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On the NP-completeness of the perfect matching free subgraph problem

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ABSTRACT

Given a bipartite graph $G = (U \cup V, E)$ such that $| U |=| V |$ and every edge is labelled true or false or both, the perfect matching free subgraph problem is to determine whether or not there exists a subgraph of G containing, for each node u of U, either all the edges labelled true or all the edges labelled false incident to u, and which does not contain a perfect matching. This problem arises in the structural analysis of differential-algebraic systems. The purpose of this paper is to show that this problem is NP-complete. We show that the problem is equivalent to the stable set problem in a restricted case of tripartite graphs. Then we show that the latter remains NP-complete in that case. We also prove the NP-completeness of the related minimum blocker problem in bipartite graphs with perfect matching.

1. Introduction

Given a graph $G = (V, E)$, a *matching* of G is a subset of edges such that no two edges share a common node. Matchings have shown to be useful for modelling various discrete structures [4]. A graph is called *bipartite (tripartite)* if its nodes can be partitioned into two (three) disjoint sets such that every edge connects one node in a set to a node in a different set. A bipartite graph is called *complete* if there exists an edge between every pair of nodes of different sets. A complete bipartite graph is also called a *biclique*. A matching M in graph G is called *perfect* if every node of G is incident to some edge of M. Given a bipartite graph $G = (U \cup V, E)$ such that $|U| = |V| = n$, a matching M of G is then perfect if and only if $|M| = n$.

Given a graph $G = (V, E)$, a matching in G of maximum cardinality is called a *maximum matching*. Its size corresponds to the *matching number* of G which is denoted by $\nu(G)$. A *stable set* of a graph is a subset of nodes S such that no two nodes in S are adjacent. Given a graph $G = (V, E)$, the stable set problem in G consists in finding a stable set of maximum cardinality.

Theorem 1. *Given a bipartite graph, if M is a maximum cardinality matching and S is a maximum stable set, then $|M| + |S| = |U \cup V|$.* □

For more details on matching theory, the reader is referred to [4].

Let $G = (U \cup V, E)$ be a bipartite graph such that $|U| = |V| = n$. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Suppose that every edge of E is labelled *true* or *false*, where an edge may have both true and false labels. For a node $u_i \in U$, let E_i^t and E_i^f denote the sets of edges incident to u_i labelled true and false, respectively. The *perfect matching free subgraph problem*

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(PMFSP) in G is to determine whether or not there exists a subgraph containing for each node $u_i \in U$ either E_i^t or E_i^f (but not both), and which is perfect matching free. This problem arises in the structural analysis of differential-algebraic systems.

The purpose of this paper is to show that PMFSP is NP-complete. For this we first show that PMFSP is equivalent to the stable set problem in a restricted case of tripartite graphs. Then we show that the latter remains NP-compete in that case.

Differential-algebraic systems (DASs) are used for modelling complex physical systems as electrical networks and dynamic movements. Such a system can be given as $f(\dot{x}, x, t) = 0$ where t is time, x is the variable vector and $\dot{x}$ the derivative vector of x with respect to time. Establishing that a DAS definitely is not solvable can be helpful. A necessary (but not sufficient) condition for solving a DAS is that the number of variables and equations must agree, and there must exist a mapping between the equations and the variables in such a way that each equation is related to only one variable and each variable is related to only one equation. If this is satisfied, then we say that the system is *well-constrained* [6]. The *structural analysis problem* for a DAS consists of verifying if the system is well-constrained.

In many practical situations, physical systems yield differential-algebraic systems with conditional equations. A conditional equation is an equation whose form depends on the (true or false) value of a condition. A conditional equation can generate several equations. A conditional differential-algebraic system may then have different forms depending on the set of conditions that hold. Here we consider conditional DAS's such that any conditional equation may take two possible values, depending on whether the associated condition is true or false and may generate only one equation. Moreover, we suppose that the conditions are independent. Consider for example the following DAS:

$$\begin{array}{lll} eq_1: & \text{if } a > 0, \text{ then } 0 = 4x_1^2 + 2\dot{x_1} + 4x_2 + 2, & \text{else } 0 = \dot{x_2} + 4, \\ eq_2: & \text{if } b > 0, \text{ then } 0 = 6\dot{x_2} + 2, & \text{else } 0 = x_1 + \dot{x_2} + 1 \end{array} \quad (1)$$

If $a > 0, b > 0$, then system (1) is nothing but the system.

$$\begin{array}{ll} eq_1: & 0 = 4x_1^2 + 2\dot{x_1} + 4x_2 + 2, \\ eq_2: & 0 = 6\dot{x_2} + 2. \end{array} \quad (2)$$

The structural analysis problem has been considered in the literature for non-conditional DAS's. In [6], Murota introduces a formulation of the problem in terms of bipartite graphs and shows that a system of equations is well constrained if and only if there exists a perfect matching in the corresponding bipartite graph. Given a DAS, one can associate a bipartite graph $G = (U \cup V, E)$, called *incidence graph*, where U corresponds to the equations, V to the variables and there is an edge $u_i v_j \in E$ between a node $u_i \in U$ and a node $v_j \in V$ if and only if the variable corresponding to v_i appears in the equation corresponding to u_i.

Given a conditional DAS, the associated structural analysis problem consists of verifying whether or not the system is well constrained for all the possible values. The SAP for a conditional DAS thus reduces to verifying whether or not the incidence bipartite graph, related to any configuration of the system, contains a perfect matching. In [2], the equivalence between the SAP for conditional DAS and the PMFSP has been shown. Also in [2], an integer programming formulation is proposed.

2. PMFSP and stable sets

The aim of this section is to show that PMFSP is equivalent to the stable set problem in a special case of tripartite graph. Let $H = (V^1 \cup V^2 \cup V^3, F)$ be a tripartite graph where $|V^1| = |V^2| = |V^3| = n$, $V^j = \{v_1^j, \ldots, v_n^j\}$ for $j = 1, 2, 3$ and V^1 and V^2 are connected by the perfect matching $M = \{v_1^1 v_1^2, v_2^1 v_2^2, \ldots, v_n^1 v_n^2\}$. We will consider the following problem: does there exist a stable set in H of size $n+1$? We will call this problem the *tripartite stable set with perfect matching problem* (TSSPMP). In what follows we shall show that both problems TSSPMP and PMFSP are equivalent.

Theorem 2. *TSSPMP and PMFSP are polynomially equivalent.*

Proof. Some parts of this proof are shortened. For the detailed proof see [3]. Let $G = (U \cup V, E)$ and $H = (V^1 \cup V^2 \cup V^3, F)$ be the graphs on which the problems PMFSP and TSSPMS are considered, respectively. We will first show that an instance of TSSPMP can be transformed into an instance of PMFSP. For an edge $v_i^1 v_i^2$ of the perfect matching where $v_i^1 \in V^1$ and $v_i^2 \in V^2$, we consider a node u_i in U. And for a node v_i^3 of V^3 we consider a node v_i in V. Moreover, if $v_i^1 v_k^3$ (resp. $v_i^2 v_k^3$) is in F for some $i, k \in \{1, \ldots, n\}$, then we add an edge $u_i v_k$ in E with label true (resp. false). Fig. 1 illustrates this transformation. Observe that graph $H = (V^1 \cup V^2 \cup V^3, F)$ can be obtained from graph $G = (U \cup V, E)$ by doing the reverse operations. Let E_i^t (resp. E_i^f) be the set of edges incident to u_i labelled true (resp. false), for $i = 1, \ldots, n$.
In what follows, we will show that there exists a stable set in H of size $n + 1$ if and only if there exists a subgraph $G' = (U \cup V, E')$ of G such that for each node $u_i \in U$, either $E_i^t \subset E'$ or $E_i^f \subset E'$, and G' is perfect matching free. In fact, suppose first that there exists a subgraph G' of G that satisfies the required properties. Since G' is perfect matching free, this implies that a maximum cardinality matching in G' contains fewer than n edges. As $|U \cup V| = 2n$ by Theorem 1 there exists a stable set in G', say S', of size $|S'| \geq n + 1$. Now from S', we are going to construct a stable set in H with the same cardinality. Let S be the node subset of H obtained as follows. For every node $v_j \in V \cap S'$, add node v_j^3 in S. And for every node $u_i \in U \cap S'$, add node v_i^1 in S if $E_i^t \subseteq E'$ and node v_i^2 if $E_i^f \subseteq E'$. As $|S'| \geq n + 1$, we have $|S| \geq n + 1$. Now it is not hard to see that S is a stable set. Moreover from a stable set in H of size greater or equal to $n + 1$ one can obtain along the same line a stable set in G with the same cardinality. □

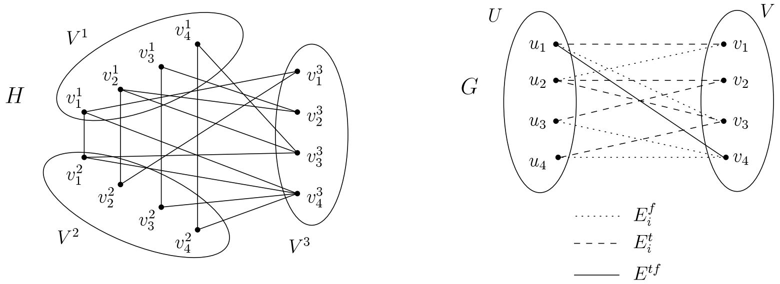

Fig. 1. Two equivalent TSSPMP and PMFSP instances.

3. The NP-completeness of PMFSP

In this section we show the NP-completeness of PMFSP. For this we shall show that TSSPMP is NP-complete. By Theorem 2, the result follows. In [7] it is shown that the stable set problem is NP-complete in tripartite graphs. (Recall that the problem is known to be polynomially solvable in bipartite graphs.) What we are going to show in the following is that the more restricted variant TSSPMP is also NP-complete. In other words, the stable set problem in tripartite graphs remains NP-complete even when the sets of the partition of the graph have the same size and that the set of edges between two of the three sets of the partition correspond exactly to a perfect matching. In order to show the NP-completeness of TSSPMP, we shall use the one-in-three 3SAT problem. An instance of *one-in-three 3SAT*(1-in-3 3SAT) consists of n literals $l_1, \ldots, l_n$ and m clauses $C_1, \ldots, C_m$. Each clause $C_j = (x_j^1, x_j^2, x_j^3)$ is the disjunction of three variables, where a variable is either a literal or its negation. Furthermore, for every $j \in \{1, \ldots, m\}$ and $k \in \{1, 2, 3\}$, $\overline{x}_j^k$ corresponds to the negation of x_j^k, that is, if $x_j^k = l_i$ (resp. $x_j^k = \bar{l}_i$) for some $i \in \{1, \ldots, n\}$, then $\overline{x}_j^k = \bar{l}_i$ (resp. $\overline{x}_j^k = l_i$). The question is whether or not there exists an assignment of truth values ("true" or "false") to the literals such that each clause has exactly one true variable. This problem has been shown to be NP-complete [1].

Theorem 3. *TSSPMP is NP-complete.*

Proof. It is clear that TSSPMP is in NP. To prove the theorem, we shall use a reduction from 1-in-3 3SAT. The proof uses ideas from [7]. So suppose we are given an instance of 1-in-3 3SAT with a set of n literals $L = \{l_1, \ldots, l_n\}$ and a set of m clauses $C = \{C_1, \ldots, C_m\}$. We shall construct an instance of TSSPMP on a graph $H = (V^1 \cup V^2 \cup V^3, F)$ where $|V^1| = |V^2| = |V^3| = p = 3n + m - 1$ and the set of edges between V^1 and V^2 correspond exactly to a perfect matching. We will show that H has a stable set of size $p + 1$ if and only if 1-in-3 3SAT admits a truth assignment. The construction will be done in 4 steps. First, with each literal $l_i \in L$, we associate the nodes $v_i^1, \overline{v}_i^1 \in V^1$, $v_i^2, \overline{v}_i^2 \in V^2$ and $v_i^3, \overline{v}_i^3 \in V^3$, along with the edges $v_i^1\overline{v}_i^2, \overline{v}_i^2 v_i^3, v_i^3\overline{v}_i^1, \overline{v}_i^1 v_i^2, v_i^2\overline{v}_i^3, \overline{v}_i^3 v_i^1$ in F. These nodes will be called *literal nodes* and the edges *literal edges*. Note that these edges form a cycle of length 6, which will be denoted by Γ_i for $i = 1, \ldots, n$. Therefore a stable set of H cannot have more than three nodes from a cycle Γ_i. Moreover, if the stable set contains three nodes, these must be either $\{v_i^1, v_i^2, v_i^3\}$ or $\{\overline{v}_i^1, \overline{v}_i^2, \overline{v}_i^3\}$. For every $j \in \{1, \ldots, m\}$ and $k \in \{1, 2, 3\}$, we associate with x_j^k the node set $\{v_i^1, v_i^2, v_i^3\}$ (resp. $\{\overline{v}_i^1, \overline{v}_i^2, \overline{v}_i^3\}$) if $x_j^k = l_i$ (resp. $x_j^k = \bar{l}_i$) for some $i \in \{1, \ldots, m\}$. As a consequence we have that $\{\overline{v}_i^1, \overline{v}_i^2, \overline{v}_i^3\}$ (resp. $\{v_i^1, v_i^2, v_i^3\}$) is associated with $\overline{x}_j^k$ if $x_j^k = l_i$ (resp. $x_j^k = \bar{l}_i$) for some $i \in \{1, \ldots, m\}$.

Next, we add for each clause $C_j, j = 1, \ldots, m$ the nodes $w_j^1 \in V^1$, $w_j^2 \in V^2$, $w_j^3 \in V^3$ along with the edges $w_j^1 w_j^2$, $w_j^2 w_j^3$, $w_j^3\ w_j^1$. These nodes will be called *clause nodes*, the edges *clause edges*. Note that these edges form a triangle, which will be denoted by T_j, for $j = 1, \ldots, m$.

In the next step, for each $C_j = (x_j^1,\ x_j^2,\ x_j^3)$, we add edges between w_j^1 (resp. w_j^2) and all the nodes associated with $\overline{x}_j^1$, x_j^2 and x_j^3 (resp. x_j^1, $\overline{x}_j^2$ and x_j^3) which belong to V^3. Moreover, we add edges between w_j^3 and all the nodes associated with x_j^1, x_j^2 and $\overline{x}_j^3$ which belong to V^1 and V^2. All these edges will be called *satisfiability edges*.

Finally, we add the nodes $z_q^1 \in V^1$, $z_q^2 \in V^2$, $z_q^3 \in V^3$ for $q = 1, \ldots, n - 1$. These nodes will be called *fictitious nodes*. For each fictitious node in $V^1 \cup V^2$, we add edges to connect this node to the all nodes in V^3. And for each fictitious node in V^3, we add edges to connect this node to the all non fictitious nodes in $V^1 \cup V^2$. We also add the edges $z_q^1 z_q^2$ for $q = 1, \ldots, n-1$. Observe that $|V^1| = |V^2| = |V^3| = p$. Moreover, the edges between V^1 and V^2 form a perfect matching given by the edges $v_i^1\overline{v}_i^2, \overline{v}_i^1 v_i^2, i = 1, \ldots, n$, $w_j^1 w_j^2, j = 1, \ldots, m$, and $z_q^1 z_q^2, q = 1, \ldots, n - 1$.

Thus, from an instance of the 1-in-3 3SAT with n literals and m clauses, we obtain a tripartite graph with $9n + 3m - 3$ nodes and $10n^2 + 4nm - 5n + 14m + 1$ edges. Fig. 2 shows an example of graph H when $L = \{l_1,\ l_2,\ l_3\}$ and $C = \{(\bar{l}_1,\ l_2,\ \bar{l}_3),\ (l_1,\ l_2,\ l_3)\}$. For the sake of clarity, only the satisfiability edges are displayed. We have given the following claim without proof. For the proof see [3].

Claim 4. *Any stable set in H cannot contain more than $3n + m$ nodes. Moreover, if a stable set contains $3n + m$ nodes, then it does not contain any fictitious node.* □

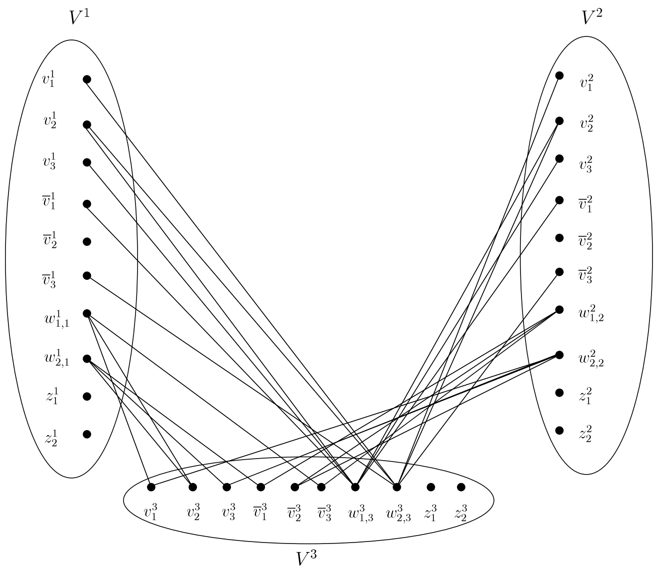

Fig. 2. A TSSPMP instance resulting from a 1-in-3 3SAT instance.

In what follows we show that there exists in H a stable set of size $3n + m$ if and only if 1-in-3 3SAT admits a solution. ($\Rightarrow$) Let S be a stable set in H of size $3n + m$. By Claim 4, S does not contain any fictitious node. Thus, as $|S| = 3n + m$, S intersects each cycle Γ_i in exactly three nodes and each triangle T_j in exactly one node. Moreover, we have either $S \cap \Gamma_i = \{v_i^1,\ v_i^2,\ v_i^3\}$ or $S \cap \Gamma_i = \{\overline{v}_i^1,\ \overline{v}_i^2,\ \overline{v}_i^3\}$, for $i = 1, \ldots, n$. Consider the solution I for 1-in-3 3SAT defined as follows. If $v_i^k \in S$ (resp. $\overline{v}_i^k \in S$), $k = 1, 2, 3$, then associate the true (resp. false) value to the literal l_i, for $i = 1, \ldots, n$. In what follows we will show that for each clause $C_j = (x_j^1,\ x_j^2,\ x_j^3)$, we have exactly one variable with true value. For this it suffices to show that a clause node of T_j is in S if and only if the corresponding variable is of true value. Indeed, suppose that $w_j^1 \in S$. We may suppose that $x_j^1 = l_i$, the case where $x_j^1 = \bar{l}_i$ is similar. By construction of H, as the satisfiability edge $w_j^1\overline{v}_i^3$ belongs to F, it follows that $\overline{v}_i^3 \notin S$. By the remark above, this implies that $v_i^1,\ v_i^2,\ v_i^3$ belong to S. Therefore literal l_i has a true value in solution I. Thus x_j^1 has a true value. It is similar for w_j^2 and w_j^3. Conversely, if $x_j^1 = l_i$, then by definition of I, v_i^1, $v_i^2, v_i^3 \in S$. Moreover, the satisfiability edges $w_j^2v_i^3,\ w_j^3v_i^2$ belong to F. As $|S \cap T_j| = 1$, it follows that $w_j^1 \in S$.

As a consequence, as S contains exactly one clause node from each T_i, it follows that each clause has exactly one true variable.

($\Leftarrow$) Suppose that there exists a solution I of 1-in-3 3SAT. Let S be the node set obtained as follows. If l_i has true value in I, then add v_i^1, v_i^2, v_i^3 to S, otherwise add $\overline{v}_i^1, \overline{v}_i^2, \overline{v}_i^3$ to S. For each clause $C_j = (x_j^1, x_j^2, x_j^3), j = 1, \ldots, m$, add node w_j^k to S if x_j^k has true value, for $k \in \{1, 2, 3\}$. We have that $|S| = 3n + m$. We now show that S is a stable set. For this, it suffices to show that no clause node in S is adjacent to a literal node. Consider $j \in \{1, \ldots, m\}$ and suppose that x_j^1 has a true value in C_j. (The case where $x_j^k, k \in \{2, 3\}$, has a true value is similar.) Node w_j^1 associated with x_j^1, which is in S, is adjacent to exactly three literal nodes, namely the ones associated with $\overline{x}_j^1, x_j^2$ and x_j^3, in V_3. As I is a solution of the 1-in-3 3SAT, and in consequence, these variables have all false values, by construction, the nodes corresponding to these variables do not belong to S. Therefore w_j^1 cannot be adjacent to one of these nodes, and hence the proof is complete. □

From Theorems 2 and 3, we deduce the following corollary.

Corollary 5. *PMFSP is NP-complete.* □

4. The minimum blocker perfect matching problem

In this section, we consider a variant of the PMFSP when there are no labels on the edges. This problem can be stated as follows. Given a graph $G = (U \cup V, E)$ with a perfect matching and $|U| = |V|$, find a perfect matching free subgraph with a maximum number of edges and covering the vertices of U. As it will turn out, this problem is nothing but a special case of the so-called minimum blocker problem [8].

Let $G = (U \cup V, E)$ be a bipartite graph with matching number $\nu(G)$. In [8], Zenklusen et al. define a *blocker* as a subset of edges $B \subset E$ such that $G' = (U \cup V, E \setminus B)$ has a matching number smaller than $\nu(G)$. They define the *minimum blocker problem* (MBP) as follows. Given a bipartite graph $G = (U \cup V, E)$ and a positive integer k, does there exist an edge subset B of E with $|B| \leq k$ such that B is a blocker? They prove that MBP is NP-complete. Here, we are interested in a special case of the MBP, hereafter called the *minimum blocker perfect matching problem* (MBPMP), where G contains a perfect matching. In what follows, we show that MBPMP is NP-complete. We also prove that it remains NP-complete in the case where $G' = (U \cup V, E \setminus B)$ must cover U, which corresponds to the PMFSP with no edge labels.

Theorem 6. *MBPMP is NP-complete.*

Proof. It is shown in [8] that MBP is NP-complete even in the case where $\nu(G) = |U|$ (see the proof of Theorem 3.3 in [8]). We consider this subproblem. Moreover, we suppose that $|U| < |V|$ (otherwise MBP would be MBPMP). Let $\deg(U) = \min_{u_i \in U}\{\deg(u_i)\}$. We suppose that $k < \deg(U)$ (in the case where $k \geq \deg(U)$, the $\deg(U)$ edges incident to the vertex with minimum degree in U clearly form a blocker).

Let $\tilde{G} = (\tilde{U} \cup \tilde{V}, \tilde{E})$ be the graph obtained from G where $\tilde{U} = U \cup \bar{U}$ and $\tilde{V} = V \cup \bar{V}$ where $\bar{U}$ and $\bar{V}$ are new vertex subsets such that $|\bar{U}| = \max\{|V| - |U|, k+1\}$, $|\bar{V}| = \max\{k+1-|V|+|U|, 0\}$ and $\tilde{E} = E \cup \{\bar{u}v : \bar{u} \in \bar{U}, v \in \tilde{V}\}$. Note that $\bar{U}$ contains at least $k+1$ nodes, $|\tilde{U}| = |\tilde{V}|$ and $(\bar{U} \cup \tilde{V}, \tilde{E} \setminus E)$ is a complete bipartite graph. Also note that, as $\nu(G) = |U|$, $\tilde{G}$ contains a perfect matching.

In what follows we will show that G contains a blocker of cardinality less or equal than k if and only if $\tilde{G}$ so does. For this we first give the following claim.

Claim 7. *Let $H = (W_1 \cup W_2, F)$ be a complete bipartite graph such that $|W_1| = |W_2| \geq k+1$ for some $k \geq 0$. Then H does not contain a blocker of size $\leq k$.*

Proof. Suppose that there is a blocker B of size $|B| \leq k$. Then the subgraph $H' = (W_1 \cup W_2, F \setminus B)$ has no perfect matching. From Hall's theorem (see [4]) there exists $i \in \{1, 2\}$ and $W \subset W_i$ such that $|W| > |\Gamma(W)|$ in H' where $\Gamma(W)$ stands for the neighbour set of W. Since H is a complete bipartite graph, we have $|B| \geq |W| \times (|W_i| - |\Gamma(W)|)$. Now, since for each triplet of nonnegative integers x, y, z with $x \geq y > z$ we have $y(x-z) \geq x$, by considering $x = |W_i|$, $y = |W|$ and $z = |\Gamma(W)|$, we conclude that $|B| \geq |W| \times (|W_i| - |\Gamma(W)|) \geq |W_i| \geq k+1$, a contradiction. □

Now consider a blocker B of G with $|B| \leq k$, and suppose that B is not a blocker of $\tilde{G}$. Thus there exists a perfect matching of $\tilde{G}$, say M, which does not intersect B. Since $|M \cap E| = |U| = \nu(G)$ and B is a blocker of G, we have a contradiction. Thus B is also a blocker of $\tilde{G}$.

Conversely, suppose that G has no blocker B with $|B| \leq k$. If $\tilde{G}$ contains a blocker, say $\tilde{B}$ with $|\tilde{B}| \leq k$, then let $\tilde{B}_1 = \tilde{B} \cap E$. Obviously, $|\tilde{B}_1| \leq k$. We claim that $\tilde{B}_1$ is a blocker of G. In fact, if this is not the case, then there must exist a matching M' in the graph $(U \cup V, E \setminus \tilde{B}_1)$ with $|M'| = |U| = \nu(G)$. Let V' be the subset of nodes of V covered by M'. Let $H = (W_1 \cup W_2, F)$ be the biclique with $W_1 = \bar{U} = \tilde{U} \setminus U$ and $W_2 = \tilde{V}'$ where $\tilde{V}' = \tilde{V} \setminus V'$. Clearly, $|W_1| = |W_2| \geq k+1$. Let $\tilde{B}_2 = \tilde{B} \cap F$. As $|\tilde{B}_2| \leq k$, by the claim above, the subgraph $(W_1 \cup W_2, F \setminus \tilde{B}_2)$ contains a perfect matching say M''. As $M' \cup M''$ is a perfect matching of $(\tilde{U} \cup \tilde{V}, \tilde{E} \setminus \tilde{B})$, this contradicts the fact that $\tilde{B}$ is a blocker of $\tilde{G}$, and the proof is complete. □

In the proof of Theorem 6, graph $\tilde{G}$ is constructed in such a way that $\deg(u) \geq k+1$ for all $u \in \tilde{U}$. Therefore, any graph obtained from $\tilde{G}$ by removing the edges of any blocker B with $|B| \leq k$ covers the vertices of $\tilde{U}$. This implies that the variant of the PMFSP without labels on the edges, considered in this section, is also NP-complete.

5. Concluding remarks

In this note we have shown that PMFSP is NP-complete. This is in connection with the SAP. For this, we have supposed that the conditions related to a DAS are independent. As the problem is NP-complete in this case, it remains NP-complete in the more general case, when several conditions may be involved in an equation of the system, and some dependences exist between the conditions. This general version of the problem has been addressed in [2,5].

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