Integer Programming Formulations for the Two 4-Hop-Constrained Paths Problem

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In this article, we consider the two 4-hop-constrained paths problem, which consists, given a graph $G = (N, E)$ and two nodes $s, t \in N$, of finding a minimum cost subgraph in G containing at least two node- (resp., edge-) disjoint paths of length at most 4 between s and t. We give integer programming formulations, in the space of the design variables, for both the node and edge versions of this problem. **NETWORKS, Vol. 49(2), 135–144 2007**

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1. INTRODUCTION

One of the main concerns when designing telecommunication networks is to build topologies that provide protection against equipment failures. This requires networks to be *survivable*, that is, networks that remain functional when links or nodes fail. However, in general, the survivability requirement is not sufficient to guarantee a cost-effective routing. Indeed, the alternative routing paths may be too long, and hence, too costly, to be suitable. In consequence, further technical constraints have to be added. In particular, one can impose a limit on the length of the rerouting paths. One of the rerouting strategies is the so-called *end-to-end rerouting strategy*. Here, if a link (node) fails, the traffic must be rerouted between its origin–destination nodes. To limit the rerouting, one must have at least two edge- (node-) disjoint paths with bounded length between each origin-destination pair, so that if one of the paths fails, the traffic may be rerouted (in a minimum time) on the second one. This applies, for instance, to ATM networks and the Internet. In many practical situations, the length of the routing path is defined as the number of links (also called hops) in the path, and we talk about a *hop-constrained path*. In this article we consider this length-constrained survivable network design problem.

Given a graph $G = (N, E)$ with $s, t \in N$, and a positive integer $L \geq 2$, an *L-st-path* in G is a path between s and t of length at most L, where the length of a path is its number of edges (also called *hops*). Given a function $c : E \to \mathbb{R}$, which associates a cost $c(e)$ to each edge $e \in E$, the *Two L-Hop-constrained Paths Problem* (THPP) is to find a minimum cost subgraph such that between s and t there exist at least two node-(edge-) disjoint L-st-paths. We will speak of the *node* THPP if those two paths must be node-disjoint, and of the *edge* THPP if they must be edge-disjoint.

In this article, we consider the node THPP and the edge THPP when $L = 4$. For both versions, we give an integer programming formulation in the space of the design variables. Such a formulation has been for the edge case an open question for the past few years [20, 21]. For this case, we will introduce a new class of valid inequalities, and show that these inequalities together with the so-called st-cut, L-path-cut and integrality constraints suffice to formulate the problem when $L = 4$. For the node case, we first extend the L-path-cut inequalities to the node THPP, and then show that these inequalities together with the st-cut, st-node-cut, and integrality constraints are sufficient to formulate the problem when $L = 4$. Unfortunately, both formulations cannot be directly extended to every value of L. In fact, as will be seen later, one needs additional inequalities for both formulations even when $L = 5$. The contribution of this article can be seen as an important step toward a formulation of the THPP in the natural variable space for all L.

The THPP can also be seen as a special case of the more general problem when more than one pair of terminals is considered. This is the case, for instance, when several

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commodities have to be routed in the network. Thus, an efficient algorithm for solving the THPP would be useful to solve (or produce upper bounds for) this more general problem. The formulations we will give in this article can be easily extended to that general problem.

In [15], Gouveia et al. discuss the node case of that general problem within the context of an MPLS (Multi-Protocol Label Switching) network design model. The authors propose two extended formulations involving one set of variables for each path between each pair of terminals. The first model uses standard flow variables, and the second uses hop-indexed variables. Each subsystem of constraints associated with a path is a flow model with additional cardinality constraints. The authors also introduce a third model involving one set of hop-indexed variables for each pair of terminals. They show that this aggregated and more compact model produces the same linear programming bound as the multipath hop-indexed model. They also present computational results for $L = 4, 5$, and 6 using these formulations. Unfortunately, as the number of variables of the resulting models grows with L (and the number of pairs of terminals), the size of the corresponding linear programming relaxation may lead to excessive computational time when more dense instances (or instances with a larger value of L or a larger number of nodes) are considered. As mentioned in [15], this points out the need for looking for formulations using only natural variables.

The edge THPP has been already investigated by Huygens et al. [20] when $L = 2, 3$. Also, the problem of finding a minimum cost subgraph that contains k edge-disjoint 2-st-paths, with $k \geq 2$, has been considered in [7]. For both problems, a complete and minimal linear description of the corresponding polytope is given. In [4], Dahl considers the hop-constrained path problem, that is the problem of finding between two distinguished nodes s and t a minimum cost path with no more than L edges when L is fixed. He gives a complete description of the dominant of the associated polytope when $L \leq 3$. Dahl and Gouveia [6] consider the directed hop-constrained path problem. They describe valid inequalities and characterize the associated polytope when $L \leq 3$. In [2], Coullard et al. investigate the structure of the polyhedron associated with the st-walks of length L of a graph, where a walk is a path that may go through the same node more than once. Dahl et al. [5] also consider the hop-constrained walk polytope in directed graphs when $L = 4$. This is the first work that addresses a polyhedral analysis for a hop-constrained network design problem with $L = 4$. By introducing extended variables in addition to the design variables, the authors characterize the polytope. They also introduce a large class of facet-defining inequalities for the dominant of that polytope, which surprisingly shows that describing the hop-constrained walk polytope for $L = 4$ is easier than describing its dominant. Moreover, one of their conclusions is that the structure of the hop-constrained path polytope for $L = 4$ is considerably more complicated.

In [8], Dahl and Johannessen consider the 2-path network design problem, which consists of finding a minimum cost subgraph connecting each pair of terminal nodes by at least one path of length at most 2. The problem of finding a minimum cost spanning tree with hop constraints is considered in [10, 11, 16]. Here, the hop constraints limit to a positive integer H the number of links between the root and any terminal in the network. Dahl [3] studies the problem for $H = 2$ from a polyhedral point of view and gives a complete description of the associated polytope when the graph is a wheel. Huygens et al. [19] consider the problem of finding a minimum cost subgraph with at least two edge-disjoint L-hop-constrained paths between each given pair of terminal nodes. They give an integer programming formulation of that problem for $L = 2, 3$, and present several classes of valid inequalities. They also devise a Branch and Cut algorithm, and discuss some computational results. In [23], Nguyen gives a complete linear description of the k-path polyhedron.

Hop constraints have been also considered by Gouveia [10, 11] for the minimum spanning tree problem. The problem is then, given a graph $G = (V, E)$ with weights on the links and a *root* node, to find a minimum spanning tree such that the (unique) path between the root and any other node in the graph has no more than L links (hops), where L is a fixed positive integer. This problem is NP-hard even for $L = 2$ (see, i.e., [3]). Gouveia [10] gives a multicommodity flow formulation for that problem and discusses a Lagrangian relaxation improving the LP bound. Gouveia [11] proposes a hop-indexed reformulation of a multicommodity flow formulation which is based on an extended description of the L-walk polyhedron. The reported computational results show that the new formulation is attractive to use when L is small. In [16], Gouveia and Requejo propose a Lagrangian relaxation for the problem that dualizes the hop-indexed flow conservation constraints. Reported results show that this relaxation is a good alternative to directly solving the corresponding linear programming relaxation. Further results, formulations, and pólyhedral analysis for the hop-constrained minimum spanning tree problem can be found in [12–14].

We assume familiarity with graphs and polyhedra. For specific details, the reader is referred to [1] and [24]. The graphs we consider are finite, undirected, loopless, and may have multiple edges. A graph is denoted by $G = (N, E)$, where N is the *node set* and E is the *edge set*. If $u, v \in N$, we will denote by uv an edge between u and v. If $W \subset N$ is a node subset of G, then the set of edges that have only one node in W is called a *cut* and is denoted by $\delta_G(W)$. When it is clear that the cut is taken with respect to G, we will simply denote it by $\delta(W)$. We will write $\delta(v)$ for $\delta(\{v\})$. A cut $\delta(W)$ such that $s \in W$ and $t \in N \backslash W$ will be called an *st-cut*. If $V, W \subset N$, $[V, W]$ is the set of edges having one endnode in V and the other one in W. Note that we will write $[v, w]$ instead of $[\{v\}, \{w\}]$. A *path* P of G is an alternating sequence of nodes and edges $(u_1, e_1, u_2, e_2, \ldots, u_{q-1}, e_{q-1}, u_q)$, where $e_i \in [u_i, u_{i+1}]$ for $i = 1, \ldots, q-1$. We will denote a path P by either its node sequence $(u_1, \ldots, u_q)$ or its edge sequence $(e_1, \ldots, e_{q-1})$. Given a partition $(V_1, \ldots, V_p)$ of N, an edge between any two nonconsecutive subsets of the partition will be called a *chord*.

Given a graph $G = (N, E)$ and an edge subset $F \subseteq E$, the $0-1$ vector $x^F \in \mathbb{R}^{|E|}$, such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ otherwise, is called the *incidence vector* of F. The x-variables are also named *design variables*. We denote by $G_F = (N, F)$ the *support graph* of x^F, that is, the subgraph of G containing only the edges e with $x^F(e) = 1$. Given an edge $e \in E$, the graph $G - e$ is the subgraph obtained from G by deleting the edge e (but not its endnodes). Given a node $z \in N$, the graph $G - z$ is the subgraph obtained from G by deleting the node z and all its incident edges (but not their other endnodes).

This article is organized as follows. In the next section, we propose an integer programming formulation, in the space of the design variables, for the node THPP with $L = 4$. This formulation will also be valid when $L = 2, 3$. In Section 3, we first present a new family of valid inequalities for the edge THPP when $L = 4$. Using this family, we give an integer programming formulation, in the space of the design variables, for this problem. In Section 4 we discuss some extensions, and in Section 5, we give some concluding remarks.

2. FORMULATION FOR THE NODE THPP WHEN $L = 4$

The incidence vector x^F of any solution (N, F) to the node (edge) THPP satisfies the following inequalities:

$$x(\delta(W)) \geq 2, \quad \text{for all } st\text{-cuts } \delta(W),$$
$$1 \geq x(e) \geq 0, \quad \text{for all } e \in E.$$

The first inequalities are called *st-cut inequalities* and the second ones are called *trivial inequalities*. Moreover, in the node case, the following so-called *st-node-cut inequalities* are also valid for the associated polytope:

$$x(\delta_{G-z}(W)) \geq 1,$$

where $z \in N \backslash \{s, t\}$ and $\delta_{G-z}(W)$ is an st-cut in the graph $G - z$.

In [4], Dahl introduces a class of valid inequalities for the hop-constrained st-path problem as follows. Let $\Pi = (V_0, V_1, \ldots, V_{L+1})$ be a partition of N such that $s \in V_0$, $t \in V_{L+1}$ and $V_i \neq \emptyset$ for all $i = 1, \ldots, L$. Let T_G be the set of the chords of the partition Π of G. Then the inequality

$$x(T_G) \geq 1$$

is valid for the L-path polyhedron. The set T_G is called an *L-path-cut*. When it is clear that the L-path-cut is taken with respect to G, we will simply denote it by T. Using the same kind of partition, these inequalities can be generalized in a straightforward way to the THPP as

$$x(T) \geq 2,$$

for any L-path-cut T. Constraints of this type will be called *L-path-cut inequalities*.

These inequalities can be extended to the node case as follows:

$$x(T_{G-z}) \geq 1,$$

where $z \in N \backslash \{s, t\}$ and T_{G-z} is an L-path-cut in the graph $G - z$. We call these constraints *L-path node-cut inequalities*.

Now, consider the following linear system consisting of the inequalities introduced above, that is

$$x(\delta(W)) \geq 2, \quad \text{for all } st\text{-cuts } \delta(W), \tag{1}$$
$$x(\delta_{G-z}(W)) \geq 1, \quad \text{for all } st\text{-cuts } \delta_{G-z}(W), \text{ for all } z \in N \backslash \{s, t\}, \tag{2}$$
$$x(T) \geq 2, \quad \text{for all } L\text{-path-cuts } T, \tag{3}$$
$$x(T_{G-z}) \geq 1, \quad \text{for all } L\text{-path-cuts } T_{G-z}, \text{ for all } z \in N \backslash \{s, t\}, \tag{4}$$
$$x(e) \leq 1, \quad \text{for all } e \in E, \tag{5}$$
$$x(e) \geq 0, \quad \text{for all } e \in E. \tag{6}$$

We will show that the system above, along with the integrality constraints, formulates the node THPP as an integer program when $L = 4$.

Theorem 1. *The node THPP for $L = 4$ is equivalent to the integer program*

$$\min \left\{cx; \text{subject to (1)–(6)}, x \in \mathbb{Z}^{|E|}\right\}.$$

Proof. By the development above, it is clear that the incidence vector of any solution to the node THPP with $L = 4$ satisfies inequalities (1)–(6).

Now consider an edge subset $F \subseteq E$ and suppose that F does not induce a solution to the node THPP with $L = 4$. Suppose that all st-cut and st-node-cut constraints are satisfied by x^F. We are going to show that there is at least one 4-path-cut or 4-path node-cut violated by x^F. Let G_F be the graph induced by F and P_0 a shortest st-path (in number of hops) in G_F. In what follows, we are going to discuss different cases with respect to the length of P_0.

If $|P_0| = 1$, that is $P_0 = (st)$ with $st \in [s, t]$, then P_0 is the only 4-st-path in G_F. In fact, if there exists a 4-st-path P different from P_0, then P would be node-disjoint from P_0, a contradiction. Therefore, in $G_F - st$, because the st-cut inequalities are satisfied, there must exist an st-path, of length at least 5. Let us define V_i, $i = 0, \ldots, 4$, as the subset of nodes at distance i from s in $G_F - st$, and $V_5 = N \backslash (\bigcup_{i=0}^{4} V_i)$. By the previous remarks, it is clear that the V_i's are nonempty, and that $s \in V_0$ and $t \in V_5$. Moreover, by construction, no edge of $G_F - st$ is in the corresponding 4-path-cut $T_{G_F - st}$, and hence, $x(T_{G_F - st}) = 0$. Let T be the corresponding 4-path-cut in G. Therefore, in G, we obtain $x^F(T) = 1$, which is a violated 4-path-cut inequality.

If $|P_0| = 2$, that is $P_0 = (su, ut)$ with $u \in N \backslash \{s, t\}$, because F is not a solution to the problem, all the other 4-st-paths of G_F must go through u. Therefore, in $G_F - u$, because the st-node-cut inequalities are satisfied, there exists an st-path, of

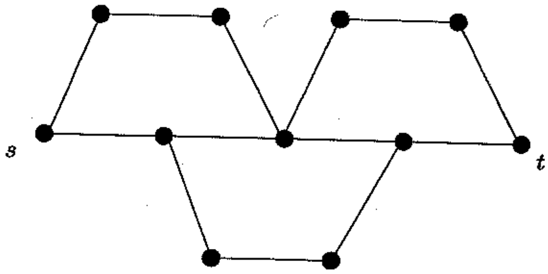

FIG. 1. Infeasible solution for the node THPP with $L = 5$.

length at least 5. Let us define a 4-path-cut T_{G_F-u} in $G_F - u$ the same way as in the previous case. By construction, no edge of $G_F - u$ is in T_{G_F-u}, and hence, $x(T_{G_F-u}) = 0$. This implies that the 4-path node-cut inequality corresponding to T_{G-u} and u is violated by x^F.

If $|P_0| = 3$, that is $P_0 = (su, uv, vt)$ with $u, v \in N\backslash\{s, t\}$, $u \neq v$, then all the other 4-st-paths of G_F must go through either u, or v, or both. Suppose that there exist at least one 4-st-path P_1 going through u, but not v, and at least one 4-st-path P_2 going through v, but not u. Let P_1^t (resp., P_1^s) be the subpath of P_1 between u and t (resp., s), and P_2^s (resp., P_2^t) the subpath of P_2 between s (resp., t) and v. Clearly, $|P_1^t| \leq 3$ and $|P_2^s| \leq 3$. Moreover, because P_0 is a shortest st-path, we have $|P_1^t| \geq 2$ and $|P_2^s| \geq 2$. Hence, $1 \leq |P_1^s| \leq 2$ and $1 \leq |P_2^t| \leq 2$. Now, because F is not feasible, P_1 and P_2 must intersect each other in a node w different from s, u, v, t. If $w \in P_1^t \cap P_2^s$, then P_0 and the 4-st-path consisting of the subpath of P_2 from s to w, and of the subpath of P_1 from w to t, are node-disjoint, a contradiction. Therefore, either $w \in P_1^s$ or $w \in P_2^t$. Suppose that $w \in P_1^s$ (the other case follows by symmetry). Then $|P_1^s| = 2$, and hence, $|P_1^t| = 2$. Let P_2^{vw} be the subpath of P_2 between v and w. Clearly, $|P_2^{vw}| \leq 2$. But then we have two node-disjoint 4-st-paths in G_F, namely su, P_1^t, and sw, P_2^{vw}, vt. This contradicts the infeasibility of the solution. Consequently, there cannot be at the same time 4-st-paths going only through u, or through v. We can suppose that u belongs to all 4-st-paths of G_F (the other case is symmetric). Therefore, there exists no 4-st-path in $G_F - u$ and, by constructing the 4-path node-cut T_{G-u} as before, we get a contradiction.

Now suppose that $|P_0| = 4$, that is $P_0 = (s, u, v, w, t)$ with $u, v, w \in N\backslash\{s, t\}$, u, v, w different from each other. Then all the other 4-st-paths of G_F must go through either u, or v, or w.

Claim. There does not exist a 4-st-path going through u which uses neither v nor w.

Proof. Assume the contrary, and let P be a 4-st-path that contains u, but neither v nor w. Because P_0 is a shortest path, P must contain three edges between u and t. Let v', w' be the nodes along this subpath, different from u and t. Now, suppose that there exists a 4-st-path P' not going through u, but through v or w. Thus, P' must also intersect P in either v' or w'. Because P_0 is a shortest st-path, it is not hard to see that we have either $P' = (s, u', v', w, t)$ or $P' = (s, u', v, w', t)$, with $u' \in N\backslash\{s, u, v, w, v', w', t\}$. But then the graph induced by $P_0 \cup P \cup P'$ contains two node-disjoint 4-st-paths, a contradiction. Consequently, there cannot exist a 4-st-path not going through u, and hence, all 4-st-paths of G_F use u. This implies that $G_F - u$ does not contain 4-st-paths. We can then get a contradiction in a similar way as before. □

By the claim above, all 4-st-paths in G_F go through v or w. Suppose there is a 4-st-path $\tilde{P}$ going through w, but not v. If $\tilde{P}$ does not contain u, we can show along the same lines that all 4-st-paths use w, and get a violated 4-path node-cut inequality.

So suppose that $\tilde{P}$ contains u. We may also suppose that there are two further 4-st-paths P_1 and P_2 such that P_1 (resp., P_2) uses u but not w (resp., w but not u). For otherwise, either each 4-st-path contains u or each 4-st-path contains w. In both cases, we get as before a violated 4-path node-cut. Moreover, P_1 and P_2 must go through v, for otherwise, we would get two node-disjoint 4-st-paths, a contradiction. Hence, P_1 is of the form (s, u, v, P_1^t), where P_1^t is a vt-path of length 2 not going through w, and P_2 is of the form (P_2^s, v, w, t), where P_2^s is a sv-path of length 2 not going through u. But then we obtain two node-disjoint 4-st-paths, namely $P_2^s \cup P_1^t$ and $\tilde{P}$, which is a contradiction.

In consequence, all 4-st-paths go through v. As before, we obtain a violated 4-path node-cut inequality.

If $|P_0| \geq 5$, there exists no 4-st-path in G_F. Thus, we can build directly an adequate partition in G_F and get a violated 4-path-cut inequality. ■

This result holds for $L = 2, 3$ by doing a similar proof. However, this is not the case for $L = 5$. Consider, indeed, the graph shown in Figure 1. Unfortunately, its incidence vector satisfies inequalities (1)–(6) when $L = 5$. However, this solution is clearly infeasible for the node THPP with $L = 5$.

3. FORMULATION FOR THE EDGE THPP WHEN $L = 4$

In [20], Huygens et al. have shown that the linear system of inequalities (1), (3), (5), and (6), along with the integrality constraints, is sufficient to formulate the edge THPP for $L = 2, 3$. However, this is not the case when $L = 4$, as illustrated by Figure 2. One can verify that all those inequalities are satisfied, while the solution is not feasible for the edge THPP with $L = 4$.

In [6], Dahl and Gouveia describe the following class of valid inequalities for the L-hop-constrained path problem. Let $V_0, \ldots, V_{L+r}$ be a partition of N such that $r \geq 1$, $s \in V_0$

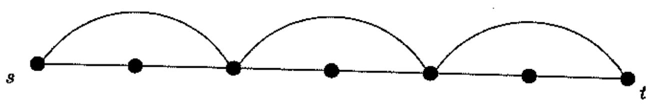

FIG. 2. Infeasible solution for the edge THPP with $L = 4$.

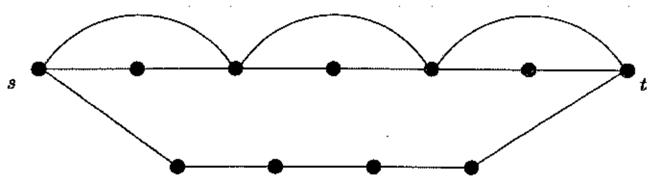

FIG. 3. A two-layered infeasible solution for the edge THPP with $L = 4$.

and $t \in V_{L+r}$. Then the *generalized jump inequality* is

$$\sum_{e \in [V_i, V_j], i \neq j} \min(|i-j| - 1, r)\, x(e) \geq r.$$

These inequalities can be easily extended to the edge THPP as follows:

$$\sum_{e \in [V_i, V_j], i \neq j} \min(|i-j| - 1, r)\, x(e) \geq 2r. \qquad (7)$$

Note that these inequalities generalize the L-path-cut inequalities (by setting $r = 1$). Moreover, it is easy to see that the incidence vector of the graph of Figure 2 does not satisfy (7) when $r = 2$ and each V_i is restricted to a single node. However, adding this class is still not sufficient to formulate the edge THPP for $L = 4$. Consider, for example, the graph of Figure 3. Clearly, this solution is not feasible for the edge THPP with $L = 4$. However, it is not difficult to check that, besides all st-cut and trivial constraints, its incidence vector also satisfies all generalized jump inequalities (7).

So, to formulate the edge THPP for $L = 4$, further inequalities are clearly needed. In what follows, we introduce a more general class of valid inequalities and show that these inequalities together with (1), (3), (5), and (6) suffice to formulate the edge THPP for $L = 4$.

Let $V_0, V_1, \ldots, V_6, W_1, \ldots, W_4$ be a two-layered partition of N such that $s \in V_0$ and $t \in V_6$, and $V_i \neq \emptyset$ for $i = 1, \ldots, 5$. See Figure 4 for an illustration. We note that the W_i's may be empty. Consider the inequality

$$ax \geq 4, \qquad (8)$$

with

$a(e) = \min(|i-j| - 1, 2)$, for all $e \in [V_i, V_j], i \neq j$,

$a(e) = 2$, for all $e \in [W_i, W_j], |i-j| \geq 2$,

$a(e) = 2$, for all $e \in [V_i, W_j], j - i \geq 2$ or $i - j \geq 3$,

$a(e) = 1$, for all $e \in [V_i, W_j], (i,j) = (2,3), (3,1), (3,4), (4,2)$,

$a(e) = 0$, otherwise.

Inequalities of type (8) will be called *two-layered 4-path-cut inequalities*. In Figure 4, the edges with coefficient 1 are in solid lines, while those with coefficient 2 are in bold. The edges with zero coefficient do not appear in the figure.

Theorem 2. *Inequalities* (8) *are valid for the edge THPP polytope when* $L = 4$.

Proof. Consider a two-layered 4-path-cut inequality $ax \geq 4$, whose coefficients can be seen as weights on the edges. It is easily seen that the total weight of any 4-st-path is at least 2. Because any feasible solution to the edge THPP with $L = 4$ must contain at least two such edge-disjoint paths, its incidence vector satisfies $ax \geq 4$. ∎

Observe that, when the W_i's are all empty, these inequalities correspond to the generalized jump inequalities with $r = 2$. Also note that the solution induced by the graph of Figure 3 can be cut off by the two-layered 4-path-cut inequality obtained from the partition with all the sets being single nodes.

We are now going to show that the linear system consisting of inequalities (1), (3), (5), (6), and (8), along with the integrality constraints, is sufficient to formulate the edge THPP with $L = 4$.

Theorem 3. *The edge THPP for* $L = 4$ *is equivalent to the integer program*

$$\min \{cx;\ \text{subject to } (1), (3), (5), (6), (8),\ x \in \mathbb{Z}^{|E|}\}.$$

Proof. By Theorem 2, together with the fact that the st-cut, L-path-cut, and trivial constraints are valid for the edge THPP polytope, the incidence vector of any solution to the edge THPP for $L = 4$ satisfies inequalities (1), (3), (5), (6), and (8).

Now suppose there exists a solution F whose incidence vector x^F satisfies the st-cut, 4-path-cut and trivial inequalities, but that is not feasible for the edge THPP with $L = 4$. We will show that there exists a two-layered 4-path-cut inequality $ax \geq 4$ violated by x^F. Let G_F be the graph induced by F and P_0 a shortest st-path in G_F. We have the following claims.

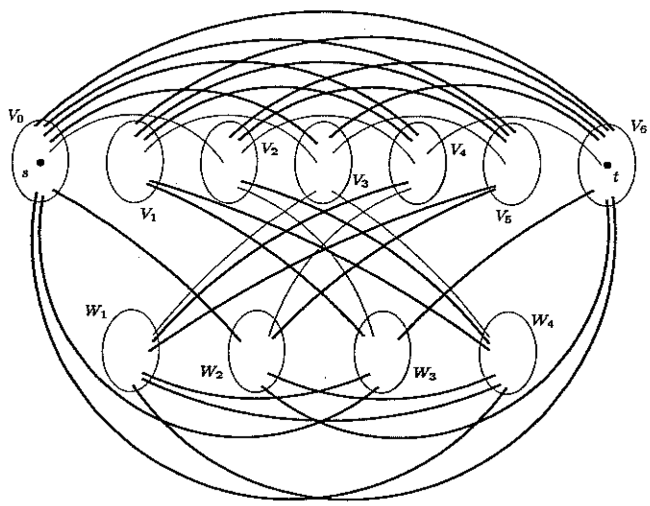

FIG. 4. Support graph of a two-layered 4-path-cut inequality.

Claim 1. For every edge e, there is at least one 4-st-path in $G_F - e$.

Proof. Suppose this is not the case. Let e_0 be an edge such that $G_F - e_0$ does not contain any 4-st-path. Because x^F satisfies all the st-cut inequalities, there must still exist an st-path in $G_F - e_0$, of length at least 5. Let us consider a partition $V_0, \ldots, V_5$ of N such that V_i, $i = 0, \ldots, 4$, contains all the nodes at distance i from s in $G_F - e_0$, and V_5 contains all the other nodes. By the previous remark, each V_i is nonempty, $s \in V_0$, and $t \in V_5$. Let T_{G-e_0} be the associated 4-path-cut. By construction, $x(T_{G-e_0}) = 0$, and hence, if T is the corresponding 4-path-cut in G, $x^F(T) \leq 1$, a contradiction. □

Claim 2. The path P_0 is of length at least 3.

Proof. Suppose this is not the case, that is, either $|P_0| = 1$, or $|P_0| = 2$. If $|P_0| = 1$, then $P_0 = (st)$. By Claim 1, there is a 4-st-path that does not contain st. But this implies that G_F contains two edge-disjoint 4-st paths, a contradiction. Thus $|P_0| = 2$. Let $P_0 = (e_1, e_2)$. By Claim 1, there is a 4-st-path P_1 that does not contain e_1. Then $e_2 \in P_1$. Let P_1^s be the subpath of P_1 between s and v, where v is the endnode of e_1 different from s. Hence, $|P_1^s| \leq 3$. Similarly, there is a 4-st-path P_2 such that $e_1 \in P_2$ and $e_2 \notin P_2$. Let P_2^t be the subpath between v and t. We also have $|P_2^t| \leq 3$.

If P_1^s and P_2^t do not intersect in some edge, then the paths e_1, P_2^t and P_1^s, e_2 are of length at most 4 and are edge-disjoint, a contradiction. Let w be the first node of P_1^s that belongs to P_2^t. Note that $w \neq v$. Let $\tilde{P}_2^t$ be the subpath of P_2^t between w and t. Clearly, $|\tilde{P}_2^t| \leq 2$. Similarly, let $\tilde{P}_1^s$ be the subpath of P_1^s between s and w. We also have $|\tilde{P}_1^s| \leq 2$. Now let P_0' be the path $\tilde{P}_1^s \cup \tilde{P}_2^t$. We have $|P_0'| \leq 4$ and $P_0 \cap P_0' = \emptyset$, a contradiction. □

Claim 3. The path P_0 is of length exactly 3.

Proof. By Claim 2, we already have $|P_0| \geq 3$. Also, it is clear that $|P_0| \leq 4$. If not, this would contradict Claim 1 for any edge e. Suppose now, by contradiction, that $|P_0| = 4$.

Let $P_0 = (sv_1, v_1v_2, v_2v_3, v_3t)$ with $v_1, v_2, v_3 \in N\backslash\{s,t\}$, v_1, v_2, v_3 different from each other. By Claim 1, there must exist a 4-st-path P_1 not containing sv_1, and another one P_2 not containing v_3t. Because P_0 is a shortest path, P_1 and P_2 are both of length exactly 4. Let P_1^s (resp., P_2^t) be the subpath of P_1 (resp., P_2) from s (resp., t) to the first node in common with P_0. Note that if this node is v_i, then $|P_1^s| = i$ (resp., $|P_2^t| = 4 - i$). Moreover, note that P_1^s and P_2^t do not intersect P_0 in any edge.

Suppose first that P_1^s and P_2^t intersect in some edge. Thus, there is a node z common to P_1^s and P_2^t, different from v_1, v_2, v_3. But then the subpaths of P_1^s, P_2^t between z and s, t, respectively, form a 4-st-path disjoint from P_0, a contradiction. Therefore, P_1^s and P_2^t are edge-disjoint.

Notice that P_1^s cannot go from s to v_3. If this was the case, we would indeed have $|P_1^s| = 3$, and hence, P_1 would be of the form $sv_1', v_1'v_2', v_2'v_3, v_3t$. But then, as $v_3t \notin P_2$, and P_1^s and P_2^t are edge-disjoint, $P_2 \backslash P_2^t$ must have an edge in common with P_1^s (and also with P_0). Clearly, the only possibility is that P_2 is of the form $(sv_1', v_1'v_2, v_2v_3, h)$ or $(sv_1, v_1v_2', v_2'v_3, h)$, with $h \in [v_3, t]\backslash\{v_3t\}$. But this creates two edge-disjoint 4-st paths, a contradiction. Similarly, P_2^t cannot go from t to v_1.

Suppose now that P_1^s goes from s to v_2. Therefore, to not create two edge-disjoint 4-st-paths, P_2^t must go from t to v_3. Recall that in this case, $P_1^s = (sv_1', v_1'v_2)$ and $P_2^t = (h)$ with $h \in [v_3, t]\backslash\{v_3t\}$. Consequently, it is clear that v_2v_3 must be common to P_0, P_1, P_2 (if not, we would immediately create a path disjoint from P_0). On the other hand, by Claim 1, there must exist a further 4-st-path not containing v_2v_3. However, because this one cannot intersect all the previous paths in at least one edge, we obtain a contradiction.

Thus, P_1^s must go from s to v_1 and, by symmetry, P_2^t must go from t to v_3. Hence, $P_1^s = (g)$ with $g \in [s, v_1]\backslash\{sv_1\}$, and $P_2^t = (h)$ with $h \in [v_3, t]\backslash\{v_3t\}$. Note that a 4-$st$-path cannot intersect at the same time sv_1 and g, or v_3t and h. Therefore, if we consider a 4-st-path not containing v_1v_2, it must use v_2v_3. In the same way, a 4-st-path not containing v_2v_3 must use v_1v_2. Once again, we obtain two edge-disjoint 4-st-paths. □

In the rest of the proof, we let $P_0 = (e_1, e_2, e_3) = (s, v_1, v_2, t)$.

Claim 4. Every 4-st-path of G_F contains at least two edges among e_1, e_2, e_3.

Proof. Let P_1 be a 4-st-path different from P_0. Let us suppose that P_1 does not intersect $\{e_1, e_2\}$. Then, $e_3 \in P_1$. Let P_1^s be the subpath of P_1 between s and the first node in common with P_0. Suppose first that this node is v_2. Because P_0 is a shortest st-path, it follows that $2 \leq |P_1^s| \leq 3$. By Claim 1, there is a 4-st-path P_2 that does not contain e_3. Thus, P_2 intersects $\{e_1, e_2\}$. Let P_2^t be the subpath of P_2 between t and the first node in common with P_0. Note that $|P_2^t| \leq 3$. If P_2^t contains another node of P_1^s than v_2, say z, then the subpaths of P_1^s between s and z, and of P_2^t between z and t form a 4-st-path edge-disjoint from P_0, a contradiction. So, P_1^s and P_2^t may only intersect in v_2. Moreover, if this is the case, because P_2 must contain e_1 or e_2, we then have $1 \leq |P_2^t| \leq 2$. But this creates two edge-disjoint 4-st-paths, namely $P_2^t \cup \{e_1, e_2\}$ and P_1, which is impossible. If P_1^s and P_2^t do not intersect in a node, then P_2^t goes from t to v_1. But, therefore, the paths $P_2^t \cup \{e_1\}$ and P_1 are of length at most 4 and edge-disjoint, a contradiction. Suppose now that the first node common to P_1^s and P_0 is v_1. Then P_1 is either of the form (P_1^s, v_1v_2, e_3), with v_1v_2 parallel to e_2 and $1 \leq |P_1^s| \leq 2$, or of the form (sv_1, v_1u, uv_2, e_3) with sv_1 parallel to e_1 and $u \in N\backslash\{s, t, v_1, v_2\}$.

CASE 1. $P_1 = P_1^s \cup \{v_1v_2, e_3\}$ where v_1v_2 is an edge of $[v_1, v_2]\backslash\{e_2\}$. By Claim 1, there is a 4-st-path not containing e_3. But it is not hard to see here that there are two edge-disjoint 4-st-paths, a contradiction.

CASE 2. $P_1 = (sv_1, v_1u, uv_2, e_3)$ with $sv_1 \in [s, v_1]\backslash\{e_1\}$ and $u \in N\backslash\{s, t, v_1, v_2\}$. Again, by Claim 1, there is a 4-st-path, say P_3, not containing e_3. Because F is not a solution to the problem, P_3 must intersect all the 4-st-paths obtained from $P_0 \cup \{sv_1, v_1u, uv_2\}$. However, this is impossible without creating a 4-st-path disjoint from P_0.

Therefore, P_1 must intersect the set $\{e_1, e_2\}$. By symmetry, P_1 must also intersect $\{e_2, e_3\}$. Now, to complete the proof of the claim, it suffices to show that P_1 also intersects $\{e_1, e_3\}$. Suppose the contrary. Then P_1 contains e_2, and hence, $P_1 = P_1^s \cup \{e_2\} \cup P_1'$, where P_1' is a path going from v_2 to t. Note that P_1^s is a path going from s to v_1. (The other case would immediately create two edge-disjoint 4-st-paths, namely, $P_1^s \cup \{e_3\}$ and $\{e_1\} \cup P_1'$.) Note that P_1^s and P_1' must be either both of length 1, or one of length 1 and the other of length 2. In both cases, by considering a 4-st-path not containing e_2, but intersecting all the previous paths, one would contradict the infeasibility of F. □

Consider now the subgraph G_F' of G_F obtained by deleting the three edges of P_0. Because F is not feasible, G_F' does not contain any 4-st-path. Thus, if P_0' is a shortest st-path in G_F', we have $|P_0'| \geq 5$. Suppose first that $|P_0'| \geq 6$ (note that this includes the case where P_0' does not exist). We will show that there exists an inequality (8), with all the W_i's empty, violated by x^F.

In G_F', let $\Pi = (V_0, V_1, \ldots, V_6)$ be a partition of N such that V_i, $i = 1, \ldots, 5$, contains the nodes at distance i from s, and V_6 contains all the other nodes. Clearly, $s \in V_0$, and by our current assumption, $t \in V_6$. Moreover, we claim that each other V_i is nonempty. Suppose this is not the case, that is, there is some $i \in \{1, \ldots, 5\}$ such that $V_i = \emptyset$. By definition, this means that there does not exist any node at distance i from s, and hence, at distance 5 from s. Therefore, $V_5 = \emptyset$ and the st-cut $\delta(V_6)$ is empty in G_F'. However, in G_F, we have by hypothesis $x^F(\delta(V_6)) \geq 2$. As any st-path intersects any st-cut an odd number of times, we obtain that e_1, e_2, and e_3 must all belong to $\delta(V_6)$. Therefore, $v_1 \in V_6$ and $v_2 \notin V_6$. Suppose now that v_2 belongs to $V_1 \cup V_2 \cup V_3$. Then, by construction, there exists an sv_2-path of length at most 3. This creates a 4-st-path containing only the edge e_3 of P_0, which contradicts Claim 4. Consequently, we have $v_2 \in V_4$. This also yields that all the V_i's, except V_5, are nonempty. Consider now the 4-path-cut T obtained from Π by collapsing V_5 and V_6. In G_F, it is clear that the only chord of T is e_1, and hence, $x^F(T) = 1$, which is again a contradiction.

Therefore, all the V_i's are nonempty, and Π is an admissible partition for a two-layered 4-path-cut inequality $ax \geq 4$ with the W_i's empty. Moreover, in G_F' we have $ax^F = 0$ by construction. Observe that we can suppose that V_6 only contains t. If not, we can put all the other nodes of V_6 in V_5 without creating a chord. Now consider this two-layered 4-path-cut in G_F. If $a(e_1) + a(e_2) + a(e_3) \leq 3$, we have $ax^F < 4$ in G_F and we have then found a violated inequality of type (8). Thus, $a(e_1) + a(e_2) + a(e_3) \geq 4$. Because by Claim 4 no 4-st-path containing only e_3 from P_0 can exist in G_F, we have $v_2 \in V_4 \cup V_5$. Suppose first that $v_2 \in V_4$. Thus, $a(e_3) = 1$, and hence, $a(e_1) + a(e_2) \geq 3$. The only possibility is that v_1 belongs to V_6. But this is impossible because V_6 only contains t. Suppose now that $v_2 \in V_5$. Therefore, $a(e_3) = 0$ and $a(e_1) + a(e_2) \geq 4$. Clearly, this is also impossible.

Finally, suppose that the shortest st-path in G_F' is of length exactly 5. We claim that there exists an inequality (8), with at least one W_i nonempty, violated by x^F.

By Claim 1, there exists in G_F a 4-st-path P_i not containing e_i, for each $i = 1, 2, 3$. Moreover, by Claim 4, the P_i's must contain $P_0 \backslash \{e_i\}$. Besides these two edges from P_0, it is clear that each P_i must contain two more edges. Indeed, if one P_i was of length 3, this would create two edge-disjoint 4-st-paths in F, a contradiction. For the same reason, the P_i's cannot have a node in common besides s, v_1, v_2, t. As a consequence, we have the graph of Figure 2 as a subgraph of G_F. Note that the st-path of this subgraph not intersecting P_0 is of length 6. Let us denote its nodes by $\{s, u_1, v_1, u_2, v_2, u_3, t\}$. Notice that, as the shortest st-path of G_F' is of length exactly 5, there must be in G_F additional edges (and nodes) forming, eventually with edges of that path, an st-path of length 5.

Consider the partition $\Pi = (V_0, \ldots, V_6, W_1, \ldots, W_4)$ in G_F defined as follows. We set $V_0 = \{s\}$, $V_1 = \{u_1\}$, $V_2 = \{v_1\}$, $V_3 = \{u_2\}$, $V_4 = \{v_2\}$, $V_5 = \{u_3\}$, and $V_6 = \{t\}$. All the other nodes are distributed to the W_i's through a breadth first search from s in G_F. See Figure 5 for an illustration. Note that some W_i's may be empty, but not all of them

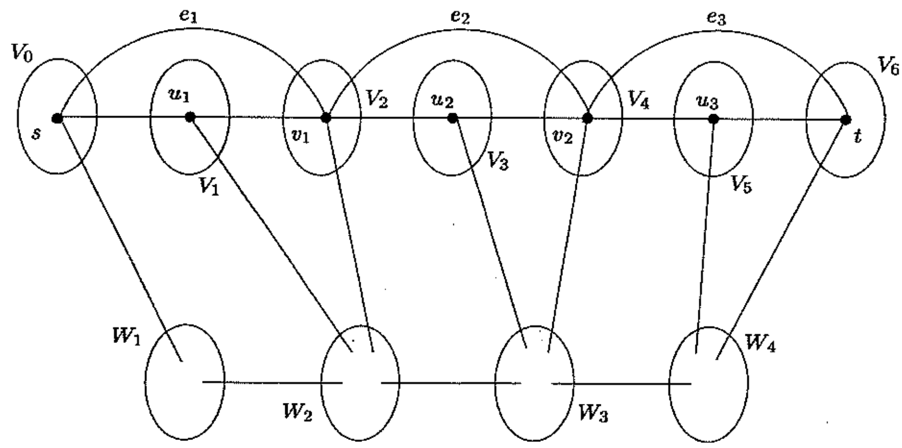

FIG. 5. Partition Π of G_F.

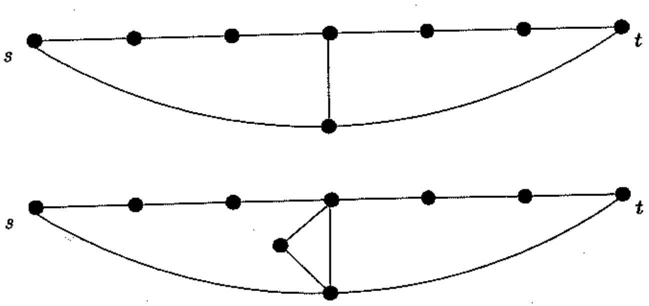

FIG. 6. Infeasible solutions for the edge THPP with $L = 5$.

by our previous remark. Let $\overline{E} = (\bigcup_{j-i\geq 2 \text{ or } i-j\geq 3}[V_i, W_j]) \cup (\bigcup_{(i,j)=(2,3),(3,1),(3,4),(4,2)}[V_i, W_j])$, that is, the set of edges between the two layers of the partition Π that have a positive coefficient in the corresponding inequality (8). We claim that G_F does not contain any edge from $\overline{E}$. Suppose this is not the case. If there was an edge of G_F in $[V_i, W_j]$ with $j - i \geq 2$, then its endnode in W_j would be at distance $i + 1$ from s, and hence, it should have been put in W_{i+1} by construction. The same contradiction holds for edges of G_F in $[V_2, W_3] \cup [V_3, W_4]$. Now, assume there is an edge e of G_F in $[V_i, W_j]$ with $i - j \geq 3$. Thus, by construction, there exists in G_F a subpath P, from s to the endnode of e in W_j, of length exactly j. But then G_F contains the graph of Figure 2, P and e, and hence, two edge-disjoint 4-st-paths, a contradiction. Finally, if G_F uses some edge from $[V_3, W_1] \cup [V_4, W_2]$, we get a similar contradiction.

Consequently, the only edges of G_F between the two layers of partition Π have coefficient zero in the corresponding inequality (8). Moreover, by construction, the lower partition cannot contain chords, and the upper one has e_1, e_2, e_3 for its only chords. Because these three edges have a coefficient 1, we obtain $ax^F = 3 < 4$, and the proof is complete. ■

Unfortunately, as pointed out in Section 1, the formulation given above for the edge THPP when $L = 4$ cannot be directly extended to an arbitrary L. In what follows we show that further inequalities are needed to formulate the edge THPP even for $L = 5$.

4. EXTENSIONS

We first describe a larger class of valid inequalities that generalizes inequalities (8) for every L.

Consider a complete graph $G = (N, E)$ and let L be a fixed positive integer. Let $V_0, V_1, \ldots, V_{2L-2}, W_1, \ldots, W_L$ be a two-layered partition of N such that $s \in V_0$ and $t \in V_{2L-2}$, and $V_i \neq \emptyset$ for $i = 1, \ldots, 2L - 1$. Note that the W_i's may be empty. Let G' be the subgraph induced by the edges that are in the same layer. Consider the inequality

$$ax \geq 2(L-2), \tag{9}$$

with

$$a(e) = \min(|i-j| - 1, L-2), \quad \text{for all } e \in [V_i, V_j], i \neq j,$$

$$a(e) = L - 2, \quad \text{for all } e \in [W_i, W_j], |i-j| \geq 2,$$

$$a(e) = \max\{L - 2 - a(P \setminus \{e\}); P \text{ is an } L\text{-st-path of } G' + e\}, \quad \text{for all } e \in [V_i, W_j].$$

Note that the coefficients of the edges between the two layers are nothing but the lifting coefficients of (9) with respect to G'. By considering $a(e)$ as a weight on each edge, it is easy to see that any L-st-path in G has a weight at least $L - 2$, implying that (9) is valid.

Inequalities (9) will be called *two-layered L-path-cut inequalities*.

Now consider the case $L = 5$ and the graphs of Figure 6.

First note that, because of the restricted number of nodes, no two-layered inequalities of type (9) arise in the first graph. There are, however, two-layered inequalities induced by the second graph, but they are not violated. Moreover, one can easily verify that the st-cut and L-path-cut inequalities are satisfied, while none of the graphs contain two edge-disjoint 5-st-paths. This implies that further inequalities, in addition to the st-cut, L-path-cut and two-layered inequalities, are needed for the formulation of the edge THPP even for $L = 5$.

In fact, the proof of Theorem 3 is based on the shortest st-path P_0 of a solution F that satisfies the st-cut and L-path-cut inequalities, but that is not feasible for the edge THPP with $L = 4$. As shown in the proof (Claim 3 of Theorem 3), P_0 has a unique length value, that is, $|P_0| = 3$. The rest of the proof consists then of showing that one of the two-layered inequalities is violated in that case. This unique length possibility for P_0 is unfortunately no longer true when $L \geq 5$. Indeed, for $L = 5$, as can be seen from the graphs of Figure 6, such a solution may have a shortest path P_0 of length 2. (Similar solutions exist for $L = 5$ where $|P_0| = 4$.) Figure 7 shows a similar solution for $L = 6$ where $|P_0| = 3$.

In general, P_0 may have different values (we conjecture that there are between 2 and $L - 1$), which makes the proof for $L \geq 5$ much harder. In fact, each possible value of $|P_0|$ should give rise to a violated inequality. We believe that there should

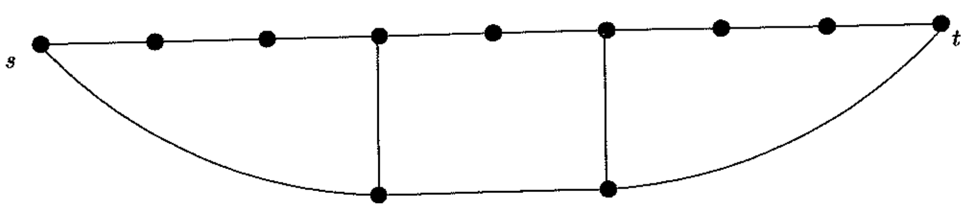

FIG. 7. Infeasible solution for the edge THPP with $L = 6$.

exist a unified framework for these inequalities, a direction that we are now exploring.

5. CONCLUDING REMARKS

In this article we have given an integer programming formulation, in the space of the design variables, for the two 4-hop-constrained paths problem in both the edge and node cases. For the edge version of the problem, we have introduced a new class of valid inequalities that we called two-layered 4-path-cut inequalities and shown that these inequalities together with the so-called st-cut and 4-path-cut inequalities yield an integer programming formulation of the problem.

A natural question that may be posed is whether or not these formulations are complete, that is whether or not their linear relaxation is integral. Unfortunately, for the node version, this is not the case as shown by the following example.

Consider the graph $G = K_6$ of Figure 8, where the edges in solid lines have value $1/2$, the ones in bold have value 1, and the remaining edges have value zero. It is easy to verify that this solution is a fractional extreme point of the polyhedron given by the linear relaxation of the node THPP with $L = 4$ and $\{s, t\} = \{1, 6\}$. This point can be cut off by the following valid inequality

$$2x(e_2) + x(e_3) + x(e_4) + 2x(e_5) + x(e_6) + x(e_7) + 2x(e_9) + x(e_{11}) + x(e_{14}) + 2x(e_{15}) \geq 3.$$

Moreover, this inequality is facet defining for the polytope on this graph.

An interesting question would be to see whether the linear relaxation of the edge version is integral.

The separation problem for a system of inequalities consists of verifying whether a given solution $x^* \in \mathbb{R}^{|E|}$ satisfies

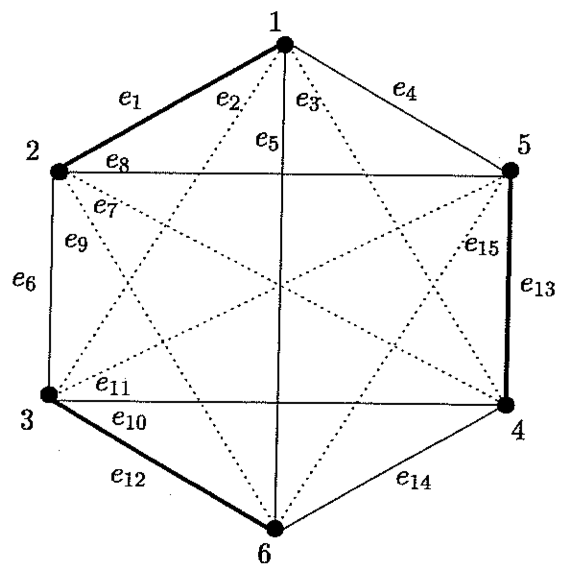

FIG. 8. A fractional extreme point of the linear relaxation of the node THPP with $L = 4$ and $(s, t) = \{1, 6\}$.

the system and, if not, of finding an inequality of the system that is violated by x^*. Grötschel et al. [17] showed that optimization over a system of inequalities can be solved in polynomial time if and only if the separation problem for that system can be solved in polynomial time. The separation problem for the st-cut and st-node-cut inequalities can be solved in polynomial time using any polynomial time max-flow algorithm [18]. The separation problem for the L-path-cut inequalities (and the L-path-node-cut inequalities) reduces to finding a minimum weight edge subset intersecting all the L-st-paths. In [9], Fortz et al. showed that, when $L \leq 3$, this problem can be reduced to a max-flow problem and can then be solved in polynomial time. The case $L = 4$ is, unfortunately, still an open question (see [22]).

Because the THPP for $L = 4$ can be solved in polynomial time (by enumeration), from the equivalence between optimization and separation, it follows that inequalities (3) and (4), as well as inequalities (8), can be separated in polynomial time among a system of inequalities describing the THPP polytope in this case. For integer solutions, inequalities (3) (and (4)) can be separated in polynomial time for all L using a breadth first search [19]. Also, from the proofs of Theorems 1 and 3, inequalities (8) can be separated in polynomial time for integer solutions. Hence, one can state the following.

Theorem 4. *Given a $0 - 1$ solution x^*, the problem of finding whether or not x^* satisfies system* (1)–(6) [*resp.*, (1), (3), (5), (6), *and* (8)] *can be solved in polynomial time.*

Theorem 4 is very important from a practical point of view. Indeed, in many approaches such as cutting planes approaches, one has to solve the feasibility problem for a given (integer) solution. If the solution is feasible for the underlying problem, then it is optimal. By Theorem 4, this problem can be solved in polynomial time for both the node and edge THPP. Also note that, by the remarks above, the classes of inequalities can be separated independently of each other.

Finally, let us mention that the formulation given in this article can be easily extended to the case where more than one pair of terminals are considered. Here, for each variant of the THPP, the formulation is given by the inequality system for each pair of terminals together with the integrality constraints. Hence, an efficient separation algorithm for inequalities (3), (4), and (8) would be of great interest for solving the multiple demands case by cutting planes.

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REFERENCES

[1] J.A. Bondy and U.S.R. Murty, Graph theory with applications, University Press, Belfast, 1976.

[2] C.R. Coullard, A.B. Gamble, and J. Liu, "The k-walk polyhedron," Advances in optimization and approximation, D.-Z. Du and J. Sun (Editors), Kluwer Academic Publishers, 1994, pp. 9–29.

[3] G. Dahl, The 2-hop spanning tree problem, Oper Res Lett 23 (1998), 21–26.

[4] G. Dahl, Notes on polyhedra associated with hop-constrained paths, Oper Res Lett 25 (1999), 97–100.

[5] G. Dahl, N. Foldnes, and L. Gouveia, A note on hop-constrained walk polytopes, Oper Res Lett 32 (2004), 345–349.

[6] G. Dahl and L. Gouveia, On the directed hop-constrained shortest path problem, Oper Res Lett 32 (2004), 15–22.

[7] G. Dahl, D. Huygens, A.R. Mahjoub, and P. Pesneau, On the k-edge disjoint 2-hop-constrained paths polytope, Oper Res Lett 34 (2006), 577–582.

[8] G. Dahl and B. Johannessen, The 2-path network problem, Networks 43 (2004), 190–199.

[9] B. Fortz, A.R. Mahjoub, S.T. McCormick, and P. Pesneau, Two-edge connected subgraphs with bounded rings: Polyhedral results and branch-and-cut, Math Prog 105 (2006), 85–111.

[10] L. Gouveia, Multicommodity flow models for spanning trees with hop constraints, Eur J Oper Res 95 (1996), 178–190.

[11] L. Gouveia, Using variable redefinition for computing lower bounds for minimum spanning and Steiner trees with hop constraints, INFORMS J Comput 10 (1998), 180–188.

[12] L. Gouveia and E. Janssen, Designing reliable tree networks with two cable technologies, Eur J Oper Res 105 (1998), 552–568.

[13] L. Gouveia and T. Magnanti, Network flow models for designing diameter-constrained minimum-spanning and Steiner trees, Networks 41 (2003), 159–173.

[14] L. Gouveia, T. Magnanti, and C. Requejo, A 2-path approach for odd-diameter-constrained minimum spanning and Steiner trees, Networks 44 (2004), 254–265.

[15] L. Gouveia, P. Patrício, and A. de Sousa, "Compact models for hop-constrained node survivable network design, an application to MPLS," Telecommunications planning: Innovations in pricing, network design and management, G. Anandalingam and S. Raghavan (Editors), Vol. 33, Springer, Berlin, 2005, pp. 167–180.

[16] L. Gouveia and C. Requejo, A new Lagrangean relaxation approach for the hop-constrained minimum spanning tree problem, Eur J Oper Res 132 (2001), 539–552.

[17] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 70–89.

[18] J. Hao and J.B. Orlin, A faster algorithm for finding the minimum cut in a directed graph, J Algor 17 (1994), 424–446.

[19] D. Huygens, M. Labbé, A. Mahjoub, and P. Pesneau, The 2-edge connected hop-constrained network design problem: Valid inequalities and branch-and-cut, Tech. Rep. RR-06:02, LIMOS, Université Blaise Pascal, Clermont-Ferrand, France, to appear in Networks.

[20] D. Huygens, A.R. Mahjoub, and P. Pesneau, Two edge-disjoint hop-constrained paths and polyhedra, SIAM J Disc Math 18 (2004), 287–312.

[21] H. Kerivin and A.R. Mahjoub, Design of survivable networks: A survey, Networks 46 (2005), 1–21.

[22] A.R. Mahjoub and S.T. McCormick, The complexity of max flow and min cut with bounded-length paths, Preprint 2004.

[23] V.H. Nguyen, A complete linear description for the k-path polyhedron, Preprint 2003.

[24] W.R. Pulleyblank, "Polyhedral combinatorics," Optimization, Handbooks Oper Research and Management Science 1, G.L. Nemhauser, A.H.G. Rinnooy Kan, and M.J. Todd (Editors), North-Holland, Amsterdam, 1989, pp. 371–446.