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On survivable network polyhedra[☆]

Hervé Kerivin[a,1], Ali Ridha Mahjoub[b,*]

[a] *Institute for Mathematics and its Applications, University of Minnesota, 357 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0436, USA*

[b] *Laboratoire LIMOS, CNRS UMR 6158, Université de Clermont-Ferrand II, Complexe Scientifique des Cézeaux, 63177 Aubière, France*

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Abstract

Given an undirected network $G = (V, E)$, a vector of nonnegative integers $r = (r(v) : v \in V)$ associated with the nodes of G and weights on the edges of G, the survivable network design problem is to determine a minimum-weight subnetwork of G such that between every two nodes u, v of V, there are at least $\min\{r(u), r(v)\}$ edge-disjoint paths. In this paper we study the polytope associated with the solutions to that problem. We show that when the underlying network is series–parallel and $r(v)$ is even for all $v \in V$, the polytope is completely described by the trivial constraints and the so-called cut constraints. As a consequence, we obtain a polynomial time algorithm for the survivable network design problem in that class of networks. This generalizes and unifies known results in the literature. We also obtain a linear description of the polyhedron associated with the problem in the same class of networks when the use of more than one copy of an edge is allowed.

Keywords: Survivable network; Polyhedron; Cut; Series–parallel graph; Polynomial algorithm

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* Corresponding author.
E-mail addresses: herve.kerivin@math.univ-bpclermont.fr (H. Kerivin), ridha.mahjoub@math.univ-bpcler mont.fr, mahjoub@ucfma.univ-bpclermont.fr (A.R. Mahjoub).

1 Present address: Laboratoire LIMOS, CNRS UMR 6158, Université de Clermont-Ferrand II, Complexe Scientifique des Cézeaux, 63177 Aubière, France.

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1. Introduction

Satisfying a suitable degree of survivability has become a major objective in the design of telecommunication networks. Survivable networks must fulfill some connectivity requirements that ensure connections between parts of the network, that is, networks that are still functional after the failure of certain links. This can be, for instance, realized by considering a sufficient number of links between every pair of nodes of the network. However, with the use of fiber-optic technology, this would be costly, which yields the need to design minimum cost networks which are survivable.

As fiber-optic cables provide a high transmission capacity and can thus carry substantially more traffic than traditional copper cables, telecommunication networks tend to be sparse. In this case, the failure of a single (or more) link might be of heavy consequences if the network does not provide alternative paths for routing. This leads to the problem of designing minimum-cost telecommunication networks with high reliability level, namely with sufficient routing paths between each pair of nodes.

More precisely, let $G = (V, E)$ be an undirected network. If we associate with each node i of G a *connectivity type* $r(i) \in \mathbb{Z}_+$ representing the importance of communication from and to node i, then G is said to be *survivable* (with respect to the connectivity types $(r(i) : i \in V)$) if it has at least $r(i, j) = \min\{r(i), r(j)\}$ edge-disjoint paths between every pair of nodes i and j.

Given a network $G=(V, E)$ with weights $(w(e) : e \in E)$ on its edges, and a connectivity type vector $(r(i) : i \in V)$, the *survivable network design problem* (SNDP) is to determine a survivable subnetwork of G (with respect to r) whose total weight is minimum.

In this paper we study the SNDP from a polyhedral point of view. We give a complete description of the polytope associated with the solutions to that problem when the underlying network is series–parallel and the connectivity types are all even. As a consequence, we obtain a polynomial time (cutting plane) algorithm for the SNDP in that class of graphs. To the best of our knowledge, this is the first polynomial time algorithm for the SNDP in that class of graphs. We also obtain a linear description of the polyhedron associated with the SNDP, in the same class of graphs, when multiple copies of an edge may be used.

1.1. *Complexity and heuristics*

The SNDP is NP-hard in general. It includes as special cases a number of well-known NP-hard combinatorial optimization problems such as the Steiner tree problem ($r(i) \in \{0, 1\}$, for all $i \in V$) and the k-edge connected network problem ($r(i) = k$, for all $i \in V$), where k is a fixed positive integer.

The SNDP was shown to be polynomially solvable in some particular cases. If $r(i) = 1$ for all $i \in V$, the SNDP is nothing but the minimum spanning tree problem which is well known to be polynomially solvable. And if the weights are restricted to be 1, Chou and Frank [10] gave a polynomial algorithm to solve the problem when G may contain parallel edges, and $r(i) \geqslant 2$ for all $i \in V$. Chou and Frank also studied a similar problem [11] when no parallel edges but additional nodes are allowed. Winter gave linear time algorithms for the SNDP with $r(i) \in \{0, 2\}$ for all $i \in V$, in series–parallel graphs [43] and Halin graphs [42]. (A Halin graph is a graph that is planar and can be drawn in the plane as a tree without nodes of degree 2 plus one cycle connecting all leaves of the tree.)

As the SNDP is NP-hard, a considerable amount of research has been conducted into the design of heuristic algorithms [34,37,39]. Steiglitz et al. [39] have proposed a heuristic based on local search for the general model. Further heuristics were given by Ko and Monma [34] for the k-edge connected subgraph problem and by Monma and Shallcross [37] for the SNDP where $r(i) \in \{1, 2\}$ for all $i \in V$.

1.2. Approximation algorithms

In the design of approximation algorithms for the SNDP, one often specifies the connectivity requirements by giving the minimum number $f(S)$ of edges crossing each cut $\delta(S)$ with $S \subseteq V$. For these very general versions of the SNDP, sometimes also called the *generalized Steiner network problem*, two variants may be considered: one in which the use of multiple copies of an edge is allowed, and one in which this is forbidden.

For the latter case, Williamson et al. [41] (see also [22]) gave a polynomial time $2f_{\max}$-approximation algorithm when the function f is proper and $f_{\max}=\max\{f(S) : S \subseteq V\}$ is the maximum requirement. (A function f is proper if $f(V)=0$, $f(S)=f(V\backslash S)$ for each $S \subseteq V$ (symmetry), and $f(A \cup B) \leqslant \max\{f(A), f(B)\}$ whenever A and B are disjoint (maximality).) In [25], Goemans et al. improved this by presenting an approximation algorithm with a performance guarantee of $2\mathscr{H}(f_{\max})$ where $\mathscr{H}(f_{\max}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{f_{\max}}$ is the harmonic function. And when the function f is weakly supermodular, Jain [32] proposed a factor 2 approximation algorithm. (A function f is weakly supermodular if $f(V) = 0$ and for every $A, B \subseteq V$ at least one of the following holds: $f(A) + f(B) \leqslant f(A\backslash B) + f(B\backslash A)$ or $f(A) + f(B) \leqslant f(A \cap B) + f(B \cup A)$.)

For the problem in which a solution may include multiple copies of an edge, Goemans and Bertsimas [24] gave a $\min\{2\mathscr{H}(r_{\max}), 2q\}$-approximation algorithm for the SNDP where q denotes the number of distinct connectivity requirement values. Using a primal-dual approach, Agrawal et al. [2] obtained a $2 \log_2 r_{\max}$-approximation algorithm for the SNDP, and Goemans and Williamson [26] devised a $2\mathscr{H}(f_{\max})$-approximation algorithm for the multiple-copy generalized Steiner network problem with arbitrary proper function. Recently, Aggarwal and Garg [1] improved this result by giving a $2 \log_2 d$-approximation algorithm for the SNDP where d is the number of nodes $v \in V$ with $r(v) > 0$.

Many variants of the SNDP have been given particular attention and have been extensively investigated. For a complete survey of the SNDP, see [31] (see also [40]).

1.3. Formulation

Let $G = (V, E)$ be a graph and $r \in \mathbb{Z}_+^V$ be a connectivity type vector associated with the nodes of V. For $W \subseteq V$, let $r(W) = \max\{r(i) : i \in W\}$ and $\mathrm{con}(W) = \min\{r(W), r(V\backslash W)\}$. $r(W)$ will be called the *connectivity type* of W. We notice that r is a nondecreasing function, that is r satisfies $r(X) \leqslant r(Y)$ for all $X \subseteq Y \subseteq V$.

If $W \subseteq V$, the set of edges having exactly one node in W is called a *cut* and denoted by $\delta(W)$. If $W = \{v\}$ then we write $\delta(v)$ for $\delta(\{v\})$.

The SNDP is equivalent to the following integer linear program:

$$\text{Min} \sum_{e \in E} w(e)x(e)$$

$$x(e) \geqslant 0 \quad \text{for all } e \in E, \tag{1.1}$$

$$x(e) \leqslant 1 \quad \text{for all } e \in E, \tag{1.2}$$

$$x(\delta(W)) \geqslant \text{con}(W) \quad \text{for all } W \subseteq V, \emptyset \neq W \neq V, \tag{1.3}$$

$$x(e) \in \{0, 1\} \quad \text{for all } e \in E. \tag{1.4}$$

Inequalities (1.1) and (1.2) are called *trivial inequalities* and inequalities (1.3) are called *cut inequalities*.

Let SNDP(G,r) be the convex hull of the solutions of (1.1)–(1.4). SNDP(G,r) will be called the *survivable network polytope* of G.

The separation problem for the cut inequalities (i.e., the problem that consists in deciding whether or not a given vector $y \in \mathbb{R}^E$ satisfies (1.3) and if not in finding a violated inequality (1.3)) can be solved in polynomial time using a polynomial time max-flow algorithm [19,21]. Hence from [27], this implies that the SNDP can be solved in polynomial time in the class of networks G where SNDP(G,r) is completely described by inequalities (1.1)–(1.3). In this paper we show that series–parallel networks belong to that class of networks when the connectivity types are all even. This was an open question, posed first by Pulleyblank [38], and partially proved in some special cases [4,17,18,35].

1.4. The polytope SNDP(G,r)

The polytope SNDP(G,r) has been the subject of substantial research in the past decade. Grötschel and Monma [28] considered a more general model where node connectivity conditions are added to the problem. They gave an integer programming formulation of the model and describe basic facets. Grötschel et al. [29,30] studied further families of valid inequalities along with experiment results for both the low ($r(i) \in \{0, 1, 2\}$, for all $i \in V$) and the high connectivity cases. A complete survey of the polyhedral aspects of this model can be found in [29,40].

The polytope SNDP(G,r) has been extensively investigated when the requirements are uniform, that is $r(i) = k$ for all $i \in V$. In this case, the SNDP reduces to the problem of designing a minimum-weight k-edge connected network. Grötschel and Monma [28] (see also [30]) showed that the so-called partition inequalities together with the trivial inequalities suffice to describe SNDP(G,r) when $r(i) = 1$ for all $i \in V$. Barahona and Mahjoub [5] characterized the SNDP(G,r) when G is a Halin graph and $r(i) = 2$ for all $i \in V$. Boyd and Hao [6] studied a general class of facets for the SNDP(G, r) when $r(i) = 2$ for all $i \in V$. Mahjoub [35] showed that when G is series–parallel and $r(i) = 2$ for all $i \in V$, the SNDP(G,r) is given by the trivial and the cut inequalities. Baïou and Mahjoub [4] generalized this to the case where $r(i) \in \{0, 2\}$ for all $i \in V$. And recently Didi Biha and Mahjoub [18] extended this to the case where $r \in \{0, k\}^V$ and k is even. The purpose of this paper is to generalize these results on series–parallel graphs to the case where $r(i)$ is even for all $i \in V$. The proofs presented in those papers cannot, unfortunately, be easily extended. Many new developments have been necessary to handle this more general case.

Chopra [7] studied the SNDP when $r(i) = k$ for all $i \in V$ and multiple copies of an edge are allowed, and he characterized the associated polyhedron $P(G, r)$ for outerplanar graphs when k is odd. (A graph is outerplanar if it can be drawn in the plane as one cycle with noncrossing chords.) The polyhedron $P(G, r)$ was previously studied by Cornuéjols et al. [12]. They showed that when the graph is series–parallel and $r(i) = k$ for all $i \in V$ and even, $P(G, r)$ is completely described by the nonnegative inequalities and the cut inequalities. Baïou [3] showed that this also holds if $r(i) \in \{0, 2\}$ for all $i \in V$. Didi Biha and Mahjoub [17] gave a complete description of $P(G, r)$ when G is series–parallel and $r(i) = k$ for all $i \in V$ where k is arbitrary.

The polytope SNDP(G,r) when $r(i) \in \{0, 1\}$ for all $i \in V$ is closely related to the Steiner tree polytope, the extreme points of which are the incidence vectors of the Steiner trees of G. During the last two decades, extensive research has been done on this polytope [8,9,16,23,36]. Chopra and Rao [8,9] described several classes of facets for the dominant of the Steiner tree polytope in both the directed and undirected cases. Didi Biha et al. [16] studied further facets of this polyhedron. Margot et al. [36] gave an extended formulation for the Steiner tree problem and showed that it is a complete linear description of the associated polytope when the graph is a 2-tree (i.e., a maximal series–parallel graph). Goemans [23] discussed an extended formulation of the Steiner tree problem and characterized the associated polytope when the underlying graph is series–parallel. He also described some classes of facets of the Steiner tree polytope.

The node version of the SNDP has also been investigated. Here, the problem is to determine a minimum-weight subgraph such that between every two nodes s, t of V there are at least $\min\{r(s), r(t)\}$ node-disjoint paths. Grötschel and Monma [28] described several classes of facets of the polytope associated with that problem. For more details on that model see [29,40]. Coullard et al. [13–15] studied the Steiner 2-node connected subgraph problem, that is when $r(v) \in \{0, 2\}$ for all $v \in V$. In [13] they described the associated polytope for series–parallel graphs. In [14] they gave a linear time algorithm for that problem on Halin graphs and the graphs noncontractible to W_4 (the wheel on five nodes). In [15] they described the dominant of that polytope for the graphs noncontractible to W_4.

1.5. Contents of the paper

The paper is organized as follows. In the next section we give a complete description of the polytope SNDP(G,r) when G is series–parallel and $r(i)$ is even for all $i \in V$. This is a consequence of a series of claims whose proofs are given in Section 3. In Section 4 we characterize the polyhedron $P(G, r)$ in the same class of graphs when $r(i)$ is even for all $i \in V$ but multiple copies of edges are allowed. In Section 5 we give some concluding remarks.

1.6. Definitions and notations

In the rest of this section, we give more definitions and notations.

The graphs we consider are finite, undirected, connected and may have multiple edges. Given a node subset $W \subseteq V$, we let $G(W)$ (resp. $E(W)$) denote the subgraph of G (resp. the edge subset of G) induced by W. If W_1, W_2 are disjoint subsets of V, then $[W_1,W_2]$ denotes the set of edges having one node in W_1 and the other in W_2. If u and v are two nodes, then we write $[u, v]$ for $[\{u\}, \{v\}]$. If $G=(V, E)$ then $\overline{W}$ denotes $V \setminus W$. If $F \subseteq E$ is a subset of edges, then $V(F)$ denotes the set of nodes induced by F.

A graph G is said to be *contractible* to a graph H, if H may be obtained from G by a sequence of elementary removals and contractions of edges. A contraction consists of identifying a pair of adjacent vertices and of preserving all other vertices as well as all other adjacencies between vertices. Contracting a set of edges $F \subseteq E$ consists of contracting all the edges of F. Note that contraction preserves connectivity.

Given a constraint $ax \geqslant \alpha$, $a \in \mathbb{R}^E$, and a solution x^*, we will say that $ax \geqslant \alpha$ is *tight* for x^* if $ax^* = \alpha$.

2. The SNDP(G,r) on series–parallel graphs

A *homeomorph* of K_4 is a graph obtained from K_4 (the complete graph on four nodes) when its edges are subdivided into paths by inserting new nodes. A graph is called *series–parallel* if it contains no homeomorph of K_4 as a subgraph.

Series–parallel graphs have the following properties [20].

Lemma 1. *If $G=(V, E)$ is a connected series–parallel graph with $|V| \geqslant 3$, then G contains a node that is adjacent to at most two nodes.*

Lemma 2. *If G is a series–parallel graph contractible to a graph H, then H is series–parallel.*

Throughout we consider a graph $G=(V, E)$ and let $r \in \mathbb{Z}_+^V$ be a connectivity type vector. We shall suppose that there are at least two nodes having maximum connectivity types. Note that the SNDP can always be reduced to this case. From this assumption, it follows that

$$\operatorname{con}(v) = r(v) \quad \text{for all } v \in V. \tag{2.1}$$

Moreover we have the following properties which will be frequently used in the paper.

Lemma 3. *Let W be a node subset of V.*

(i) *If W_1, W_2 is a partition of W (that is $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 = W$) and $r(W_1) \leqslant r(W_2)$, then* $\operatorname{con}(W) \leqslant \operatorname{con}(W_2)$.

(ii) *Let $v \in W$ and $W' = W \setminus \{v\}$.*

  (1) *If $r(v) > r(W')$, then* $\operatorname{con}(W') = r(W')$.

  (2) *If $r(\overline{W}) < r(v)$, then* $\operatorname{con}(W') = r(v)$.

Proof. (i) First note that as $r(W_1) \leqslant r(W_2)$ and W_1, W_2 is a partition of W, we have $r(W) = r(W_2)$. Moreover,

$$\begin{aligned}\mathrm{con}(W) &= \min\{r(W), r(\overline{W})\}\\ &= \min\{r(W_2), r(\overline{W})\}\\ &\leqslant \min\{r(W_2), r(\overline{W}_2)\}\\ &= \mathrm{con}(W_2).\end{aligned}$$

(ii) (1) As $v \in \overline{W}'$, $r(v) \leqslant r(\overline{W}')$. Since $r(v) > r(W')$, it then follows that $r(\overline{W}') > r(W')$. Therefore $\mathrm{con}(W') = r(W')$.

(2) As $r(\overline{W}) < r(v)$, we have that $r(\overline{W}') = r(v)$. Moreover as G contains at least two nodes of maximum connectivity types, it also follows that W' contains at least one node of maximum connectivity type, and hence $r(W') \geqslant r(v)$. Thus $\mathrm{con}(\overline{W}') = r(v)$. □

If $F \subseteq E$ is an edge subset inducing a connected subgraph of G, then $G/F = (V', E')$ will denote the subgraph obtained by contracting F and r_F will denote the vector of $\mathbb{Z}_+^{V'}$ such that $r_F(w) = \mathrm{con}(V(F))$ and $r_F(i) = r(i)$ if $i \in V' \backslash \{w\}$, where w is the node that arises from the contraction of F. Let $Q(G, r)$ be the polytope given by the inequalities (1.1)–(1.3). In what follows we are going to show that if G is series–parallel and $r(i)$ is even for all $i \in V$, then SNDP(G,r) $= Q(G, r)$. To this end, we first discuss some structural properties of the polytope $Q(G, r)$.

2.1. *Structural properties of Q(G,r)*

Let x be a solution of $Q(G, r)$. We have the following lemmas.

Lemma 4. *Let $G = (V, E)$ be a graph and $r \in \mathbb{Z}_+^V$. Let $F \subseteq E$ be an edge subset of E that induces a connected subgraph of G. Let $x' \in \mathbb{R}^{E\backslash F}$ be the restriction of x on $E\backslash F$. Then x' is a solution of $Q(G/F, r_F)$.*

Proof. Easy. □

Now we introduce two properties which will be useful throughout the paper. Two subsets $X, Y \subseteq V$ are said to be *intersecting* if none of $X\backslash Y$, $Y\backslash X$ and $X \cap Y$ is empty. Moreover if X and Y are intersecting and $X \cup Y \neq V$ then they are said to be *crossing*.

Lemma 5. *Let $\delta(W_1)$ and $\delta(W_2)$ be two cuts tight for x such that W_1 and W_2 are crossing and $r(W_1 \cap W_2) \leqslant \min\{\mathrm{r}(W_1\backslash W_2),\mathrm{r}(W_2\backslash W_1)\}$. Then*

(a) $\mathrm{con}(W_1) = \mathrm{con}(W_1\backslash W_2)$,
$\mathrm{con}(W_2) = \mathrm{con}(W_2\backslash W_1)$.

(b) *$\delta(W_1\backslash W_2)$ and $\delta(W_2\backslash W_1)$ are tight for x, and*
$x[W_1 \cap W_2, \overline{W_1 \cup W_2}] = 0$.

Proof. (a) As $r(W_1 \cap W_2) \leqslant r(W_1 \backslash W_2)$, by Lemma 3 (i) it follows that $\text{con}(W_1) \leqslant \text{con}(W_1 \backslash W_2)$. Also as $r(W_1 \cap W_2) \leqslant r(W_2 \backslash W_1)$ and $\overline{W}_1 \supset W_2 \backslash W_1$, since r is increasing with respect to inclusion, we have $r(W_1 \cap W_2) \leqslant r(\overline{W}_1)$. Since $\overline{W_1 \backslash W_2} = (W_1 \cap W_2) \cup \overline{W}_1$, by Lemma 3 (i) it also follows that $\text{con}(W_1 \backslash W_2) = \text{con}(\overline{W_1 \backslash W_2}) \leqslant \text{con}(\overline{W}_1) = \text{con}(W_1)$. Therefore $\text{con}(W_1) = \text{con}(W_1 \backslash W_2)$.

Similarly we can show that $\text{con}(W_2) = \text{con}(W_2 \backslash W_1)$.

(b) From (a), we have

$$\begin{aligned}\text{con}(W_1 \backslash W_2) + \text{con}(W_2 \backslash W_1) &= \text{con}(W_1) + \text{con}(W_2)\\ &= x(\delta(W_1)) + x(\delta(W_2))\\ &= x(\delta(W_1 \backslash W_2)) + x(\delta(W_2 \backslash W_1))\\ &\quad + 2x[W_1 \cap W_2, \overline{W_1 \cup W_2}].\end{aligned}$$

By inequalities (1.1) and (1.3), this implies that

$$\begin{aligned}&x(\delta(W_1 \backslash W_2)) = \text{con}(W_1 \backslash W_2),\\ &x(\delta(W_2 \backslash W_1)) = \text{con}(W_2 \backslash W_1),\\ &x[W_1 \cap W_2, \overline{W_1 \cup W_2}] = 0. \qquad \square\end{aligned}$$

Lemma 6. *Suppose that x is an extreme point of $Q(G, r)$. If u, v are two nodes of G, then $[u, v]$ contains at most one edge with fractional value.*

Proof. The lemma holds vacuously if $|[u, v]| = 1$. Suppose that $|[u, v]| \geqslant 2$, and that there are two edges e_1, e_2 such that $0 < x(e_1) < 1$ and $0 < x(e_2) < 1$. Let $x' \in \mathbb{R}^E$ such that

$$x'(e) = \begin{cases} x(e) + \varepsilon & \text{if } e = e_1,\\ x(e) - \varepsilon & \text{if } e = e_2,\\ x(e) & \text{if } e \in E \backslash \{e_1, e_2\},\end{cases}$$

for $\varepsilon \neq 0$ arbitrarily small. Since any cut of G either contains $[u, v]$, or does not intersect this set, all cuts that are tight for x are also tight for x'. As $x'(e)$ is integer if $x(e)$ is also for $e \in E \backslash \{e_1, e_2\}$, this implies that every inequality of $Q(G, r)$ that is satisfied with equality by x, is also satisfied with equality by x'. But this contradicts the extremality of x. $\square$

Lemma 7. *Suppose that $x(e) > 0$ for all $e \in E$. If $\delta(W)$ is a cut tight for x, then $G(W)$ and $G(\overline{W})$ are both connected.*

Proof. Suppose for instance that $G(W)$ is not connected. Let W^1, W^2 be a partition of W such that $[W^1, W^2] = \emptyset$. Since G is connected, it follows that $[W^1, \overline{W}] \neq \emptyset$ and $[W^2, \overline{W}] \neq \emptyset$. From the hypothesis, we then have

$$x[W^1, \overline{W}] > 0, \quad x[W^2, \overline{W}] > 0. \tag{2.2}$$

In addition, since $\delta(W)$ is tight for x and $x(e) > 0$ for all $e \in E$, we must have $\mathrm{con}(W) > 0$. Thus at least one of the subsets W^1, W^2 has a positive connectivity type. Without loss of generality, we may suppose that $r(W^1) > 0$ and $r(W^2) \leqslant r(W^1)$. Hence by Lemma 3 (i)

$$\mathrm{con}(W) \leqslant \mathrm{con}(W^1). \tag{2.3}$$

As $\delta(W^1)$ is a cut of G, we have

$$x(\delta(W^1)) = x[W^1, \overline{W}] \geqslant \mathrm{con}(W^1). \tag{2.4}$$

Consequently, by (2.2) and (2.4), we obtain that

$$\begin{aligned} \mathrm{con}(W) &= x(\delta(W)) \\ &= x[W^1, \overline{W}] + x[W^2, \overline{W}] \\ &> \mathrm{con}(W^1), \end{aligned}$$

contradicting (2.3). □

2.2. The SNDP(G,r) on series–parallel graphs

We now state the main result of the paper.

Theorem 8. *If $G = (V, E)$ is series–parallel and $r(i)$ is even for all $i \in V$, then* SNDP $(G,r) = Q(G, r)$.

Proof. The proof is by induction on $|E|$. It is not hard to see that the statement holds for any graph with no more than two edges. Suppose it holds for any series–parallel graph with no more than m edges, and suppose that G contains exactly m+1 edges. We may suppose that G is connected. In fact, if G has only one component with positive connectivity, then the polytope reduces to the one associated with that component. And if this is not the case, then both polytopes are empty and the theorem trivially holds.

Now let us assume that, on the contrary, SNDP(G,r) $\neq Q(G, r)$. As SNDP(G,r) $\subseteq Q(G, r)$ and any integer solution of $Q(G, r)$ is a solution of SNDP(G,r), there must exist a fractional extreme point $x \in \mathbb{R}^E$ of $Q(G, r)$. From the induction hypothesis, it follows that

$$x(e) > 0 \qquad \text{for all } e \in E. \tag{2.5}$$

Let E_1 be the set of edges $e \in E$ such that $x(e) = 1$. As x is an extreme point of $Q(G, r)$, it then follows that there exists a family of cuts $\{\delta(W_i) : i = 1, \dots, s\}$ such that x is the unique solution of the system

$$\begin{aligned} &x(e) = 1 && \text{for all } e \in E_1, \\ &x(\delta(W_i)) = \mathrm{con}(W_i) && \text{for } i = 1, \dots, s, \end{aligned} \tag{2.6}$$

where $|E| = |E_1| + s$.

The proof of the theorem proceeds by successively establishing the following sequences of claims that build on each other. For the sake of clarity, their proofs are deferred.

Claim 1. *Let $\delta(W)$ be a cut tight for x. Then system* (2.6) *can be chosen so that if $\delta(W_i)$, $i \in \{1, \ldots, s\}$, is such that W and W_i are crossing, then $r(W \cap W_i) > \min\{r(W \backslash W_i), r(W_i \backslash W)\}$ and $r(W \backslash W_i) > \min\{r(W \cap W_i), r(\overline{W \cup W_i})\}$.*

Claim 2. *Each variable $x(f)$ has a nonzero coefficient in at least two equations of system* (2.6).

Claim 3. *G contains no node having less than two neighbors.*

Since G is series–parallel and contains at least three nodes, by Lemma 1 together with Claim 3, there must exist a node v that is adjacent to exactly two nodes v_1, v_2. Let F_1 (resp. F_2) be the set of edges between v and v_1 (resp. v_2). Without loss of generality, we may suppose that $x(F_1) \geqslant x(F_2)$.

Claim 4. *There exists a cut $\delta(W)$ tight for x such that $v \in W$, $F_1 \subseteq \delta(W)$, $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$.*

By Claim 4, there must exist a cut $\delta(W)$, tight for x, such that $F_1 \subseteq \delta(W)$ and, without loss of generality, $v \in W$. In consequence we may suppose that $x(\delta(W)) = \mathrm{con}(W)$ is a constraint of system (2.6). Furthermore, by (2.5) together with Lemma 7, $G(W)$ and $G(\overline{W})$ are both connected, and thus $F_2 \subseteq E(W)$. For the rest of the proof we suppose that system (2.6) verifies Claim 1 with respect to W. We also make the following hypothesis:

H1. $\delta(W)$ is chosen such that $r(W \backslash \{v\})$ is minimum among all the cuts $\delta(Z)$ of system (2.6) satisfying $F_1 \subseteq \delta(Z)$ and $F_2 \subseteq E(Z)$, i.e., $r(Z \backslash \{v\}) \geqslant r(W \backslash \{v\})$.

Let $W' = W \backslash \{v\}$ (see Fig. 1).

Claim 5. $r(v) > r(W')$.

Claim 6. *The equation $x(\delta(v)) = r(v)$ does not belong to system* (2.6).

Claim 7. $x(F_1) - x(F_2) < r(v) - r(W')$.

Now by Claims 2 and 6 together with Lemma 7, there exists a cut $\delta(W_{i_0})$ of system (2.6) such that $F_2 \subseteq \delta(W_{i_0})$ and $F_1 \subseteq E(W_{i_0})$. Note that $v \in W_{i_0}$. We claim

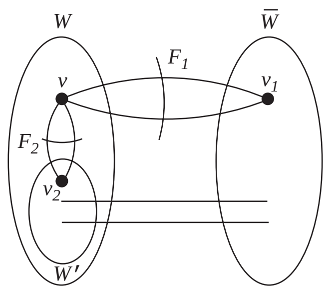

Fig. 1.

that $\mathrm{con}(W_{i_0}) \leqslant \mathrm{con}(W')$. To prove this, we first assume that $\overline{W}_{i_0} \subseteq W'$. Since r is a nondecreasing function, we have that $r(\overline{W}_{i_0}) \leqslant r(W')$ and $r(v) \leqslant r(W_{i_0})$. As by Claim 5 $r(v) > r(W')$, it follows that $r(W_{i_0}) > r(\overline{W}_{i_0})$. Note also that by Claim 5 together with Lemma 3 (ii) (1), we have that $\mathrm{con}(W') = r(W')$. Thus, $\mathrm{con}(W_{i_0}) = \min\{r(W_{i_0}), r(\overline{W}_{i_0})\} = r(\overline{W}_{i_0}) \leqslant r(W') = \mathrm{con}(W')$. Now suppose that $\overline{W}_{i_0} \cap \overline{W} \neq \emptyset$ and $\mathrm{con}(W_{i_0}) > \mathrm{con}(W')$. We claim that $r(W \backslash W_{i_0}) < r(\overline{W \cup W_{i_0}})$. In fact, if this is not the case, then we would have $r(W \backslash W_{i_0}) = r(\overline{W}_{i_0}) \leqslant r(W')$. Moreover by Claim 5 together with Lemma 3 (ii) (1), we also have $\mathrm{con}(W') = r(W')$. Therefore, we would obtain $\mathrm{con}(W_{i_0}) = \mathrm{con}(\overline{W}_{i_0}) \leqslant r(\overline{W}_{i_0}) = r(W \backslash W_{i_0}) \leqslant r(W') = \mathrm{con}(W')$, a contradiction. As $v \in W \cap W_{i_0}$, and by Claim 5, $r(v) > r(W')$,we also have $r(W \backslash W_{i_0}) \leqslant r(W') < r(v) \leqslant r(W \cap W_{i_0})$. Consequently, $r(W \backslash W_{i_0}) < \min\{r(\overline{W \cup W_{i_0}}), r(W \cap W_{i_0})\}$. As system (2.6) verifies Claim 1 with respect to W, W and W_{i_0} are then noncrossing, and thus either $W_{i_0} \subseteq W$ or $W \subseteq W_{i_0}$. As $F_1 \subseteq E(W_{i_0}) \backslash E(W)$ and $F_2 \subseteq E(W) \backslash E(W_{i_0})$, this is impossible. Therefore, $\mathrm{con}(W_{i_0}) \leqslant \mathrm{con}(W') \leqslant r(W') < r(v)$ by Claim 5. Since $v \in W_{i_0}$, it then follows that $\mathrm{con}(W_{i_0}) = r(\overline{W}_{i_0}) < r(v)$. Let $W'_{i_0} = W_{i_0} \backslash \{v\}$. By Lemma 3 (ii) (2) with respect to W_{i_0} and v, it follows that $\mathrm{con}(W'_{i_0}) = r(v)$. As $\mathrm{con}(W') = r(W')$, by Claim 5 we then have

$$\begin{aligned} x(\delta(W'_{i_0})) &= x(\delta(W_{i_0})) - x(F_2) + x(F_1) \\ &= \mathrm{con}(W_{i_0}) - x(F_2) + x(F_1) \\ &\leqslant \mathrm{con}(W') - x(F_2) + x(F_1) \\ &= r(W') - x(F_2) + x(F_1). \end{aligned}$$

This together with Claim 7 imply that

$$\begin{aligned} x(\delta(W'_{i_0})) &< r(W') + r(v) - r(W') \\ &< r(v) \\ &= \mathrm{con}(W'_{i_0}), \end{aligned}$$

a contradiction, which completes the proof of our theorem. □

Thus by Theorem 8, the trivial and cut inequalities suffice to describe the polytope SNDP(G, r) if G is series–parallel and $r(v)$ is even for all $v \in V$. As the separation problem for constraints (1.3) can be solved in polynomial time using any polynomial max-flow algorithm, an immediate consequence of Theorem 8 is the following.

Corollary 9. *The SNDP can be solved in polynomial time in series–parallel graphs if the connectivity types are all even.*

3. Proofs of the claims

In order to allow a better understanding and readability of the proof of Theorem 8, we have presented it without giving the proofs of the various used claims. This section is thus devoted to prove these claims.

Throughout this section, and as it has been considered in the proof of Theorem 8, x will denote a fractional extreme point of $Q(G,r)$, which is a unique solution of system (2.6), and which, by (2.5), has all its values positive. Moreover, we have that $G=(V,E)$ is a connected graph and, by the induction hypothesis, $Q(G',r)$ is integral for any graph G' having less edges than G.

Claim 1. *Let $\delta(W)$ be a cut tight for x. Then system* (2.6) *can be chosen so that if $\delta(W_i)$, $i \in \{1,\ldots,s\}$, is such that W and W_i are crossing, then $r(W\cap W_i) > \min\{r(W\backslash W_i), r(W_i\backslash W)\}$ and $r(W\backslash W_i) > \min\{r(W\cap W_i), r(\overline{W\cup W_i})\}$.*

Proof. First of all, note that we may assume that $x(\delta(W))=\mathrm{con}(W)$ is one of the equations of system (2.6). Now suppose for instance that $r(W\cap W_i) \leqslant \min\{r(W\backslash W_i), r(W_i\backslash W)\}$. By Lemma 5, we have that the cuts $\delta(W\backslash W_i)$ and $\delta(W_i\backslash W)$ are tight for x, and

$$\mathrm{con}(W)=\mathrm{con}(W\backslash W_i),$$

$$\mathrm{con}(W_i)=\mathrm{con}(W_i\backslash W),$$

$$x[W\cap W_i, \overline{W\cup W_i}]=0.$$

Thus,

$$x(\delta(W_i))=x(\delta(W\backslash W_i))+x(\delta(W_i\backslash W))-x(\delta(W)).$$

In consequence, the equation $x(\delta(W_i))=\mathrm{con}(W_i)$ is redundant with respect to the equalities

$$x(\delta(W))=\mathrm{con}(W),$$

$$x(\delta(W_i\backslash W))=\mathrm{con}(W_i\backslash W),$$

$$x(\delta(W\backslash W_i))=\mathrm{con}(W\backslash W_i).$$

One may then replace in system (2.6) $x(\delta(W_i))=\mathrm{con}(W_i)$ by the last two equations and get a system still having x as a unique solution. Moreover, clearly one can extract from this new system a (nonsingular) system of $|E_1|+s$ equations. As $W\backslash W_i \subseteq W$ and $W_i\backslash W \subseteq \overline{W}$, the statement follows. □

Claim 2. *Each variable $x(f)$ has a nonzero coefficient in at least two equations of system* (2.6).

Proof. It is clear that $x(f)$ must have a nonzero coefficient in at least one of the equations of system (2.6). For otherwise, one can increase $x(f)$ and obtain a solution still satisfying system (2.6), which is impossible.

Now let us suppose that for an edge $f=uv$, $x(f)$ has a nonzero coefficient in exactly one equation of system (2.6). Let (2.6)$'$ be the system obtained from (2.6) by deleting this equation as well as the equations $x(e)=1$ where $e\in[u,v]$. Note that none of the variables $x(e)$, $e\in[u,v]$ is involved in system (2.6)$'$. Let $F=[u,v]$. Let x' be the restriction of x on $E\backslash F$. By Lemma 4, $x'\in Q(G/F, r_F)$. Furthermore x' is a solution of system (2.6)$'$. Since system (2.6)$'$ is nonsingular and every equation of this system comes from a constraint of $Q(G/F, r_F)$, this implies that x' is an extreme point of $Q(G/F, r_F)$.

We claim that x' is fractional. Indeed as x is a fractional solution of a system whose right-hand side is integer and all the coefficients are either 0 or 1, x must have at least two fractional components. Since, by Lemma 6, $[u, v]$ may have at most one edge with fractional value, this implies that x' is fractional. As by Lemma 2, G/F is series–parallel, this contradicts the induction hypothesis. □

Claim 3. *G contains no node having less than two neighbors.*

Proof. Assume, on the contrary, that such a node, say $v_0 \in V$, exists. As G is connected, v_0 has then only one neighbor, say u. From Lemma 7, there doesn't exist a cut $\delta(W)$ tight for x such that $[u, v_0] \subseteq \delta(W)$, $v_0 \in W$ and $|W| \geqslant 2$. Moreover the cut $\delta(v_0)$ $(=[u, v_0])$ does not belong to system (2.6). In fact, since $x(e) > 0$ for all $e \in [u, v_0]$, this is clear if $r(v_0) = 0$. So assume that $r(v_0) > 0$. Obviously we have $r(v_0) \leqslant |[u, v_0]|$ since otherwise, both polytopes $Q(G, r)$ and SNDP(G, r) would be empty. As $r(v_0)$ $(=\text{con}(v_0))$ is integer, Lemma 6 implies that $[u, v_0] \subseteq E_1$. If $r(v_0) < |[u, v_0]|$, then clearly the constraint $x(\delta(v_0)) \geqslant \text{con}(v_0)$ cannot be tight for x. If $r(v_0) = |[u, v_0]|$, then $x(\delta(v_0)) = \text{con}(v_0)$ is redundant with respect to the equation $x(e) = 1$ for all $e \in [u, v_0]$. Consequently, the variable $x(e)$, $e \in [u, v_0]$, may have at most one nonzero coefficient in system (2.6), which contradicts Claim 2. □

For the rest of the claims, we need the following lemmas.

Lemma 10. $|V| \geqslant 4$.

Proof. By Claim 3 we have $|V| \geqslant 3$. So suppose that $|V| = 3$. Let $F' = [v_1, v_2]$ (see Fig. 2).

Without loss of generality, we may assume that there is an edge of F_1, say f_1, such that $0 < x(f_1) < 1$. By Claim 2, $x(f_1)$ has a nonzero coefficient in at least two equations of system (2.6). Thus $\delta(v)$ and $\delta(v_1)$ are both tight for x, $r(v) > 0$, and $r(v_1) > 0$. As by Lemma 6, F_1 contains at most one edge with fractional value, there must exist two edges $f_2 \in F_2$ and $f' \in F'$ such that $0 < x(f_2) < 1$ and $0 < x(f') < 1$. Moreover, these should

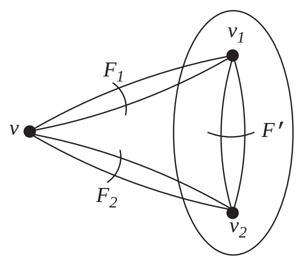

Fig. 2.

be the only edges of F_2 and F' that are fractional. Thus

$$\begin{aligned}x(\delta(v)) &= x(F_1) + x(F_2)\\ &= |F_1| - 1 + |F_2| - 1 + x(f_1) + x(f_2)\\ &= \mathrm{con}(v).\end{aligned}$$

As $x(f_1) + x(f_2) = 1$, this yields

$$x(\delta(v)) = |F_1| + |F_2| - 1. \tag{3.1}$$

Similarly, we obtain

$$\begin{aligned}x(\delta(v_1)) &= |F_1| + |F'| - 1\\ &= \mathrm{con}(v_1).\end{aligned} \tag{3.2}$$

Now by interchanging f_1 and f_2, we deduce that $\delta(v_2)$ is also tight for x and $r(v_2) > 0$. We then get along the same line

$$\begin{aligned}x(\delta(v_2)) &= |F_2| + |F'| - 1\\ &= \mathrm{con}(v_2).\end{aligned} \tag{3.3}$$

Now as $r(u)$ is even for all $u \in \{v, v_1, v_2\}$, it follows from (3.1) and (3.2) that $|F_2|$ and $|F'|$ have the same parity. However by (3.3) this implies that $\mathrm{con}(v_2)$ is odd, a contradiction. □

Lemma 11. *If $F_1 \subseteq E_1$, then the equation induced by $\delta(v)$ does not belong to system* (2.6).

Proof. Assume that $x(e) = 1$ for all $e \in F_1$ and $x(\delta(v)) = \mathrm{con}(v)$ appear in system (2.6). We have

$$\begin{aligned}x(\delta(v)) &= x(F_1) + x(F_2)\\ &= \mathrm{con}(v)\end{aligned}$$

and by Lemma 6, it then follows that $F_2 \subseteq E_1$. Since E_1 is maximal, this implies that $x(\delta(v)) = \mathrm{con}(v)$ is redundant with respect to the equations $x(e) = 1$ for all $e \in \delta(v)$, and hence cannot be among the equations of system (2.6). □

Lemma 12. *If the equation induced by $\delta(v)$ does not appear in system* (2.6), *then there exists a cut $\delta(W_{i_1})$, $i_1 \in \{1, \ldots, s\}$, such that $F_1 \subset \delta(W_{i_1})$ and the inequality induced by the cut $(\delta(W_{i_1}) \setminus F_1) \cup F_2$ is not tight for x.*

Proof. Assume that the result does not hold. Let $I_1 = \{i \in \{1, \ldots, s\} : F_1 \subset \delta(W_i)\}$. Since the equation induced by $\delta(v)$ does not belong to system (2.6), from Lemma 7, we obtain that $F_2 \cap \delta(W_i) = \emptyset$ for all $i \in I_1$. Without loss of generality, we may suppose that $v \in W_i$

and thus $|W_i| \geqslant 2$ for all $i \in I_1$. Then the subset of edges $(\delta(W_i)\backslash F_1) \cup F_2$ corresponds to the cut induced by $W_i\backslash\{v\}$, i.e.,

$$(\delta(W_i)\backslash F_1) \cup F_2 = \delta(W_i\backslash\{v\}).$$

By the assumption that the result is not true, we have

$$\begin{aligned} x(\delta(W_i\backslash\{v\})) &= \mathrm{con}(W_i\backslash\{v\}) \\ &= x(\delta(W_i)) - x(F_1) + x(F_2) \\ &= \mathrm{con}(W_i) - x(F_1) + x(F_2) \end{aligned}$$

for all $i \in I_1$, and thus

$$x(F_1) - x(F_2) = \mathrm{con}(W_i) - \mathrm{con}(W_i\backslash\{v\}). \tag{3.4}$$

Now let (2.6)* be the system obtained from (2.6) by replacing each cut $\delta(W_i)$ by the cut $\delta(W_i\backslash\{v\})$ for all $i \in I_1$, and deleting the equations $x(e) = 1$, for all $e \in F_1 \cap E_1$. We notice that if the equation induced by $W_i\backslash\{v\}$ already belongs to system (2.6), we only have to delete the equation $x(\delta(W_i)) = \mathrm{con}(W_i)$. Clearly system (2.6)* does not contain any equation involving edges of F_1. Let x^* be the restriction of x on $E\backslash F_1$. Obviously, x^* is a (fractional) solution of system (2.6)*. Moreover the graph G/F_1 is series–parallel with fewer edges. Thus, by the induction hypothesis, $Q(G/F_1, r_{F_1})$ is integer. In consequence, as all the equations of system (2.6)* corresponds to constraints of $Q(G/F_1, r_{F_1})$, there exists an integer solution y^* of $Q(G/F_1, r_{F_1})$ which is at the same time a solution of system (2.6)*. We shall consider two cases.

Case 1: $x(e) = 1$ for all $e \in F_1$. As, by (3.4), $x(F_1) - x(F_2)$ is integer, and by Lemma 6, F_2 can have at most one fractional edge, it follows that $x(e) = 1$ for all $e \in F_2$. Let $y \in \mathbb{R}^E$ be the solution given by

$$y(e) = \begin{cases} y^*(e) & \text{if } e \in E\backslash F_1, \\ 1 & \text{if } e \in F_1. \end{cases}$$

We will show that y satisfies system (2.6). In fact, it is clear that y satisfies the equations of system (2.6) corresponding to trivial constraints and cuts not containing F_1. Now let $\delta(W_i)$, $i \in I_1$, be a cut of system (2.6) containing F_1. Without loss of generality, we may suppose that $v \in W_i$. Thus, by the remarks above, $\delta(W_i\backslash\{v\})$ is tight for x, and in consequence, the equation $x(\delta(W_i\backslash\{v\})) = \mathrm{con}(W_i\backslash\{v\})$ would belong to system (2.6)* and hence be tight for y^*. Thus

$$\begin{aligned} y(\delta(W_i)) &= y^*(\delta(W_i\backslash\{v\})) + y(F_1) - y^*(F_2) \\ &= \mathrm{con}(W_i\backslash\{v\}) + y(F_1) - y(F_2) \\ &= \mathrm{con}(W_i\backslash\{v\}) + x(F_1) - x(F_2) \\ &= \mathrm{con}(W_i), \end{aligned}$$

where the last equation follows from (3.4). Hence $\delta(W_i)$ is tight for y.

Case 2: There is an edge $f_1 \in F_1$ with $0 < x(f_1) < 1$. As $x(F_1) - x(F_2)$ is integer, by Lemma 6, there must exist exactly one edge $f_2 \in F_2$ with $0 < x(f_2) < 1$ and $x(f_2) = x(f_1)$. We also have $x(F_1) - x(F_2) = |F_1| - |F_2|$. Let $y \in \mathbb{R}^E$ such that

$$y(e) = \begin{cases} y^*(e) & \text{if } e \in E \backslash F_1, \\ 1 & \text{if } e \in F_1 \backslash \{f_1\}, \\ y^*(f_2) & \text{if } e = f_1. \end{cases}$$

Clearly, y satisfies the trivial equations of system (2.6) as well as the cut constraints $\delta(W_j)$ with $F_1 \cap \delta(W_j) = \emptyset$. If $F_1 \subseteq \delta(W_i)$ with $i \in I_1$ (and without loss of generality, $v \in W_i$), then $\delta(W_i \backslash \{v\})$ is a cut of system (2.6)* and thus $y^*(\delta(W_i) \backslash \{v\}) = \mathrm{con}(W_i \backslash \{v\})$. Therefore

$$\begin{aligned} y(\delta(W_i)) &= y^*(\delta(W_i \backslash \{v\})) - y^*(F_2) + y(F_1) \\ &= \mathrm{con}(W_i \backslash \{v\}) - y^*(F_2) + y(F_1) \\ &= \mathrm{con}(W_i \backslash \{v\}) - (|F_2| - 1 + y^*(f_2)) + (|F_1| - 1 + y^*(f_2)) \\ &= \mathrm{con}(W_i \backslash \{v\}) - |F_2| + |F_1| \\ &= \mathrm{con}(W_i \backslash \{v\}) - x(F_2) + x(F_1) \\ &= \mathrm{con}(W_i). \end{aligned}$$

And hence $\delta(W_i)$ is tight for y.

Thus in both cases y is a solution of system (2.6). As $y \neq x$ this is a contradiction. □

Claim 4. *There exists a cut $\delta(W)$ tight for x such that $v \in W$, $F_1 \subseteq \delta(W)$, $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$.*

Proof. Assume the contrary. We shall consider two cases.

Case 1: $x(e) = 1$ for all $e \in F_1$. By Lemma 11 the equation $x(\delta(v)) = \mathrm{con}(v)$ cannot belong to system (2.6). The hypothesis that the claim is false together with Claim 2 imply that $\delta(v_1)$ is a cut of system (2.6) that contains F_1. Moreover, it is the only cut of system (2.6) containing F_1. Let $F' = [v_1, V \backslash \{v, v_1\}]$ (see Fig. 3). Then $x(\delta(V \backslash \{v, v_1\})) = x(F_2) + x(F') \geqslant \mathrm{con}(V \backslash \{v, v_1\})$. Since $x(F_1) \geqslant x(F_2)$ and $x(\delta(v_1)) = x(F_1) + x(F') = \mathrm{con}(v_1)$, we then have

$$\mathrm{con}(V \backslash \{v, v_1\}) \leqslant \mathrm{con}(v_1). \tag{3.5}$$

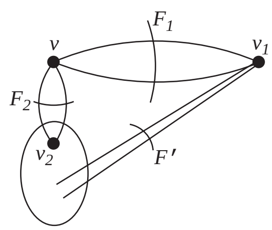

Fig. 3.

By (2.1) we have $\mathrm{con}(v_1) = r(v_1)$. Thus by inequality (3.5) we obtain that $r(v_1) \geqslant \min\{r(v), r(V\backslash\{v, v_1\})\}$.

(a) We claim that $r(v) = r(v_1) > r(V\backslash\{v, v_1\})$. In fact suppose that $r(v) \leqslant r(V\backslash\{v, v_1\})$. Then $r(v_1) \geqslant r(v)$ and $r(v_1) \leqslant r(V\backslash\{v, v_1\})$. In fact, if $r(v_1) > r(V\backslash\{v, v_1\})$ we then deduce that $r(v_1) > r(u)$ for all $u \in V\backslash\{v_1\}$ which contradicts the fact that G contains at least two nodes of maximum connectivity type. This implies that $\mathrm{con}(V\backslash\{v, v_1\}) = r(v_1) = \mathrm{con}(v_1)$, and hence

$$\begin{aligned} x(\delta(V\backslash\{v, v_1\})) &= x(F_2) + x(F') \\ &= x(\delta(v_1)) - x(F_1) + x(F_2) \\ &= \mathrm{con}(v_1) - x(F_1) + x(F_2) \\ &\geqslant \mathrm{con}(V\backslash\{v, v_1\}) \\ &= \mathrm{con}(v_1). \end{aligned}$$

In consequence, $x(F_1) \leqslant x(F_2)$. As $x(F_1) \geqslant x(F_2)$, we then have that $x(F_1) = x(F_2)$ and therefore the cut $\delta(V\backslash\{v, v_1\})$ is tight for x. As $\delta(v_1)$ is the only cut containing F_1 in system (2.6), this contradicts Lemma 12. Thus $r(v) > r(V\backslash\{v, v_1\})$. Since G contains at least two nodes of maximum connectivity type, we also have $r(v_1) = r(v)$.

(b) By Claim 2, there must exist a cut $\delta(W_2)$ in system (2.6) such that $F_2 \subseteq \delta(W_2)$. It is clear that $F_1 \cap \delta(W_2) = \emptyset$. For otherwise $\delta(W_2)$ would be the cut $\delta(v)$. But as $F_1 \subseteq E_1$, this contradicts Lemma 11. Without loss of generality, we may suppose that $v_2 \in W_2$. Let $F_1' = [v_1, \overline{W}_2\backslash\{v, v_1\}]$, $F_2' = [v_1, W_2]$ and $F'' = [W_2, \overline{W}_2\backslash\{v, v_1\}]$ (see Fig. 4).

For the remainder of the proof for Case 1, as by Lemma 10, $|V| \geqslant 4$, we shall suppose that $W_2 \neq V\backslash\{v, v_1\}$, that is $\overline{W}_2\backslash\{v, v_1\} \neq \emptyset$. If $W_2 = V\backslash\{v, v_1\}$, the proof is similar (by setting $x(F_1') = x(F'') = 0$). Since $\delta(v_1)$ and $\delta(W_2)$ are tight for x and $\mathrm{con}(v_1) = r(v_1)$ we have that

$$x(F_1) + x(F_1') + x(F_2') = r(v_1), \tag{3.6}$$

and

$$x(F_2) + x(F_2') + x(F'') = \mathrm{con}(W_2). \tag{3.7}$$

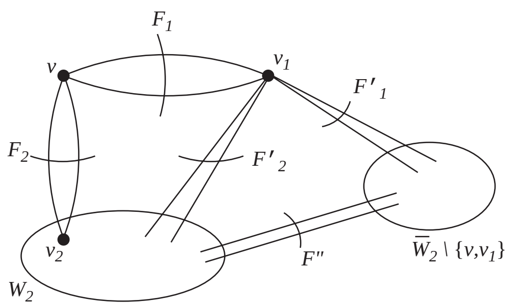

Fig. 4.

As by (a), v and v_1 have maximum connectivity types, it follows that $\mathrm{con}(W_2 \cup \{v\}) = r(v_1)$. Thus

$$\begin{aligned} x(\delta(W_2 \cup \{v\})) &= x(F_1) + x(F_2') + x(F'') \\ &\geqslant \mathrm{con}(W_2 \cup \{v\}) \\ &= r(v_1). \end{aligned}$$

By (3.6), this yields

$$x(F'') \geqslant x(F_1'). \tag{3.8}$$

Moreover, from (a) we also have $\mathrm{con}(V \setminus \{v, v_1\}) \geqslant \mathrm{con}(W_2)$. Consequently,

$$\begin{aligned} x(\delta(V \setminus \{v, v_1\})) &= x(F_2) + x(F_1') + x(F_2') \\ &\geqslant \mathrm{con}(V \setminus \{v, v_1\}) \\ &\geqslant \mathrm{con}(W_2), \end{aligned}$$

which by (3.7) implies that $x(F'') \leqslant x(F_1')$. By (3.8) we then obtain that $x(F'') = x(F_1')$. Thus equation (3.6) can also be written as

$$x(F_1) + x(F_2') + x(F'') = r(v_1). \tag{3.9}$$

By combining (3.7) and (3.9) we get

$$x(F_1) - x(F_2) = r(v_1) - \mathrm{con}(W_2). \tag{3.10}$$

Therefore

$$\begin{aligned} \mathrm{con}(V \setminus \{v, v_1\}) &\leqslant x(\delta(V \setminus \{v, v_1\})) \\ &= x(F_2) + x(F') \\ &= x(\delta(v_1)) - x(F_1) + x(F_2) \\ &= r(v_1) - x(F_1) + x(F_2) \\ &= \mathrm{con}(W_2) \\ &\leqslant \mathrm{con}(V \setminus \{v, v_1\}), \end{aligned}$$

where the last equation comes from (3.10). Thus $\mathrm{con}(W_2) = \mathrm{con}(V \setminus \{v, v_1\})$ and $\delta(V \setminus \{v, v_1\})$ is tight for x. Since the only cut containing F_1 in system (2.6) is $\delta(v_1)$, we obtain a contradiction to Lemma 12.

Case 2: There exists an edge $f_1 \in F_1$ such that $0 < x(f_1) < 1$. Then from Lemma 6, it follows that $x(e) = 1$ for all $e \in F_1 \setminus \{f_1\}$. Moreover, by Claim 2 together with the hypothesis that the claim does not hold, the cuts $\delta(v)$ and $\delta(v_1)$ must be tight for x and in system (2.6).

(a) We claim that $x(F_1) > x(F_2)$. In fact, suppose not, that is $x(F_1) = x(F_2)$. As $\delta(v)$ is tight for x, we have

$$\begin{aligned} x(\delta(v)) &= \mathrm{con}(v) \\ &= x(F_1) + x(F_2) \\ &= 2x(F_1) \\ &= 2(|F_1| - 1) + 2x(f_1). \end{aligned}$$

However, as $0 < x(f_1) < 1$, $2(|F_1| - 1) + 2x(f_1)$ cannot be even, a contradiction.

(b) Next we show that $r(v) = r(v_1) > r(V\setminus\{v, v_1\})$. Indeed, since $\delta(v_1)$ is tight for x and therefore $x(F_1) + x(F') = \mathrm{con}(v_1)$ where, we recall, $F' = [v_1, V\setminus\{v, v_1\}]$, by (a) we have that $x(F_2) + x(F') < \mathrm{con}(v_1)$. As $x(\delta(V\setminus\{v, v_1\})) = x(F_2) + x(F') \geqslant \mathrm{con}(V\setminus\{v, v_1\})$, and by (2.1) $\mathrm{con}(v_1) = r(v_1)$ we get

$$\mathrm{con}(V\setminus\{v, v_1\}) < \mathrm{con}(v_1) = r(v_1).$$

Thus $r(v_1) > \min\{r(V\setminus\{v, v_1\}), \max\{r(v), r(v_1)\}\}$, and hence $r(v_1) > r(V\setminus\{v, v_1\})$. As G contains at least two nodes of maximum connectivity type, $r(v_1)$ must be equal to $r(v)$ and thus the Case 2 (b) follows.

Now from (b) together with (2.1) it follows that $\mathrm{con}(v_1) = \mathrm{con}(v)$. As $\delta(v)$ and $\delta(v_1)$ are tight for x and hence $x(F_1) + x(F_2) = \mathrm{con}(v)$ and $x(F_1) + x(F') = \mathrm{con}(v_1)$, we obtain that

$$x(F_2) = x(F'). \tag{3.11}$$

Moreover, as f_1 has a fractional value and $\delta(v)$ is tight, there must exist an edge, say f_2, of F_2 such that $0 < x(f_2) < 1$. Thus by Claim 2 there must exist in system (2.6) two cuts containing f_2 and hence F_2. We may then consider again the cut $\delta(W_2)$ introduced in Case 1 (b). Suppose that $v_2 \in W_2$ and let F_1', F_2' and F'' be as defined in Case 1. Suppose also that $W_2 \neq V\setminus\{v, v_1\}$ (the case where $W_2 = V\setminus\{v, v_1\}$ is similar). We claim that $x(F_1') = x(F'')$. In fact as $\delta(v)$ and $\delta(W_2)$ are tight for x, Eqs. (3.6) and (3.7) hold. Now by considering the cuts $\delta(W_2 \cup \{v\})$ and $\delta(V\setminus\{v, v_1\})$ we get

$$x(F_1) + x(F'') + x(F_2') \geqslant \mathrm{con}(W_2 \cup \{v\})$$

and

$$x(F_2) + x(F_2') + x(F_1') \geqslant \mathrm{con}(V\setminus\{v, v_1\}).$$

As by (b), v and v_1 have maximum connectivity types, it follows that $\mathrm{con}((W_2 \cup \{v\}) = r(v_1)$ and $\mathrm{con}(V\setminus\{v, v_1\}) = r(V\setminus\{v, v_1\})$. Hence

$$x(F_1) + x(F'') + x(F_2') \geqslant r(v_1), \tag{3.12}$$

$$x(F_2) + x(F_2') + x(F_1') \geqslant r(V\setminus\{v, v_1\}). \tag{3.13}$$

From (3.6) and (3.12), it follows that $x(F_1') \leqslant x(F'')$. And, as by (b), $r(V\setminus\{v, v_1\}) \geqslant r(W_2) = \mathrm{con}(W_2)$, (3.7) and (3.13) yield $x(F_1') \geqslant x(F'')$. Therefore $x(F_1') = x(F'')$.

Consequently,

$$\begin{aligned} x(\delta(W_2)) &= x(F_2) + x(F_2') + x(F'') \\ &= x(F_2) + x(F_2') + x(F_1') \\ &= x(F_2) + x(F'). \end{aligned}$$

By (3.11) we then have

$$\begin{aligned} x(\delta(W_2)) &= 2x(F_2) \\ &= 2(|F_2| - 1) + 2x(f_2) \\ &= \mathrm{con}(W_2). \end{aligned}$$

Since $0 < x(f_2) < 1$, $2(|F_2| - 1) + 2x(f_2)$ cannot be even. But this contradicts the fact that $\mathrm{con}(W_2)$ is even, which ends the proof of our claim. □

For the proof of the next claim we need the following.

Lemma 13. *For any cut $\delta(Z)$ in system* (2.6) *such that $F_1 \subseteq \delta(Z)$ and $F_2 \subseteq E(Z)$, we have $x(F_1) - x(F_2) \leqslant \mathrm{con}(Z) - \mathrm{con}(Z\backslash\{v\})$.*

Proof. Since $\delta(Z)$ is tight for x and $\{v, v_2\} \subseteq Z$, we have

$$\begin{aligned} x(\delta(Z)) &= x(F_1) + x[Z\backslash\{v\}, \overline{Z}] \\ &= \mathrm{con}(Z). \end{aligned}$$

Moreover we have

$$\begin{aligned} x(\delta(Z\backslash\{v\})) &= x(F_2) + x[Z\backslash\{v\}, \overline{Z}] \\ &\geqslant \mathrm{con}(Z\backslash\{v\}). \end{aligned}$$

We thus deduce $x(F_1) - x(F_2) \leqslant \mathrm{con}(Z) - \mathrm{con}(Z\backslash\{v\})$. □

Claim 5. $r(v) > r(W')$.

Proof. Suppose $r(v) \leqslant r(W')$. Thus $r(W) = r(W')$ and hence $\mathrm{con}(W) \leqslant \mathrm{con}(W')$. By Lemma 13, it then follows that $x(F_1) \leqslant x(F_2)$. As $x(F_1) \geqslant x(F_2)$, we then obtain that $x(F_1) = x(F_2)$ and $\mathrm{con}(W) = \mathrm{con}(W')$. Now we claim that the equation $x(\delta(v)) = r(v)$ does not belong to system (2.6). (Note that by (2.1), $\mathrm{con}(v) = r(v)$.) Indeed, suppose the contrary. Since

$$x(\delta(W)) = x(F_1) + x[W', \overline{W}] = \mathrm{con}(W) = \mathrm{con}(W'), \tag{3.14}$$

we have

$$x(\delta(W')) = x(F_2) + x[W', \overline{W}] = \mathrm{con}(W'). \tag{3.15}$$

Thus

$$\begin{aligned}2x[W', \overline{W}] &= 2\,\mathrm{con}(W') - (x(F_1) + x(F_2))\\ &= 2\,\mathrm{con}(W') - x(\delta(v))\\ &= 2\,\mathrm{con}(W') - r(v),\end{aligned}$$

which implies that

$$x[W', \overline{W}] = \mathrm{con}(W') - \frac{r(v)}{2}.$$

As $r(v)$ is even, we obtain that $x[W', \overline{W}]$ is integer. Therefore, by (3.14) and (3.15) it follows that $x(F_1)$ and $x(F_2)$ are integer. As by Lemma 6, both F_1 and F_2 cannot have more than one edge with fractional value, we then have $x(e) = 1$ for all $e \in F_1 \cup F_2$, contradicting Lemma 11.

Consequently, $x(\delta(v)) = r(v)$ is not an equation of system (2.6). Now let $\delta(W_i)$ be a cut of system (2.6) containing F_1 such that $F_2 \subseteq E(W_i)$, ($\delta(W_i)$ may be $\delta(W)$). By the minimality type hypothesis H1 on $W\backslash\{v\}$, we have that $r(W_i\backslash\{v\}) \geqslant r(W\backslash\{v\}) = r(W')$. Since we have supposed that $r(v) \leqslant r(W')$, it then follows that $r(W_i\backslash\{v\}) \geqslant r(v)$. Hence $\mathrm{con}(W_i) \leqslant \mathrm{con}(W_i\backslash\{v\})$. Furthermore, applying Lemma 13 with respect to W_i and $W_i\backslash\{v\}$ yields $x(F_1) - x(F_2) \leqslant \mathrm{con}(W_i) - \mathrm{con}(W_i\backslash\{v\})$. Since $x(F_1) = x(F_2)$, it follows that $\mathrm{con}(W_i) \geqslant \mathrm{con}(W_i\backslash\{v\})$, and therefore $\mathrm{con}(W_i) = \mathrm{con}(W_i\backslash\{v\})$. In consequence, as $x(F_1) = x(F_2)$, $\delta(W_i\backslash\{v\})$ is tight for *x*. Since the latter holds for all cuts containing F_1 in system (2.6), this contradicts Lemma 12. □

Claim 6. *The equation $x(\delta(v)) = r(v)$ does not belong to system* (2.6).

Proof. Assume that, on the contrary, $\delta(v)$ is a cut of system (2.6). By Lemmas 11 and 6, there must exist an edge f_1 of F_1 and an edge f_2 of F_2 such that $0 < x(f_1) < 1$ and $0 < x(f_2) < 1$. Note that by Lemma 6 we have $x(e) = 1$ for all $e \in F_1\backslash\{f_1\}$ and $x(e) = 1$ for all $e \in F_2\backslash\{f_2\}$.

On the other hand, from Claim 5 together with the fact that *G* contains at least two nodes of maximum connectivity type, it follows that $\mathrm{con}(W) = r(v)$. As $\delta(W)$ and $\delta(v)$ are tight for *x*, we have

$$\begin{aligned}&x(\delta(W)) = x(F_1) + x[W', \overline{W}] = \mathrm{con}(W) = r(v),\\ &x(\delta(v)) = x(F_1) + x(F_2) = r(v),\end{aligned}$$

which yields

$$\begin{aligned}&x[W', \overline{W}] = x(F_2),\\ &x(\delta(W')) = 2x(F_2) \geqslant \mathrm{con}(W').\end{aligned}$$

Also, by Claim 5 it follows that $\mathrm{con}(W') = r(W')$. Since $r(W')$ is even, and, as $x(F_2)$ is fractional, $2x(F_2)$ is not even, we obtain that

$$x(F_2) > \frac{r(W')}{2}. \tag{3.16}$$

Since $x(\delta(v)) = x(F_1) + x(F_2) = r(v)$, this implies that

$$x(F_1) < r(v) - \frac{r(W')}{2}. \tag{3.17}$$

Now by Claim 2, there must exist a further cut, say $\delta(W_2)$ (different from $\delta(v)$) of system (2.6) that contains f_2. Without loss of generality, we may suppose that $v \in \overline{W}_2$. Thus, by Lemma 7, $F_1 \subseteq E(\overline{W}_2)$. Let $W_2' = \overline{W}_2 \backslash \{v\}$ (see Fig. 5). **We claim that**

$$x(\delta(W_2')) \geqslant r(v). \tag{3.18}$$

In fact suppose, on the contrary, that $x(\delta(W_2')) < r(v)$. As $x(\delta(W_2')) \geqslant \mathrm{con}(W_2')$, we then have $\mathrm{con}(W_2') < r(v)$, and therefore $r(W_2') < r(v)$. Since G contains at least two nodes with maximum connectivity type, it follows that $r(W_2) \geqslant r(v)$, and hence $\mathrm{con}(W_2) = r(v)$. So we have

$$\begin{aligned} x(\delta(W_2)) &= x(F_2) + x[W_2, W_2'] \\ &= \mathrm{con}(W_2) \\ &= r(v). \end{aligned}$$

As

$$x(\delta(W_2')) = x(F_1) + x[W_2, W_2'] < r(v),$$

this implies that $x(F_2) > x(F_1)$, a contradiction.

Consequently, inequality (3.18) holds. Now, since $\delta(W_2)$ is tight for x, by (3.16) we get

$$x[W_2, W_2'] < \mathrm{con}(W_2) - \frac{r(W')}{2}.$$

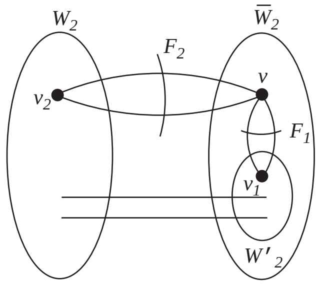

Fig. 5.

Therefore by (3.17) we obtain that

$$x(\delta(W_2')) = x(F_1) + x[W_2, W_2']$$
$$< \mathrm{con}(W_2) + r(v) - r(W').$$

From (3.18) it thus follows that

$$\mathrm{con}(W_2) > r(W'). \tag{3.19}$$

Now we claim that

$$r(W \cap W_2) < \min\{r(W \backslash W_2), r(W_2 \backslash W)\}. \tag{3.20}$$

In fact as $v \in W \backslash W_2$, from Claim 5 it follows that

$$r(W \cap W_2) \leqslant r(W') < r(v) = r(W \backslash W_2).$$

As by (3.19) we also have $r(W') < r(W_2)$ and hence $r(W \cap W_2) < r(W_2)$, it follows that $r(W_2) = r(W_2 \backslash W)$. This implies that $r(W \cap W_2) < r(W_2 \backslash W)$, and consequently, (3.20) holds.

Since $v \in \overline{W}_2 \cap W$ and $v_2 \in W_2 \cap W$, by Claim 1, we have either $\overline{W}_2 \subseteq W$ or $W_2 \subseteq W$. In the first case, one would have $F_1 \subseteq E(W)$ which contradicts the definition of $\delta(W)$. So assume that $W_2 \subseteq W$. This implies that $W_2 \subseteq W'$ and thus $r(W_2) \leqslant r(W')$. Since, by inequality (3.19), we know that $r(W_2) > r(W')$, we get a contradiction. □

Claim 7. $x(F_1) - x(F_2) < r(v) - r(W')$.

Proof. As V contains at least two nodes with a maximum connectivity type, and by Claim 5, $r(v) > r(W')$, We have

$$x(\delta(W)) = x(F_1) + x[W', \overline{W}] = \mathrm{con}(W) = r(v).$$

Also,

$$x(\delta(W')) = x(F_2) + x[W', \overline{W}] \geqslant \mathrm{con}(W') = r(W').$$

Hence $x(F_1) - x(F_2) \leqslant r(v) - r(W')$.

Suppose now that, on the contrary, the statement does not hold, that is

$$x(F_1) - x(F_2) = r(v) - r(W'). \tag{3.21}$$

Let $\delta(W_i)$ be a cut of system (2.6) containing F_1 such that $v \in W_i$. By the minimality type hypothesis H1 on $W \backslash \{v\}$, we have $r(W_i \backslash \{v\}) \geqslant r(W \backslash \{v\}) = r(W')$. Moreover, since $r(\overline{W_i \backslash \{v\}}) = \max\{r(\overline{W}_i), r(v)\} > r(W')$ by Claim 5, we then have

$$\mathrm{con}(W_i \backslash \{v\}) > r(W') \qquad \text{if } r(W_i \backslash \{v\}) > r(W'), \tag{3.22}$$

$$\mathrm{con}(W_i \backslash \{v\}) = r(W') \qquad \text{if } r(W_i \backslash \{v\}) = r(W'). \tag{3.23}$$

We now claim that $\mathrm{con}(W_i)\leqslant r(v)$. Indeed, if $\mathrm{con}(W_i)>r(v)$, then $\mathrm{con}(W_i)=\mathrm{con}(W_i\backslash\{v\})$. Therefore

$$\begin{aligned}x(\delta(W_i\backslash\{v\})) &= x(\delta(W_i)) - x(F_1) + x(F_2)\\ &= \mathrm{con}(W_i) - r(v) + r(W')\\ &= \mathrm{con}(W_i\backslash\{v\}) - r(v) + r(W').\end{aligned}$$

By Claim 5, this implies that $x(\delta(W_i\backslash\{v\}))<\mathrm{con}(W_i\backslash\{v\})$, which is impossible. Thus, $\mathrm{con}(W_i)\leqslant r(v)$.

Suppose now that $r(W_i\backslash\{v\})>r(W')$. From (3.22), we have $\mathrm{con}(W_i\backslash\{v\})>r(W')$ and therefore,

$$\begin{aligned}x(F_2)+x[W_i\backslash\{v\},\overline{W}_i] &= x(\delta(W_i\backslash\{v\}))\\ &\geqslant \mathrm{con}(W_i\backslash\{v\})\\ &> r(W').\end{aligned}$$

As $\delta(W_i)$ is a tight cut of system (2.6) and $\mathrm{con}(W_i)\leqslant r(v)$, we then have

$$\begin{aligned}x(F_1)+x[W_i\backslash\{v\},\overline{W}_i] &= x(\delta(W_i))\\ &= \mathrm{con}(W_i)\\ &\leqslant r(v)\end{aligned}$$

and it immediatly follows that $x(F_1)-x(F_2)<r(v)-r(W')$, contradicting (3.21).

Thus, $r(W_i\backslash\{v\})=r(W')$ and from (3.23) we get $\mathrm{con}(W_i\backslash\{v\})=r(W')$. Since there exist at least two nodes having maximum connectivity types and $r(v)>r(W')$ by Claim 5, we deduce $\mathrm{con}(W_i)=r(v)$. As $\delta(W_i)$ is a tight cut of system (2.6) and using (3.21), we then have

$$\begin{aligned}x(\delta(W_i\backslash\{v\})) &= x(\delta(W_i)) - x(F_1) + x(F_2)\\ &= \mathrm{con}(W_i) - x(F_1) + x(F_2)\\ &= r(v) - r(v) + r(W')\\ &= r(W')\\ &= \mathrm{con}(W_i\backslash\{v\}).\end{aligned}$$

Hence, $\delta(W_i\backslash\{v\})$ is tight for x. As $\delta(W_i)$ is an arbitrary cut of system (2.6) containing F_1 with $v\in W_i$ and, by Claim 6, $\delta(v)$ is not among the cuts of that system, we obtain a contradiction with Lemma 12. $\square$

4. The polyhedron $P(G,r)$

Let $G=(V,E)$ be a graph with weights $w(e)$, $e\in E$ and let $r\in\mathbb{Z}_+^V$ be a connectivity type vector. Here we consider the SNDP when more than one copy of an edge may be used.

More precisely, the problem here is to determine an integer vector $x \in \mathbb{N}^E$ such that

(i) the graph $H=(V, E(x))$ is survivable, and
(ii) $\sum_{e\in E} w(e)x(e)$ is minimum.

Here $E(x)$ is the set of edges obtained by replacing each edge e of E by $x(e)$ edges. This relaxation of the SNDP is important because it may provide a lower cost solution than the case where at most one copy of an edge may be used.

In this section we shall discuss the polyhedron $P(G,r)$ associated with the solutions to that problem. Clearly inequalities (1.1) and (1.3) are valid for $P(G,r)$. Using Theorem 8 we are going to show, in what follows, that these inequalities are sufficient to describe $P(G,r)$ when G is series–parallel and $r(v)$ is even for all $v \in V$.

Theorem 14. *Let $G=(V,E)$ be a series–parallel graph. If the connectivity types are all even, then $P(G,r)$ is completely described by inequalities* (1.1) *and* (1.3).

Proof. Let $P^*(G,r)$ be the polyhedron described by inequalities (1.1) and (1.3). It suffices to show that the extreme points of $P^*(G,r)$ are integral. Suppose that, on the contrary, there exists a fractional extreme point $x \in \mathbb{R}^E$ of $P^*(G,r)$. Let $G'=(V,E')$ be the graph obtained from G by replacing each edge e of E by $\lceil x(e)\rceil$ edges $e_1,\ldots,e_{\lceil x(e)\rceil}$. Clearly, G' is series–parallel and by Theorem 8, $Q(G',r)$ is integral. Let $x' \in \mathbb{R}^{E'}$ be the solution given by

$$\left.\begin{cases} x'(e_i)=1 & \text{for } i=1,\ldots,\lceil x(e)\rceil-1,\\ x'(e_i)=x(e)-\lceil x(e)-1\rceil & \text{for } i=\lceil x(e)\rceil, \end{cases}\right\} \quad \text{if } x(e)\neq 0.$$

It is easily seen that $x' \in Q(G',r)$. Moreover, x' is an extreme point of $Q(G',r)$. In fact, if this is not the case, as $Q(G',r)$ is integral, there must exist t integer solutions $(t\geqslant 2)$ $y'_1,\ldots,y'_t$ of $P(G',r)$ and $\lambda_1,\ldots,\lambda_t \in \mathbb{R}^+$ such that

$$x'=\sum_{j=1}^{t}\lambda_j y'_j \quad \text{and} \quad \sum_{j=1}^{t}\lambda_j=1.$$

Now, let $y_1,\ldots,y_t \in \mathbb{R}^E$ be the solutions such that

$$y_i(e)=\sum_{j=1}^{\lceil x(e)\rceil} y'_i(e_j)$$

for $e\in E$ and $i=1,\ldots,t$. It is clear that $y_1,\ldots,y_t \in P^*(G,r)$. Moreover we have that

$$x=\sum_{j=1}^{t}\lambda_j y_j.$$

But this contradicts the fact that x is an extreme point of $P^*(G,r)$.

Consequently x' is an extreme point of $Q(G',r)$. Since x' is fractional and G' is series–parallel this contradicts Theorem 8. $\square$

5. Concluding remarks

We have studied the survivable network design problem and have given a complete linear description of the associated polytope when the underlying graph is series–parallel and the node connectivity types are all even. We have shown that in this case, the trivial and the cut inequalities suffice to describe the polytope. Since the cut inequalities can be separated in polynomial time, this provides a polynomial time cutting plane algorithm for solving the survivable network design problem in series–parallel graphs. As a consequence we also obtained that the nonnegativity inequalities together with the cut inequalities characterize the polyhedron when multiple copies of an edge are allowed.

The trivial and the cut inequalities do not, unfortunately, suffice to describe the survivable network polytope of a series–parallel graph if the node connectivity types may be (even and) odd. The polytope in this case is an extension of the widely studied Steiner tree polytope [8,9,16,23,36] whose complete description in series–parallel graphs, although it contains further classes of facets, is still unknown. However, when the connectivity types are uniform, say equal to k, and k is odd and the graph is series–parallel, as shown by Didi Biha and Mahjoub [17], the corresponding polytope can be described by the trivial, cut and the so-called series–parallel inequalities.

In general graphs, further classes of facets are needed for describing the survivable network polytope even when the node types are uniform and equal to 2. Mahjoub [35] introduced for this case a large class of valid inequalities called the F-partition inequalities. These inequalities can be extended in a straightforward manner to the case where $r(i) \in \mathbb{Z}_+$ for all $i \in V$. Kerivin et al. [33] investigated a generalization of the class of F-partition inequalities and discussed a branch-and-cut algorithm based on these inequalities, the trivial and the cut inequalities for both the 2-edge and the 2-node connected subgraph problems. The algorithm is also used to solve the SNDP when $r(v) \in \{1, 2\}$ for all $v \in V$.

Goemans and Bertsimas [24] showed that if the weights $(w(e) : e \in E)$ satisfy the triangle inequalities (i.e., $w(e) + w(f) \geqslant w(g)$ for every three edges $e,\ f,\ g$ defining a triangle), then the linear programs $\min\{wx : x \in P^*(G, r)\}$ and $\min\{wx : x \in P^*_S(G, r)\}$ have the same optimal values. Here $P^*_S(G, r)$ is obtained from $P^*(G, r)$ by adding the constraints $\{x(\delta(i)) = r(i) : i \in S\}$ and S is an arbitrary node subset of G. They referred to this property as the *parsimonious property*.

If $G = (V, E)$ is series–parallel and $r(v)$ is even for all $v \in V$, then by Theorem 14, $P^*(G, r)$ is integral. As $P^*_S(G, r)$ is a face of $P^*(G, r)$, it follows that $P^*_S(G, r)$ is also integral and thus the SNDP in G where edges may be used repeatedly is equivalent to the linear program $\min\{wx : x \in P^*_V(G, r)\}$. As a consequence one can delete any vertex $i \in V$ with $r(i) = 0$ when solving this linear program.

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