w_{l+1} \sim w_{p}$, respectively. Then, by Claim 2 w.r.t. $R_{2}$ and $R_{3}$, there is a path $R$ between $u_{2}$ and $w_{p}$ having opposite parity to that of $u_{2} \sim w_{p}$. Since the faces ( $u u_{2}, u_{2} \sim u_{3}, u_{3} u$ ) and ( $\left.u u_{1}, u_{1} \sim w_{0}, P^{*}, w_{p} \sim u_{3}, u_{3} u\right)$ are odd, then the cycle ( $u u_{1}, R_{1}, P^{*}, w_{p} \sim u_{2}, u_{2} u$ ) is even. Hence, the cycle ( $u u_{1}, R_{1}, P^{*}, R, u_{2} u$ ) is odd, a contradiction.

Case 2.3.2: $w_{k} \in \bar{B}$ and $\left|V\left(P_{k-1}\right) \cap \bar{B}\right|=1$.
Thus, by Claim 7, $w_{k-1} \in \bar{B}$. Suppose that either $w_{l-1} \notin \bar{B}$ or $w_{l-1} \in \bar{B}$ and $V\left(P_{l}\right) \cap \bar{B}=\emptyset$. From Claim 5, one can easily prove that there are two paths $R_{1}$ and $R_{2}$ having the same extremities and parities as $u_{1} \sim w_{k-2}$ and $w_{k+1} \sim u_{3}$, respectively. So, by Claim 2 , there is a path $R$ between $u_{1}$ and $u_{3}$ having opposite parity to that of $u_{1} \sim u_{3}$. Hence, the cycle ( $u u_{1}, u_{1} \sim u_{3}, u_{3} u$ ) is even, and the cycle $\left(u u_{1}, R, u_{3} u\right)$ is odd, a contradiction.

In consequence $w_{l-1} \in \bar{B}$ and $\left|V\left(P_{l}\right) \cap \bar{B}\right|=1$. Thus, from Claim $7, w_{l} \in \bar{B}$. From Claim 5 together with Claim 1 , it is easy to see that there are three paths $R_{1}, R_{2}$ and $R_{3}$ having the same endnodes and parities as $w_{0} \sim w_{k-2}$, $w_{k+1} \sim w_{l-2}$ and $w_{l+1} \sim w_{p}$, respectively. Then, by Claim 2 w.r.t. $R_{1}$ and $R_{2}$, there is a path $R$ between $w_{0}$ and $w_{l-2}$ having opposite parity to that of $w_{0} \sim w_{l-2}$. Similarly, from Claim 2 w.r.t. $R$ and $R_{3}$, there is a path, say $\widetilde{R}$, between $w_{0}$ and $w_{p}$ having the same parity as $P^{\prime}$. As path $\widetilde{R}$ together with $P^{*}$ form an odd cycle whose node set is in $B$, this is a contradiction, and the proof of our theorem is complete.

## 3. Construction of facets

In what follows, we describe three lifting methods that permit to construct "facets from facets". But first, let us give the following remark.

Remark 3.1. If $a x \leqslant \alpha$ is facet defining for $P(G)$, different from a trivial inequality, then $a(u) \geqslant 0$ for all $u \in V$ and $\alpha>0$.

Proof. Suppose there exists a node $v$ with $a_{v}<0$. As $a x \leqslant \alpha$ is different from the facet induced by $x(v) \geqslant 0$, there must exist a set $B \in \beta(G)$ such that $u \in B$ and $a x^{B}=\alpha$. Let $B^{\prime}=B \backslash\{v\}$. Obviously, $B^{\prime} \in \beta(G)$. However, as $a_{v}<0$, we have $a x^{B^{\prime}}>\alpha$, which is impossible.

On the other hand, since $a x \geqslant \alpha$ is facet defining, at least one of the coefficients of $a$, say $a_{u}$, is positive. As $\{u\} \in \beta(G)$, it follows that $0<a_{u} \leqslant \alpha$.

Our first method consists in adding an universal node.

### 3.1. Adding an universal node

Theorem 3.2. Let $G=(V, E)$ be a graph and ax $\leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $\delta=$ $\max \left\{a x^{S} \mid S\right.$ is a stable set of $\left.G\right\}$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be a graph obtained from $G$ by adding an universal node $v_{0}$. Let $\bar{a} x \leqslant \bar{\alpha}$ such that

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { if } u \in V, \\
& \bar{a}_{v_{0}}=\alpha-\delta, \\
& \bar{\alpha}=\alpha .
\end{aligned}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. Since $a x \leqslant \alpha$ is not a trivial inequality, then by Remark $3.1 \alpha>0$. We first show that $\bar{a} x \leqslant \bar{\alpha}$ is valid for $P\left(G^{\prime}\right)$. Let $B^{\prime} \in \beta\left(G^{\prime}\right)$. If $v_{0} \notin B^{\prime}$, then $B^{\prime} \in \beta(G)$ and thus $\bar{a} x^{B^{\prime}}=a x^{B^{\prime}} \leqslant \alpha=\bar{\alpha}$. If $v_{0} \in B^{\prime}$, then $B=B^{\prime} \backslash\left\{v_{0}\right\}$ is a stable set of $G$, and hence $a x^{B} \leqslant \delta$. Thus, $\bar{a} x^{B^{\prime}}=a x^{B}+(\alpha-\delta) \leqslant \delta+\alpha-\delta=\bar{\alpha}$.

Next, we shall exhibit $\left|V^{\prime}\right|$ bipartite induced node sets whose incidence vectors satisfy $\bar{a} x \leqslant \bar{\alpha}$ with equality and are linearly independent, which shows that $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$. Indeed, since $a x \leqslant \alpha$ defines a facet of $P(G)$ with $\alpha>0$, there are $n=|V|$ node sets $B_{1}, \ldots, B_{n} \in \beta(G)$ such that $a x^{B_{i}}=\alpha$, for $i=1, \ldots, n$ and $x^{B_{1}}, \ldots, x^{B_{n}}$ are linearly independent. Now consider the sets

$$
\begin{aligned}
& B_{i}^{\prime}=B_{i} \quad \text { for } i=1, \ldots, n, \\
& B_{n+1}^{\prime}=S^{*} \cup\left\{v_{0}\right\},
\end{aligned}
$$

where $S^{*}$ is a stable set of $G$ such that $a x^{S^{*}}=\delta$. Clearly, $B_{1}^{\prime}, \ldots, B_{n+1}^{\prime} \in \beta\left(G^{\prime}\right)$. Moreover, their incidence vectors satisfy $\bar{a} x \leqslant \bar{\alpha}$ with equality and are linearly independent.


Fig. 6.
An immediate consequence of Theorem 3.2 is the following:
Corollary 8 (Adding an universal complete graph). Let $G=(V, E)$ be a graph and $a x \leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $\delta=\max \left\{a^{t} x^{S} \mid S\right.$ is a stable set of $\left.G\right\}$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ by adding a complete graph $h=(W, F)$ and linking each node of $H$ to every node in $G$. Let $\bar{a} x \leqslant \bar{\alpha}$ such that

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { if } u \in V, \\
& \bar{a}_{v}=\alpha-\delta \quad \text { if } v \in W, \\
& \bar{\alpha}=\alpha .
\end{aligned}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. The statement follows by repeated applications of Theorem 3.2 for the nodes of $W$.
Our second procedure of construction of facets consists in splitting a node. The following theorem describes this procedure and shows that the underlying operation can be reversed.

### 3.2. Splitting and contracting

Theorem 3.3 (Splitting a node). (a) Let $G=(V, E)$ be a graph and ax $\leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $v \in V$ be such that $a_{v}>0$ and let $F$ be an edge subset of $\delta(v)$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ by splitting the node $v$ into two nodes $v_{1}, v_{2}$ such that $v_{1}$ is incident to all edges of $F$ and $v_{2}$ is incident to all edges of $\delta(v) \backslash F$. Let $a_{u}^{\prime}=a_{u}$ for all $u \in V \backslash\{v\}$ and $a_{v_{1}}^{\prime}=a_{v_{2}}^{\prime}=a_{v}$. Suppose that every maximal bipartite induced subgraph of $G^{\prime}$ intersects $\left\{v_{1}, v_{2}\right\}$ and that $\alpha+2 a_{v}=\max \left\{a^{\prime} x^{B} \mid B \in \beta\left(G^{\prime}\right)\right\}$. Let $\bar{G}=(\bar{V}, \bar{E})$ be the graph obtained from $G^{\prime}$ by adding a new node $v_{3}$ and the edges $v_{1} v_{3}$ and $v_{3} v_{2}$ (see Fig. 6). Set

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { if } u \in V \backslash\{v\}, \\
& \bar{a}_{v_{1}}=\bar{a}_{v_{2}}=\bar{a}_{v_{3}}=a_{v}, \\
& \bar{\alpha}=\alpha+2 a_{v} .
\end{aligned}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P(\bar{G})$.
(b) (Contracting a path of length two). Let $G=(V, E)$ be a graph and ax $\leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $v_{1}, v_{2}, v_{3}$ be three nodes of $G$ such that $\left(v_{1}, v_{2}, v_{3}\right)$ is a path, $v_{3}$ is of degree two and $a_{v_{1}}=a_{v_{2}}=a_{v_{3}}>0$. Suppose that every maximal induced bipartite subgraph of $G$ intersects $\left\{v_{1}, v_{2}\right\}$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ by contracting $v_{1} v_{3}$ and $v_{3} v_{2}$. Let $v$ be the node that arises from the contraction. Set

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { if } u \in V \backslash\{v\}, \\
& \bar{a}_{v}=a_{v_{1}}, \\
& \bar{\alpha}=\alpha-2 a_{v_{1}} .
\end{aligned}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. (a) Let $\bar{B} \in \beta(\bar{G})$. If $\left|\left\{v_{1}, v_{2}, v_{3}\right\} \cap \bar{B}\right| \leqslant 2$, as $\bar{B} \backslash\left\{v_{1}, v_{2}, v_{3}\right\} \in \beta(G)$, it follows that $\bar{a} x^{\bar{B}} \leqslant \alpha+2 a_{v}$. If $\left\{v_{1}, v_{2}, v_{3}\right\} \subset \bar{B}$, then let $B=\left(\bar{B} \backslash\left\{v_{1}, v_{2}, v_{3}\right\}\right) \cup\{v\}$. Obviously $B \in \beta(G)$ and hence $\bar{a} x^{\bar{B}}=a x^{B}-a_{v}+3 a_{v} \leqslant \alpha+2 a_{v}$. Therefore $\bar{a} x \leqslant \bar{\alpha}$ is valid for $P(\bar{G})$.

Now we shall show that $\bar{a} x \leqslant \bar{\alpha}$ is facet defining. Let $b x \leqslant \gamma$ be a facet defining inequality of $P(\bar{G})$ such that

$$
\{x \in P(\bar{G}) \mid \bar{a} x=\bar{\alpha}\} \subset\{x \in P(\bar{G}) \mid b x=\gamma\}
$$

To show that $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P(\bar{G})$, it suffices to show that $b=\rho \bar{a}$ for some $\rho>0$.
Let $B^{*} \in \beta\left(G^{\prime}\right)$ such that $a^{\prime} x^{B^{*}}=\alpha+2 a_{v}$. Then we claim that $v_{1}, v_{2} \in B^{*}$. In fact, suppose, for instance, that $v_{1} \notin B^{*}$. Let $B=B^{*} \backslash\left\{v_{2}\right\}$ if $v_{2} \in B^{*}$ and $B=B^{*}$ if not. Thus $B \in \beta(G)$ and $a x^{B} \geq \alpha$, a contradiction. Now consider the sets $B_{1}^{*}=\left(B^{*} \backslash\left\{v_{1}\right\}\right) \cup\left\{v_{3}\right\}$ and $B_{2}^{*}=\left(B^{*} \backslash\left\{v_{2}\right\}\right) \cup\left\{v_{3}\right\}$. Then $B^{*}, B_{1}^{*}, B_{2}^{*} \in \beta(\bar{G})$ with $\bar{a} x^{B^{*}}=\bar{a} x^{B_{1}^{*}}=\bar{a} x^{B_{2}^{*}}=\bar{\alpha}$. This implies that $b x^{B^{*}}=b x^{B_{1}^{*}}=b x^{B_{2}^{*}}=\gamma$, and thus $b_{v_{1}}=b_{v_{2}}=b_{v_{3}}$.
Since $a x \leqslant \alpha$ defines a facet of $P(G)$, there are $n=|V|$ node sets $B_{1}, \ldots, B_{n} \in \beta(G)$ such that $x^{B_{1}}, \ldots, x^{B_{n}}$ satisfy $a x \leqslant \alpha$ with equality and are linearly independent. As $a x \leqslant \alpha$ is different from a trivial inequality, there is at least one set among $B_{1}, \ldots, B_{n}$ that contains (does not contain) $v$. So suppose that $v$ belongs to $k, k<n$, of these sets. W.l.o.g., we may assume that $v \in B_{1}, \ldots, B_{k}$. Since each maximal bipartite node set of $G^{\prime}$ intersects $\left\{v_{1}, v_{2}\right\}$, it follows that for each set $B_{i}, i=k+1, \ldots, n$, there exists $v_{i} \in\left\{v_{1}, v_{2}\right\}$ such that $B_{i}^{\prime}=B_{i} \cup\left\{v_{i}\right\} \in \beta\left(G^{\prime}\right)$. Consider the following node sets

$$
\begin{aligned}
\bar{B}_{i} & =B_{i} \backslash\{v\} \cup\left\{v_{1}, v_{2}, v_{3}\right\} \quad \text { for } i=1, \ldots, k \\
\bar{B}_{i} & =B_{i}^{\prime} \cup\left\{v_{3}\right\} \quad \text { for } i=k+1, \ldots, n
\end{aligned}
$$

which belong to $\beta(\bar{G})$ and verify $\bar{a} x^{\bar{B}_{i}}=\bar{\alpha}$, for $i=1, \ldots, n$. We have that $b x^{\bar{B}_{i}}=\gamma$, for $i=1, \ldots, n$. Let $b^{\prime} \in \mathbb{R}^{V}$ such that $b_{u}^{\prime}=b_{u}$, for $u \in V \backslash\{v\}$ and $b_{v}^{\prime}=b_{v_{1}}$. We have that the incidence vectors $x^{B_{1}}, \ldots, x^{B_{n}}$ satisfy the inequality $b^{\prime} x \leqslant \gamma-2 b_{v_{1}}$ with equality and are linearly independent. In consequence, $b^{\prime} x \leqslant \gamma-2 b_{v_{1}}$ defines a facet for $P(G)$. Note that this facet contains that defined by $a x \leqslant \alpha$. Therefore, $b_{v_{1}}=\rho a_{v}$ and $b_{u}=\rho a_{u}$, for all $u \in V \backslash\{v\}$, for some $\rho \in \mathbb{R}$. This yields $b_{v_{2}}=b_{v_{3}}=\rho a_{v}$, and, consequently, $b=\rho \bar{a}$.

Moreover, since $a x \leqslant \alpha$ is nontrivial, $b x \leqslant \gamma$ is also nontrivial and then $\rho>0$. Hence $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P(\bar{G})$.
(b) The validity of the new inequality can be shown easily as in (a) by elementary constructions. Next we shall show that $\bar{a} x \leqslant \bar{\alpha}$ is facet defining.

Since $a x \leqslant \alpha$ defines a facet of $P(G)$, there are $n=|V|$ node sets $B_{1}, \ldots, B_{n} \in \beta(G)$ whose incidence vectors verify $a x^{B_{i}}=\alpha$, for $i=1, \ldots, n$ and are linearly independent. Set

$$
B_{i}^{\prime}= \begin{cases}B_{i} \backslash\left\{v_{1}, v_{2}, v_{3}\right\} \cup\{v\} & \text { if }\left\{v_{1}, v_{2}, v_{3}\right\} \subset B_{i}, \\ B_{i} \backslash\left\{v_{1}, v_{2}, v_{3}\right\} & \text { if not, }\end{cases}
$$

for $i=1, \ldots, n$. Clearly, $B_{1}^{\prime}, \ldots, B_{n}^{\prime} \in \beta\left(G^{\prime}\right)$. Moreover, we have that $\bar{a} x^{B_{i}^{\prime}}=\bar{\alpha}$ for $i=1, \ldots, n$. Let $M\left(M^{\prime}\right)$ be the matrix whose columns are the vectors $x^{B_{1}}, \ldots, x^{B_{n}}\left(x^{B_{1}^{\prime}}, \ldots, x^{B_{n}^{\prime}}\right)$. To prove our claim, it suffices to show that $M^{\prime}$ is of rank $n-2$. We first claim that for $i=1, \ldots, n,\left|B_{i} \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right| \geqslant 2$. In fact, if this is not the case, then by hypothesis, it follows that there exists $B_{i}, i \in\{1, \ldots, n\}$, such that $v_{3} \notin B_{i}$ and $B_{i}$ contains exactly one of the nodes $v_{1}, v_{2}$, say $v_{1}$. However, since $B_{i} \cup\left\{v_{3}\right\}$ still induces a bipartite subgraph and $a_{v_{3}}>0$, this yields $a x^{B_{i} \cup\left\{v_{3}\right\}}>\alpha$, a contradiction. In addition, there must exist at least one set $B_{i}$ that contains exactly two nodes among $\left\{v_{1}, v_{2}, v_{3}\right\}$ and at least one set $B_{i}$ that contains $\left\{v_{1}, v_{2}, v_{3}\right\}$. For otherwise, the facet defined by $a x \leqslant \alpha$ would be included either in the face $\left\{x \in P(G) \mid x\left(v_{1}\right)+x\left(v_{2}\right)+x\left(v_{3}\right)=2\right\}$ or in the face $\left\{x \in P(G) \mid x\left(v_{1}\right)+x\left(v_{2}\right)+x\left(v_{3}\right)=3\right\}$. Since $x\left(v_{1}\right)+x\left(v_{2}\right)+x\left(v_{3}\right) \leqslant 2$ is not valid (note that $\left\{v_{1}, v_{2}, v_{3}\right\} \in \beta(G)$ ), this contradicts the former case. If the latter case occurs then it follows that the facet is included in the trivial face $\left\{x \in P(G) \mid x\left(v_{1}\right)=1\right\}$. But this is again a contradiction. In consequence, we may suppose, w.l.o.g, that the sets $B_{1}, \ldots, B_{n}$ are ordered in such a way that $B_{1}, \ldots, B_{k}, 1 \leqslant k<n$, contain $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $B_{k+1}, \ldots, B_{n}$ contain exactly two nodes from $\left\{v_{1}, v_{2}, v_{3}\right\}$. Therefore $M^{\prime}$ looks as follows

$$
M^{\prime}=\left[\begin{array}{llllll} 
& & & & \\
& & & & & \\
& & & & \\
\hline 1 \ldots & 1 & & \ldots & \ldots & 0
\end{array}\right]
$$

where the last line corresponds to $v$ and $A$ is the submatrix of $M$ given by the lines corresponding to the nodes of $V \backslash\left\{v_{1}, v_{2}, v_{3}\right\}$.


Fig. 7.

Now, let $L=(2, \ldots, 2)$ be the line vector which contains twos only. Note that $L=(2 / \alpha) a^{t} . M$ (recall that $\alpha>0$ ). If we substract $L$ from the sum of the lines of $M$ corresponding to $v_{1}, v_{1}, v_{3}$, we obtain the last line of $M^{\prime}$. This implies that the lines of $M^{\prime}$ are nothing but linear combinations of lines of $M$. This show that $M^{\prime}$ is of rank $n-2$.

Our last operation consists in subdividing a star.

### 3.3. Star subdivision

Theorem 3.4. Let $G=(V, E)$ be a graph and ax$\leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $v \in V$ such that $a_{v}>0$. Suppose that for each edge $e \in E$ incident to $v$, there is a node set $B$ which induces a bipartite subgraph of $G \backslash e$ and such that $a x^{B}=\alpha+a_{v}$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ by subdividing the edges incident to $v$ (see Fig. 7). Set

$$
\begin{array}{ll}
\bar{a}_{u}=a_{u} & \text { if } u \in V, \\
\bar{a}_{u}=a_{v} & \text { if } u \in V^{\prime} \backslash V, \\
\bar{\alpha}=\alpha+|N(v)| a_{v} .
\end{array}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. Let $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}, k \geqslant 1$. For $j=1, \ldots, k$, let $v_{j}$ be the node added on the edge $v u_{j}$.
Let us show first that $\bar{a} x \leqslant \bar{\alpha}$ is valid for $P\left(G^{\prime}\right)$. Let $B^{\prime} \in \beta\left(G^{\prime}\right)$. If $v \notin B^{\prime}$, then $B^{\prime} \backslash\left\{v_{1}, . . v_{k}\right\} \in \beta(G)$ and hence $\bar{a} x^{B^{\prime}} \leqslant \alpha+k a_{v}$. Now suppose that $v \in B^{\prime}$. We consider two cases:

Case $1:\left|\left\{v_{1}, \ldots, v_{k}\right\} \cap B^{\prime}\right|<k$. As $B^{\prime} \backslash\left\{v, v_{1}, \ldots, v_{k}\right\} \in \beta(G)$, it follows that $\bar{a} x^{B^{\prime}} \leqslant \alpha+k a_{v}$.
Case 2: $\left\{v_{1}, \ldots, v_{k}\right\} \subset B^{\prime}$. Let $B=B^{\prime} \backslash\left\{v_{1}, \ldots, v_{k}\right\}$. We claim that $B \in \beta(G)$. In fact, suppose this is not the case, and let $C$ be an odd cycle of ( $B, E(B)$ ). Clearly, $v \in V(C)$, for otherwise $C$ would be an odd cycle in the graph obtained by removing $v$, which is impossible. Let $u_{1}, u_{2}$ be the nodes of $V(C)$ adjacent to $v$ and let $P$ be the path in $C$ joining $u_{1}$ and $u_{2}$ and not containing $v$, that is $P=C \backslash\left\{u_{1} v, v u_{2}\right\}$. Now let $C^{\prime}$ be the cycle obtained from $P$ by adding the edges $\left\{u_{1} v_{1}, v_{1} v, v v_{2}, v_{2} u_{2}\right\}$. Obviously, $C^{\prime}$ is odd. Since $C^{\prime}$ is a cycle of the subgraph of $G^{\prime}$ induced by $B^{\prime}$, this is a contradiction. Thus $B \in \beta(G)$ and in consequence, $\bar{a} x^{B^{\prime}}=a x^{B}+k a_{v} \leqslant \alpha+k a_{v} \leqslant \bar{\alpha}$. Therefore, $\bar{a} x \leqslant \bar{\alpha}$ is valid for $P\left(G^{\prime}\right)$.

Since $a x \leqslant \alpha$ defines a facet of $P(G)$, there are $n=|V|$ node sets $B_{1}, \ldots, B_{n} \in \beta(G)$ such that $a x^{B_{i}}=\alpha$ for $i=1, \ldots, n$ and $x^{B_{1}}, \ldots, x^{B_{n}}$ are linearly independent. For $u_{j} \in N(v)$, let us denote by $D_{j}$ the set of $\beta\left(G \backslash v u_{j}\right)$ such that $a x^{D_{j}}=\alpha+a_{v}$. Consider the sets

$$
\begin{aligned}
& B_{i}^{\prime}=B_{i} \cup\left\{v_{1}, \ldots, v_{k}\right\} \text { for } i=1, \ldots, n, \\
& B_{n+j}^{\prime}=D_{j} \cup\left\{v_{1}, \ldots, v_{k}\right\} \backslash\left\{v_{j}\right\} \text { for } j=1, \ldots, k
\end{aligned}
$$

which induce bipartite subgraphs in $G^{\prime}$. Moreover, we have that $\bar{a} x^{B_{i}^{\prime}}=\bar{\alpha}$ for $i=1, \ldots, n+k$. Now, let $M\left(M^{\prime}\right)$ be the matrix whose columns are the vectors $x^{B_{i}}, i=1, \ldots, n\left(x^{B_{i}^{\prime}}, i=1, \ldots, n+k\right)$. Then $M^{\prime}$ looks as follows:

$$
M^{\prime}=\left[\begin{array}{ccccccccc} 
& M & & & & A & & \\
& & & & & & & \\
1 & . . & 1 & 0 & 1 & . & . & 1 \\
& & & 1 & . & & & \cdot \\
: & & : & . & & . & & . \\
1 & & & . & & & . & 1 \\
1 & . . & 1 & 1 & . & . & 1 & 0
\end{array}\right],
$$

where the $k$ last lines of $M^{\prime}$ correspond to the nodes $v_{1}, \ldots, v_{k}$ and $A$ is a $0 / 1$ matrix. Note that line $n$ of $M^{\prime}$ corresponds to node $v$. Also note that $a_{v}>0$. Now perform the following operations on $M^{\prime}$ : replace the line $n$ by $\sum_{i=1}^{i=n} a_{i} M_{i}^{\prime}$, where $M_{i}^{\prime}$ is the line $i$ of $M^{\prime}$, and multiply each line $i$ by $\alpha$, for $i=n+1, \ldots, n+k$. Then substract the line $n$ from each line $i$, for $i=n+1, \ldots, n+k$. This yields a matrix of the form

$$
\left[\begin{array}{cc}
\bar{M} & \bar{A} \\
0 & -D
\end{array}\right],
$$

where $\bar{M}$ is a matrix obtained from $M$ by linear combinations and $D$ is a matrix which can be written as

$$
\left[\begin{array}{ccccc}
\alpha+a_{v} & a_{v} & \cdot & \cdot & a_{v} \\
a_{v} & \alpha+a_{v} & & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & & \cdot & a_{v} \\
a_{v} & \cdot & & a_{v} & \alpha+a_{v}
\end{array}\right] .
$$

Thus $\bar{M}$ is nonsingular. Furthermore, as $\alpha, a_{v}>0$, it can be easily seen that $D$ is a nonsingular matrix. In consequence, $M^{\prime}$ is nonsingular and thus $x^{B_{1}^{\prime}}, \ldots, x^{B_{n+k}^{\prime}}$ are linearly independent, which implies that $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.

### 3.4. Applications and examples

First of all let us remark that the validity of most of the inequalities introduced in the previous sections can also be shown using these lifting operations. In fact, a clique inequality (3) can be obtained from a triangle by adding universal nodes (Theorem 3.2). Also a wheel inequality can be obtained from a cycle by adding a complete graph (Corollary 8).

Further valid and facet defining inequalities can also be generated using the lifting operations. Consider for instance a hole $H_{1}=\left(W_{1}, E_{1}\right)$ on three nodes, see Fig. 8(a). From [3] we have that the inequality $x\left(W_{1}\right) \leqslant 2$ defines a facet of $P\left(H_{1}\right)$. By applying twice Theorem 3.2, we get the complete graph $H_{2}=\left(W_{2}, E_{2}\right)$ of Fig. 8(b) and the facet defining inequality $x\left(W_{2}\right) \leqslant 2$ of $P\left(H_{2}\right)$. Now, pick a node $v$, take any set $F$ of two edges incident to $v$ and apply Theorem 3.3(a) with respect to $v$ and $F$. This yields the graph $H_{3}=\left(W_{3}, F_{3}\right)$ of Fig. 8(d) and the associated facet defining inequality $x\left(W_{3}\right) \leqslant 4$. By repeated applications of Theorem 3.3(a) for the remaining nodes of $W_{2}$, we get the graph $H_{4}=\left(W_{4}, E_{4}\right)$ of Fig. 8(c). The resulting facet inequality is $x\left(W_{4}\right) \leqslant 12$. Now, if we repeatedly apply Theorem 3.4 for the nodes of the exterior cycle of $H_{4}$, we obtain the graph $H_{5}=\left(W_{5}, E_{5}\right)$ of Fig. 8(e), and the associated facet defining inequality $x\left(W_{5}\right) \leqslant 27$. Finally, by using Theorem 3.3(b), we get the graph $H_{6}=\left(W_{6}, E_{6}\right)$ of Fig. 8(f) and the facet defining inequality $x\left(W_{6}\right) \leqslant 7$. Note that graph $H_{5}$ is the well known Petersen graph.

Given a wheel of order 1, the node that is universal to the exterior cycle of the wheel is called the hub. A lifted $p$-wheel $H=(W, F)$ is a subdivision of a wheel $H^{\prime}=\left(W^{\prime}, F^{\prime}\right)$ of order 1 with $p+1$ nodes such that the cycles obtained from the triangles of $H^{\prime}$ by the subdivisions are all odd. $H$ is said to be odd (even) if $p$ is odd (even). The hub of $H^{\prime}$ will also be called the hub of $H$.

Now let $G=(V, E)$ be a graph and $H=(W, F)$ an induced lifted $p$-wheel of $G$ with hub $v_{0}$. Suppose that $p$ is odd. Consider the inequality

$$
\begin{equation*}
\sum_{u \in W \backslash\left\{v_{0}\right\}} x(u)+\frac{p-1}{2} x\left(v_{0}\right) \leqslant|W|-2 . \tag{8}
\end{equation*}
$$



Fig. 8. Construction of facets.


Fig. 9. A lifted 5-wheel.

It is not hard to see that inequality (8) can be obtained from a wheel inequality by repeated applications of the operations: contracting a path and subdividing a star. This implies that inequalities (8) are valid for $P(G)$. Moreover, these inequalities may define facets. Fig. 9 shows the construction of a lifted 5 -wheel (Fig. 9(c)) from a wheel on 6 nodes (Fig. 9(a)). The graph of Fig. 9(b) is obtained by subdivision operations on the stars $\delta\left(v_{i}\right), i=1,2,4,5$. Note that $\delta\left(v_{1}\right)$ and $\delta\left(v_{2}\right)$ are subdivided twice. The graph of Fig. 9(c)) is obtained by contracting some paths of three nodes. Observe that the edge between $v_{1}$ and $v_{2}$ is obtained by two contractions.

Cheng and Cunnigham [7] introduced a large class of valid inequalities for the stable set polytope, called general wheel inequalities. Using the same ideas, we can describe similar inequalities, which will also be called general wheel inequalities, valid for $P(G)$. Inequalities ( 8 ) are nothing but a special case of these inequalities. Moreover, Cheng and Cunnigham devised a polynomial time separation algorithm for the general wheel inequalities. One can easily extend this algorithm to separate the general wheel inequalities for the bipartite induced subgraph polytope. We preferred not to introduce these inequalities here for a sake of clarity. Also these inequalities are not going to be used in the experiments we present in the sequel.

## 4. Computational study

In this section we present a Branch-and-Cut algorithm for the bipartite induced subgraph problem. Our aim is to address the algorithmic application of the polyhedral results described in the previous sections. So let us assume that we are given a graph $G=(V, E)$ and a weight vector $c \in \mathbb{R}_{+}^{V}$ associated with the nodes of $G$. The algorithm starts by solving a linear relaxation program of the form

$$
\begin{aligned}
\max & \sum_{v \in V} c(v) x(v) \\
\text { s.t. } & 0 \leqslant x(v) \leqslant 1 \quad \text { for all } v \in V, \\
& x(T) \leqslant 2 \quad \text { for all } T \in \zeta .
\end{aligned}
$$

where $\zeta$ is a set of triangles. The optimal solution $y \in \mathbb{R}^{V}$ of this relaxation is feasible for the problem if $y$ is integer and satisfies all the odd cycle inequalities. Usually the solution $y$ is not feasible, and thus, in each iteration of the Branch-and-Cut algorithm, it is necessary to generate further inequalities that are valid for the BIS polytope but violated by the current solution $y$.

In what follows, we present separation algorithms for the different classes of inequalities used by the algorithm as well as the way the separation of these inequalities is organized. In the second part of this section we present our experimental results.

### 4.1. Separation procedures

The separation of the inequalities is performed in the following order.

1. Clique inequalities,
2. Cycle inequalities,
3. Lifted odd wheel inequalities,
4. Path-cycle inequalities.

We remark that all inequalities are global (i.e. valid in all the Branch-and-Cut tree) and several constraints may be added at each iteration. For each class of inequalities, $i=1, \ldots, 4$, we associate a bound $p_{i}$ on the number of constraints that may be generated from that class in each iteration. We go to the next class of inequalities $i+1$ (in the same iteration) only if less than $50 p_{i} / 100$ new constraints of class $i$ could be found.

Moreover some inequalities may belong to more than one of the classes $1, \ldots, 4$ (like a clique inequality induced by $K_{4}$ is at the same time a wheel inequality). In order not to generate such an inequality more than once in the same iteration, we remove one of the nodes of the graph producing it the first time in the separation.
Now we describe the separation routines used in the algorithm. These may be either exact algorithms or heuristics depending on the class of inequalities. All the separation algorithms are applied on the graph $G$ with weights $y(v), v \in$ $V$, where $y$ is the optimal solution of the current linear relaxation. In what follows, we will denote by $y^{\prime}$ the vector $\mathbf{1}-y$, where $\mathbf{1}=(1, \ldots, 1)^{\mathrm{T}}$. We also denote by $G_{y}$ the support graph of $y$, that is the graph induced by the nodes $u$ with $y(u) \neq 0$.
The separation of the odd cycle inequalities (2) can be performed in $\mathrm{O}\left(|V|^{3}\right)$ time [10]. These inequalities define facets for $P(G)$ only when the cycles are odd holes [3]. By slightly modifying the algorithm in [10], the separation of the odd holes inequalities can also be done in $\mathrm{O}\left(|V|^{3}\right)$ time. The main idea of the algorithm developed in [10] consists in considering a new graph $G_{y}^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ with two nodes $u^{\prime}$ and $u^{\prime \prime}$ for every node $u$ of $G_{y}$. For every edge $u v$ of $G_{y}$ we put the edges $u^{\prime} v^{\prime \prime}$ and $u^{\prime \prime} v^{\prime}$ with weight $\left(y^{\prime}(u)+y^{\prime}(v)\right) / 2$. Clearly, the graph $G_{y}^{\prime}$ with the node set $V^{\prime}=V_{1} \cup V_{2}$ is bipartite where $V_{1}=\left\{u^{\prime} \mid u \in V\right\}$ and $V_{2}=\left\{u^{\prime \prime} \mid u \in V\right\}$.

Now observe that a path $P_{u^{\prime} v^{\prime \prime}}^{\prime}$ in $G_{y}^{\prime}$ from a node $u^{\prime}$ to a node $v^{\prime \prime}$ corresponds to an odd walk $P_{u v}$ in $G_{y}$ between $u$ and $v$. For each node $u$ in $G_{y}$ we compute a shortest path $P_{u^{\prime} u^{\prime \prime}}^{\prime \prime}$ in $G^{\prime}$ between $u^{\prime}$ and $u^{\prime \prime}$. The path $P_{u u}$ corresponds to an odd (closed) walk containing $u$. Since every closed odd walk contains at least one odd hole, we compute a shortest odd hole $C_{u}$ in $P_{u u}$ w.r.t. the weight vector $y^{\prime}$. This can be handled using a simple labeling technique.

Now let $C_{u_{0}}$ such that $y^{\prime}\left(C_{u_{0}}\right)=\min _{u \in V^{\prime}} y^{\prime}\left(C_{u}\right)$. If $y^{\prime}\left(C_{u_{0}}\right) \geqslant 1$, then $y(C) \leqslant|C|-1$ for all odd cycles, and hence inequalities (2) are all satisfied by $y$. Otherwise, the constraint $x\left(C_{u_{0}}\right) \leqslant\left|C_{u_{0}}\right|-1$ would be violated by $y$.

The shortest path computation is handled by Dijkstra's algorithm. In order to speed up this computation we make use of the well known trick to start the shortest path trees from both endpoints. Due to symmetries in the graph, both shortest path trees starting from $u^{\prime}$ and $u^{\prime \prime}$ are identical and the computation of only one suffices.

Also the separation of odd cycle inequalities is implemented so that if an odd hole $C_{u}$ is generated, then all the odd holes already computed have a weight (w.r.t $y^{\prime}$ ) greater than or equal to that of $C_{u}$. This means that sometimes there is no need to explore all the nodes of the graph $G_{y}$.

Our separation of the clique inequalities (3) uses a simple greedy heuristic introduced by Nemhauser and Sigismondi [15] for the stable set problem. The heuristic goes as follows: we pick a node, say $v$, with maximum weight (w.r.t $y$ ). Let $K=\{v\}$. Then iterate the following:

Determine a maximum weight node among the nodes universal to $K$, if there is any, and add this node to $K$.
If $y(K)>2$ then the clique inequality corresponding to $K$ is violated.
Now, we turn our attention to the separation of lifted wheel inequalities (8). We recall that, as mentioned before, these inequalities can be separated in polynomial time within the framework of the general wheel inequalities. However that algorithm, which is in $\mathrm{O}\left(|V|^{4}\right)$, requires that all the cycle inequalities are satisfied. For this we decided to develop a heuristic with a more efficient time complexity. This can be presented as follows: we arbitrarily pick a node, say $u$, of degree $k \geqslant 3$, and $l$ nodes $u_{1}, \ldots, u_{l} \in N(u)$ with $l \leqslant k$ and odd. For every $i=1, \ldots, l$, we compute a shortest odd walk $P_{i}$ (w.r.t $y^{\prime}$ ) between $u_{i}$ and $u_{i+1}$ such that $u \notin V\left(P_{i}\right)$ and every node of $P_{i}$ appears in at most one of the walks $P_{1}, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{l}$. The idea behind this is that every walk $P_{i}$ may correspond to a cycle of the lifted odd wheel, and every node different from $u$ may belong to at most two of these cycles. Let $W=\bigcup_{i=1, \ldots, l} V\left(P_{i}\right) \cup\{u\}$. If all the $P_{i}$ are paths, then $W$ induces a lifted odd wheel. Otherwise $W$ would produce a redundant inequality. If the former case holds, then we test whether the corresponding lifted odd wheel inequality is violated.

This heuristic is applied as far as violated lifted odd wheel inequalities can be found. If no more constraints can be generated by this heuristic, then the exact algorithm of Section 2 is applied to find violated wheel inequalities.

We have also developed a separation heuristic for the path-cycle inequalities (6) using ideas similar to those used in the separation heuristic for the lifted wheel inequalities. This procedure can be described as follows. Let $w(e)=$ $\left(y^{\prime}(u)+y^{\prime}(v)\right) / 2$ for all edge $e=u v$ of $G_{y}$. We first compute a minimum spanning tree $T$ in $G_{y}$ w.r.t. $w$. We then arbitrarily select three nodes, say $u, v, w$, of $T$ and determine the subtree $T^{\prime}$ of $T$ that covers these nodes and having, say $u$ and $w$ as leaves. Note that $T^{\prime}$ may have two forms, either a path with $u$ and $w$ as extremities or a tree with all the nodes $u, v, w$ as leaves. Once $T^{\prime}$ is computed, if it contains a sufficient number of nodes, we fix $q$ nodes in $T^{\prime}$, $u_{1}, \ldots, u_{q}$, with $q \geqslant 5$, such that $u_{1}=u, \ldots, u_{t}=v, \ldots, u_{q}=w, q$ is odd and $u_{1}, \ldots, u_{t-1}, u_{t+1}, \ldots, u_{q}$ are in this order on the path joining $u$ to $w$. For each node $u_{j}, j=2, \ldots, q-1$, let $u_{j}^{\prime}$ be a neighbour of $u_{j}$ in $V \backslash V\left(T^{\prime}\right)$ and let $u_{1}^{\prime}=u_{1}, u_{q}^{\prime}=u_{q}$. At this step we obtain a graph similar to that shown in Fig. 10(a) (Here $q=7$ ). Then we compute a shortest path $P_{i}$ between every pair of nodes $\left(u_{i}^{\prime}, u_{i+1}^{\prime}\right), i=1, \ldots, q$ (where the indices are taken modulo $q$ ). These paths are computed so that $P_{i} \cup T^{\prime}$ contains an odd cycle, for $i=1, \ldots, p$ and two paths $P_{i}$ and $P_{j}$ with $|j-i|>1$ are node-disjoint. This leads to a graph of the same type as that given in Fig. 10(b). Now the paths $P_{i}$ together with $T^{\prime}$ and the set of edges $\left\{u_{i} u_{i}^{\prime}, i=2, \ldots, q-1\right\}$ form a path-cycle $C$ of order $q-1$. This is shown in Fig. 10(c). Note that $P_{q}$ corresponds to the path $P^{*}$ of the path-cycle. By construction $C^{\prime}$ is odd. We claim that $C^{\prime \prime}$ is also odd. Indeed, the separation algorithm produces a planar graph having $q$ interior faces which are all odd. Furthermore, each edge is shared by at most two faces. As $q$ is odd, the exterior face, that is $C^{\prime \prime}$, must also be odd. In addition, if the weight of $C$ w.r.t. $y$ is $>|V(C)|-(((q-1) / 2)+1)$, then a violated path-cycle inequality has been found.

We have also developed a primal rounding heuristic in order to compute a lower bound. This heuristic works as follows. We first compute a maximum spanning tree $H$ w.r.t. to the weight vector $\omega \in \mathbb{R}^{E}$ defined as $\omega(e)=$ $\max (y(u) c(u), y(v) c(v))$ for every edge $e=u v$ of $G_{y}$. The idea behind this is to have in the solution a maximum number of nodes of large weights. We then partition $V$ into two sets $V_{1}$ and $V_{2}$ so that every edge of $H$ has one extremity in $V_{1}$ and the other in $V_{2}$. Let $\widetilde{V}=V$ and $\tilde{v}$ be a node of $\widetilde{V}$ such that $y(\tilde{v}) c(\tilde{v})$ is minimum. Let $i \in\{1,2\}$ such that $\tilde{v} \in V_{i}$. If $\tilde{v}$ has a neighbour in $V_{i}$, then an odd cycle is detected and, hence we let $\widetilde{V}:=\widetilde{V} \backslash\{\tilde{v}\}$. If the graph induced by $\widetilde{V}$ is bipartite, then we terminate, otherwise we iterate the process.

### 4.2. Computational results

We can now present some computational results obtained using our Branch-and-Cut algorithm. The algorithm has been implemented in C++ using the ABACUS framework [18,19] and CPLEX 7.1 LP solver. It was tested on a Pentium III 600 MHz with 512 MO RAM, running under Linux.


Fig. 10.

The first column of the tables gives the instances names. The other entries of the table are as follows.
$|V| \quad$ Number of nodes,
$|E| \quad$ Number of edges,
$\mathrm{Ncl} \quad$ Number of generated clique inequalities,
Noc Number of generated odd cycle inequalities,
Now Number of generated lifted odd wheel inequalities,
Npc Number of generated path cycle inequalities,
SB Number of generated nodes in the branch-and-cut tree,
Gap The relative error between the optimal value and the upper bound achieved before branching,
Copt The optimal value,
TT Total CPU time in minutes to solve problem instance to optimality.
For all instances we allowed a maximum of 5 h of CPU time. If for an instance the time limit exceeds, then the primal heuristic is applied. The value of the solution given by this heuristic as well as the gap between this solution and the upper bound obtained within the time limit, are given in italic.

Our first series of experiments, given in Table 1, concerns some via minimization instances. The first five instances are randomly generated, however the last ones come from industry. The instances have been first transformed to equivalent maximum bipartite induced subgraph problems. The corresponding graphs have 999 to 73851 nodes and a maximum degree not exceeding 4 . Also, they do not have clique. This is, in fact, a consequence of the model introduced in [9].

We can see that all the instances except r200-47 have been solved to optimality in the cutting plane phase. For that instance, we applied the primal heuristic and obtained a very small gap. We can also note that many lifted odd wheel inequalities have been generated for the random instances. However, there are a few inequalities of that type added for the industrial ones. Also path-cycle inequalities have not been used for these instances. This may be explained by the particular structure of the industrial instances, which seem to be easier to solve. Finally we can observe that the odd cycle inequalities play a central role in the resolution of all the instances.

Table 2 reports experimental results for some BIS problem instances that are randomly generated. First of all let us note that the size of these instances is much smaller than that of those corresponding to, the via minimization problems given in Table 1. Moreover, for all the graphs, the maximum degree is greater than 4 and the first ten instances are more dense than the last ones. We noticed that when the graphs get more dense, the problem gets much harder to

Table 1
Experimental results: via minimization instances

| Instance | $\|V\|$ | $\|E\|$ | Ncl | Noc | Now | Npc | SB | Gap | Copt | TT |
| :--- | ---: | ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| r33-9 | 999 | 1222 | 0 | 334 | 96 | 0 | 0 | 0 | 2050 | $0: 09$ |
| r70 -45 | 5211 | 6453 | 0 | 3457 | 194 | 36 | 0 | 0 | 10527 | $4: 43$ |
| r100 - 987 | 13293 | 26572 | 0 | 4861 | 286 | 38 | 0 | 0 | 26646 | $13: 26$ |
| r150 -100 | 32012 | 39964 | 0 | 9585 | 1312 | 369 | 0 | 0 | 64030 | 190 |
| r200 - 47 | 57072 | 71370 | 0 | 14522 | 1932 | 156 | 0 | 0.003 | 113970 | $-*$ |
| indc2 | 13764 | 17032 | 0 | 1983 | 1 | 0 | 0 | 0 | 27490 | $3: 18$ |
| indc5 | 23520 | 29042 | 0 | 3083 | 8 | 0 | 0 | 0 | 46998 | $8: 30$ |
| indm1.f1 | 44408 | 55162 | 0 | 4205 | 9 | 0 | 0 | 0 | 88658 | $24: 24$ |
| indt1.0 | 73851 | 91985 | 0 | 4333 | 73 | 0 | 0 | 0 | 147404 | $42: 40$ |

* : time limit of 5 CPU hours exceeded.

Table 2
Experimental results: randomly generated graphs

| Instance | $\|V\|$ | $\|E\|$ | Ncl | Noc | Now | Npc | SB | Gap | Copt | TT |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| c20a | 20 | 169 | 21 | 9 | 2 | 0 | 0 | 0 | 20844 | 0:01 |
| c50a | 50 | 266 | 340 | 501 | 890 | 258 | 420 | 14.57 | 70026 | 3:17 |
| c50b | 50 | 1082 | 353 | 32 | 3 | 0 | 8 | 1.56 | 75123 | 0:19 |
| c80a | 80 | 2005 | 43945 | 138 | 9 | 2 | 2350 | 6.47 | 156748 | -* |
| c100a | 100 | 200 | 1 | 326 | 1631 | 38 | 100 | 2.89 | 80208 | 4:19 |
| c100b | 100 | 477 | 317 | 3567 | 11539 | 131 | 5388 | 12.8 | 119964 | -* |
| c150a | 150 | 295 | 1 | 902 | 6542 | 98 | 642 | 2.42 | 122627 | 61:40 |
| c150b | 150 | 392 | 16 | 2025 | 14351 | 205 | 2404 | 5.51 | 131266 | -* |
| c200a | 200 | 347 | 0 | 1484 | 11838 | 119 | 2672 | 1.83 | 153992 | -* |
| n200a | 200 | 299 | 0 | 196 | 276 | 42 | 0 | 0 | 138427 | 0:08 |
| n300a | 300 | 399 | 0 | 426 | 1397 | 52 | 78 | 0.39 | 192174 | 14:12 |
| n500a | 500 | 600 | 0 | 442 | 930 | 74 | 34 | 0.16 | 291922 | 15:10 |
| n700a | 700 | 800 | 0 | 543 | 1140 | 54 | 122 | 0.18 | 380339 | 56:54 |
| n800a | 800 | 900 | 0 | 240 | 146 | 33 | 0 | 0 | 428223 | 0:10 |
| n1000a | 1000 | 1100 | 0 | 462 | 529 | 79 | 4 | 0.03 | 542411 | 7:22 |
| n1000b | 1000 | 1200 | 0 | 1353 | 1714 | 394 | 166 | 1.43 | 551834 | -* |

_* : time limit of 5 CPU hours exceeded.
solve. In fact, only three of the instances have been solved to optimality in the cutting plane phase. Also, some of the instances (with no more than 100 nodes) have not been solved within the time limit of 5 h . As shown in the table, these instances have been solved by the primal heuristic with sometimes a very small gap. We can also note that the odd cycle inequalities play here a minor role. However, the lifted odd wheel and path-cycle inequalities seem to be more useful for this type of instances. In fact, in more dense graphs, cycle inequalities may be dominated by clique, wheel and path-cycle inequalities.
Finally we can note that a large number of lifted odd wheel and path-cycle inequalities have been added for most of the instances. This shows that the separation procedures we developed for these inequalities are effective. Moreover, our Branch-and-Cut algorithm seems to be more efficient for via minimization instances than for general BIS problems, because of the special structure of those instances.

## 5. Concluding remarks

We have studied the bipartite induced subgraph problem. We have described two classes of valid inequalities of the associated polytope called, respectively, lifted odd wheel and path-cycle inequalities and given necessary and sufficient conditions for these inequalities to be facet defining. We have devised a polynomial time separation algorithm for the wheel inequalities. We have also described some lifting operations that permit to construct facet defining inequalities
from known ones. We have provided separation heuristics for these lifted inequalities as well as for the path-cycle inequalities. Using these results we have described a Branch-and-Cut algorithm for the bipartite induced subgraph problem. Our computational results show that the algorithm is quite efficient for solving via minimization instances. They also show that the lifted odd wheel and the path-cycle inequalities are effective for the more general instances. We could also measure the performance of our separation techniques. In particular our heuristics for the lifted odd wheel and the path-cycle inequalities have shown to be efficient.

Now it would be interesting to identify further classes of facet defining inequalities of the bipartite induced subgraph polytope. It would also be interesting to study a column generation method that can be incorporated in our Branch-and-Cut algorithm. This may permit to solve larger instances of the BIS problem in a very efficient way. We are now making investigations in this direction.

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[^0]:    E-mail addresses: Pierre.Fouilhoux @isima.fr (P. Fouilhoux), Ridha.Mahjoub@math.univ-bpclermont.fr (A.R. Mahjoub).

