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# On the independent dominating set polytope

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**Abstract**

In this paper, we consider the independent dominating set polytope. We give a complete linear description of that polytope when the graph is reduced to a cycle. This description uses a general class of valid inequalities introduced in [T.M. Contenza, Some results on the dominating set polytope, Ph.D. Dissertation, University of Kentucky, 2000]. We devise a polynomial time separation algorithm for these inequalities. As a consequence, we obtain a polynomial time cutting plane algorithm for the minimum (maximum) independent dominating set problem on a cycle. We also introduce a lifting operation called twin operation, and discuss some polyhedral consequences. In particular, we show that the above results can be extended to a more general class of graphs.


## 1. Introduction

Given a graph $G = (V, E)$, a node subset $D \subseteq V$ of $G$ is called a *dominating set* if every node of $V \setminus D$ is adjacent to at least one node of $D$. An *independent set* of $G$ is a node set $T \subseteq V$ such that there is no edge with both endnodes in $T$. Given a weight system $w(j),\ j \in V$, associated with the nodes of $G$, the *minimum weight independent dominating set problem* (MWIDSP for short) is to find an independent dominating set $S \subseteq V$ of $G$ such that $\sum_{i \in S} w(i)$ is minimum.

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The MWIDSP has applications in social network theory [7] and game theory [1,17]. It is NP-complete in general [15]. It has been shown to be polynomially solvable in some special classes of graphs such as strongly chordal graphs [11], permutation graphs [12,6], interval graphs [21] and cocomparability graphs [18]. Most algorithms developed for these classes of graphs are linear time algorithms.

The complexity aspect of the cardinality version of the problem has been intensively studied as well. Corneil and Perl [9] show that the minimum cardinality independent dominating set problem is NP-complete in the bipartite graphs and the comparability graphs. Farber [13] shows that this problem is polynomially solvable in chordal graphs. In [13], it is surprisingly shown that the MWIDSP is however NP-complete for this class of graphs. It has also been shown that it is polynomial in cographs [12] and split graphs [13]. Further complexity and combinatorial results on the MWIDSP can be found in [2,3,14,23].

In this paper, we study the MWIDSP from a polyhedral point of view. We give a complete linear description of the associated polytope when the graph is reduced to a cycle. This description uses a general class of valid inequalities introduced by Contenza [8]. We also show that this class of inequalities can be separated in polynomial time. In consequence, we obtain a polynomial time cutting plane algorithm for the MWIDSP on a cycle. To the best of our knowledge, this is the first polynomial time algorithm for the MWIDSP on these graphs. We also introduce a lifting operation called twin operation and discuss some polyhedral consequences. In particular we show that the above results can be extended to a more general class of graphs.

The closely related dominating set problem has been the subject of extensive research in the past three decades. A complete survey of the algorithmic complexity of this problem and the MWIDSP can be found in [7,17].

If $G = (V, E)$ is a graph and $S \subseteq V$ a node set of $G$, then the 0–1 vector $x^S \in \mathbb{R}^V$ with $x^S(u) = 1$ if $u \in S$ and $x^S(u) = 0$ otherwise is called the incidence vector of $S$. The convex hull of the incidence vectors of all independent dominating sets of $G$, denoted by $P_{ID}(G)$, is called the independent dominating set polytope of $G$, i.e.,

$$P_{ID}(G) = \text{conv}\{x^S \in \mathbb{R}^V \mid S \subseteq V \text{is an independent dominating set of } G\}.$$

Hence, the MWIDSP is equivalent to the linear programming problem

$$\min\{wx \mid x \in P_{ID}(G)\}.$$

Since the MWIDSP is NP-complete, we cannot expect to find a complete characterization of $P_{ID}(G)$ for all graphs. It may however be that for certain classes of graphs $G$, the polytope $P_{ID}(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities, polynomial time separation algorithms can be designed so that the MWIDSP on these graphs can be solved in polynomial time.

Let $G = (V, E)$ be a graph. If $u \in V$ is a node of $G$, the neighborhood of $u$ in $G$, denoted by $N_G(u)$, is the node set consisting of $u$ together with the nodes which are adjacent to $u$. If $u \in V$, let $N_G{}^*(u)$ denote the set $N_G(u) \setminus \{u\}$. If the context prevents any ambiguity, we will omit the subscript and simply write $N(u)$ and $N^*(u)$. If $S \subseteq V$ and $b : V \longrightarrow \mathbb{R}$, $b(S)$ will denote $\sum_{u \in S} b(u)$.

If $S \subseteq V$ is an independent dominating set, then $x^S$, the incidence vector of $S$, satisfies the following inequalities:

$$x(u) + x(v) \leq 1 \qquad \text{for all } (u, v) \in E \tag{1}$$

$$x(N(u)) \geq 1 \qquad \text{for all } u \in V \tag{2}$$

$$x(u) \geq 0 \qquad \text{for all } u \in V. \tag{3}$$

Inequalities (1), called *edge inequalities*, imply that $S$ is an independent set. And inequalities (2), called *neighborhood inequalities*, imply that $S$ is a dominating set. Inequalities (3) are called *trivial inequalities*.

In contrast to many NP-hard combinatorial optimization problems, such as the independent set problem [22], the polyhedral aspect of the MWIDSP has not received much attention. To the best of our knowledge, the polytope $P_{ID}(G)$ has been characterized only in the class of strongly chordal graphs [11] within the framework of totally balanced matrices. Actually, Farber [11] showed that inequalities (2) and (3) together with the so-called *clique inequalities* (which are valid inequalities and generalize inequalities (1) for the independent set polytope) suffice to describe $P_{ID}(G)$ when $G$ is strongly chordal.

If $C_n$ is a chordless cycle on $n$ nodes, then the following inequalities are also valid for $P_{ID}(C_n)$:

$$x(C_n) \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{4}$$

$$x(C_n) \geq \left\lceil \frac{n}{3} \right\rceil . \tag{5}$$

Inequality (4) must be satisfied by every independent set and inequality (5) must be satisfied by every dominating set of $C_n$. Inequalities (4) and (5) will be called *cycle inequalities*.

In [8], Contenza shows that $P_{ID}(C_n)$ is full dimensional if $n \geq 8$. It is also characterized when inequalities (1), (2), (4) and (5) define facets for $P_{ID}(C_n)$. Observe that inequalities (3) are redundant with respect to inequalities (1) and (2) when $G$ is a cycle.

Contenza [8] also introduces a class of valid inequalities for $P_{ID}(C_n)$ if $n \geq 8$. In this paper, we show that these inequalities together with inequalities (1), (2), (4) and (5) completely describe the polytope $P_{ID}(C_n)$.

Related work can also be found in [19,4,5]. In [19], Mahjoub gives a description of the dominating set polytope, $P_D(G)$, in the class of threshold graphs. In [4], Bouchakour and Mahjoub study $P_D(G)$ in the graphs that decompose by one-node cutsets. It is shown that if $G$ decomposes into $G_1$ and $G_2$, then the dominating set polytope of $G$ can be described from two linear systems related to $G_1$ and $G_2$. In [5], Bouchakour et al. discuss the dominating set polytope in cactus graphs. As a consequence, they obtain a characterization of the polytope when the graph is a cycle.

The paper is organized as follows. In the next section, we give a description of the polytope $P_{ID}(C_n)$ and present some structural properties of its facets. In Section 3 we prove our main result. In Section 4 we study the separation problem for the system describing $P_{ID}(C_n)$. In Section 5, we study a lifting operation and discuss some consequences.

In the rest of this section we give some definitions and notation. We consider finite, undirected and loopless graphs. We denote a graph by $G = (V, E)$ where $V$ is the *node set* and $E$ the *edge set*. If $G = (V, E)$ is a graph and $e \in E$ is an edge whose endnodes are $u$ and $v$, then we write $e = (u, v)$. A *path* $P$ of $G = (V, E)$ is a sequence of nodes $v_0, v_1, \ldots, v_k$, such that $(v_i, v_{i+1})$ is an edge for $i = 0, \ldots, k-1$ and no node appears more than once in $P$. The nodes $v_0$ and $v_k$ are the endnodes of $P$ and we say that $P$ links $v_0$ and $v_k$. If $(v_0, v_k) \in E$ and $k \geq 2$, then the sequence $v_0, v_1, \ldots, v_k$ is also called a *cycle*. Throughout the paper, we will denote by $C_n$ a cycle on $n$ nodes and by $1, \ldots, n$ its nodes. If $i, j \in C_n$, we will denote by $C_n(i, j)$ the path $i, i+1, \ldots, i+t = j$ of $C_n$ between $i$ and $j$, where the integers are modulo $n$.

## 2. The polytope $P_{ID}(C_n)$

### 2.1. Description

In [8], Contenza introduces a general class of valid inequalities for $P_D(C_n)$ as follows. Let $C_n = \{1, \ldots, n\}$. Let $s \in \{1, \ldots, n-2\}$, $i_1 < \cdots < i_s$, be nodes of $C_n$ and $I_1, \ldots, I_s$ disjoint node subsets of $C_n$ such that $I_l = \{i_l, \ldots, i_l + 3k_l + 1\}$ for some integer $k_l \geq 0$, and $i_{l+1} \geq i_l + 3(k_l + 1) + 1$ for $l = 1, \ldots, s$ (the indices are taken modulo $s$). Let $J_l$ be the set of nodes of $C_n$ between $I_l$ and $I_{l+1}$ different from $i_l + 3k_l + 2$ and $i_{l+1} - 1$, and let $r_l = |J_l|$ for $l = 1, \ldots, s$. Consider the inequality

$$\sum_{l=1}^{s} \sum_{j \in I_l} x(j) - \sum_{l=1}^{s} \sum_{j \in J_l} x(j) \geq \sum_{l=1}^{s} (k_l + 1) - \left\lfloor \frac{\sum_{l=1}^{s} (r_l + 1)}{2} \right\rfloor. \tag{6}$$

For example, for $n = 12$, if $s = 2$, $I_1 = \{1, 2, 3, 4, 5\}$ and $I_2 = \{8, 9\}$, we have the following valid inequality for $P_{ID}(C_{12})$:

$$\begin{aligned} &x(1) + x(2) + x(3) + x(4) + x(5) + x(8) + x(9) - x(11) \\ &\quad \geq 2 + 1 - \left\lfloor \frac{(1+2)}{2} \right\rfloor = 2. \end{aligned}$$

**Theorem 2.1** ([8]). *Inequality* (6) *is valid for* $P_{ID}(C_n)$.

**Proof.** The following inequalities are valid for $P_{ID}(C_n)$.

$$\begin{aligned} x(N(i)) &\geq 1 \qquad \text{for } i = i_l + 3q, i_l + 3q + 1, \\ &\qquad\qquad q = 0, \ldots, k_l;\ l = 1, \ldots, s, \\ -x(i) - x(i+1) &\geq -1 \qquad \text{for } i = i_l + 3k_l + 2, \ldots, i_{l+1} - 2, \\ &\qquad\qquad l = 1, \ldots, s. \end{aligned}$$

By summing these inequalities we obtain

$$2 \sum_{l=1}^{s} \sum_{j \in I_l} x(j) - 2 \sum_{l=1}^{s} \sum_{j \in J_l} x(j) \geq 2 \sum_{l=1}^{s} (k_l + 1) - \sum_{l=1}^{s} (r_l + 1).$$

By dividing by 2 and rounding up the right hand side to the next highest integer we obtain inequality (6). □

We can now state our main result.

**Theorem 2.2.** *$P_{ID}(C_n)$ is completely described by inequalities (1), (2), (4)–(6).*

The proof of Theorem 2.2 will be given in Section 3. In what follows, we are going to discuss some structural properties of the facets of $P_{ID}(C_n)$ which will be useful for that proof.

### 2.2. Structural properties

Consider a cycle $C_n$ with $n \geq 8$. Hence, $P_{ID}(C_n)$ is full dimensional. Let $ax \geq a_0$ be a constraint that defines a facet of $P_{ID}(C_n)$ different from constraints (1) and (2). Let $\Omega(C_n)$ be the set of independent dominating sets of $C_n$, and let

$$S_a = \{S \in \Omega(C_n) \mid ax^S = a_0\}.$$

In what follows we will also consider $a(i)$ as a weight on $i$. Hence, any 0–1 solution $S$ of $S_a$ will have a weight $a(S)$ equal to $a_0$, and any 0–1 solution of $\Omega(C_n)$ a weight greater than or equal to $a_0$. We have the following lemmas; the first one is a direct consequence of the fact that $ax \geq a_0$ is different from inequalities (1) and (2).

**Lemma 2.3.** (i) *For every node $i \in C_n$, there is a node set $S \in S_a$ such that $|S \cap N(i)| \geq 2$.*
(ii) *For every $i \in C_n$, there is a node set $S' \in S_a$ such that $S' \cap \{i, i+1\} = \emptyset$.*

A consequence of Lemma 2.3 is that for every $i \in C_n$, there is a set $S \in S_a$ such that $S \cap N(i) = \{i-1, i+1\}$.

**Lemma 2.4.** *For all $i \in C_n$, we have*

(i) $a(i) \geq \min(a(i+1), a(i+1) - a(i+2))$,
(ii) $a(i) \leq \max(a(i+1), a(i+1) - a(i+2), a(i+1) - a(i+2) + a(i+3))$,
(iii) $a(i) \geq \min(a(i-1), a(i-1) - a(i-2))$,
(iv) $a(i) \leq \max(a(i-1), a(i-1) - a(i-2), a(i-1) - a(i-2) + a(i-3))$.

**Proof.** We shall show (i) and (ii), (iii) and (iv) follow by symmetry.

(i) By Lemma 2.3(ii), there is a solution $S_1 \in S_a$ such that $i-1, i \notin S_1$. By inequality (2), it follows that $i+1 \in S_1$. If $i+3 \in S_1$, as the node set $S_1' = (S_1 \setminus \{i+1\}) \cup \{i\}$ is a solution of $\Omega(C_n)$, we have that $a(S_1') \geq a_0$, and therefore $a(i) \geq a(i+1)$. If not, then $i+4 \in S_1$ and, hence, $(S_1 \setminus \{i+1\}) \cup \{i, i+2\}$ is a solution of $\Omega(C_n)$. This implies that $a(i) + a(i+2) \geq a(i+1)$, and in consequence (i) holds.

(ii) By Lemma 2.3(i), there is a set $S_2 \in S_a$ that contains $i-2, i$. If $i+3 \in S_2$, then $(S_2 \setminus \{i\}) \cup \{i+1\}$ is a solution of $\Omega(C_n)$ and hence

$$a(i) \leq a(i+1). \tag{7}$$

If this is not the case, then $i+2 \in S_2$.

– If $i + 4 \notin S_2$, then $i + 5 \in S_2$. Consequently, $(S_2 \setminus \{i, i + 2\}) \cup \{i + 1, i + 3\} \in \Omega(C_n)$, implying that

$$a(i) + a(i + 2) \leq a(i + 1) + a(i + 3). \tag{8}$$

– If $i + 4 \in S_2$, then $(S_2 \setminus \{i, i + 2\}) \cup \{i + 1\} \in \Omega(C_n)$ and we obtain that

$$a(i) + a(i + 2) \leq a(i + 1). \tag{9}$$

Combining (7)–(9) yields (ii). □

The following lemmas are given without proof; for the proof see [20].

**Lemma 2.5.** *Let $i \in C_n$. Suppose that $a(j) = \delta > 0$ for $j = i, \ldots, i + p - 1$ for some integer $p \geq 1$ and $\delta \in \mathbb{R}$. Suppose also that $a(i - 1), a(i + p) < \delta$. Then $|C_n(i + p, i - 1)| \geq 3$.*

**Lemma 2.6.** *Let $i \in C_n$. Suppose that $a(j) = \delta > 0$ for $j = i, \ldots, i + p - 1$ for some integer $p \geq 1$ and $\delta \in \mathbb{R}$. Suppose also that $a(i - 1), a(i + p) < \delta$. Then $p = 3k + 2$ for some $k \geq 0$.*

**Lemma 2.7.** (i) *Let $i \in C_n$. Suppose that $a(j) = \delta > 0$ for $j = i, \ldots, i + 3k + 1$ for some integer $k \geq 0$ and $\delta \in \mathbb{R}$. Suppose also that $a(i - 1), a(i + 3k + 2) < \delta$. Then there exists an independent dominating set of $S_a$ such that $i - 1, i + 1, i + 3 \in S$.*

(ii) *Let $i \in C_n$. Suppose that $a(j) = \delta > 0$ for $j = i - (3k + 1), \ldots, i$ for some integer $k \geq 0$ and $\delta \in \mathbb{R}$. Suppose also that $a(i - 3k - 2), a(i + 1) < \delta$. Then there exists an independent dominating set of $S_a$ such that $i - 3, i - 1, i + 1 \in S$.*

**Lemma 2.8.** (i) *If for some $i \in V_n$, $a(i) = a(i + 1) < 0$ and $a(i) < a(i - 1)$, then $a(i - 2) > 0$, and $a(i - 2) > a(i - 1)$.*

(ii) *If for some $i \in V_n$, $a(i) = a(i + 1) < 0$ and $a(i + 1) < a(i + 2)$, then $a(i + 3) > 0$, and $a(i + 3) > a(i + 2)$.*

(iii) *If for some $i \in V_n$, $a(i) = a(i + 1)$ and $a(i) > a(i - 1)$, then $a(i - 1) \geq 0$ and $a(i - 2) \leq a(i - 1)$. Moreover, there exists an integer $k \geq 0$ such that $a(j) = a(i)$ for $j = i, \ldots, i + 3k + 1$ and $a(i) > a(i + 3k + 2)$.*

(iv) *If for some $i \in V_n$, $a(i - 1) = a(i)$ and $a(i) > a(i + 1)$, then $a(i + 1) \geq 0$ and $a(i + 2) \leq a(i + 1)$. Moreover, there exists an integer $k \geq 0$ such that $a(j) = a(i)$ for $j = i - (3k + 1), \ldots, i$ and $a(i) > a(i - 3k - 2)$.*

(v) *If for some $i \in V_n$, $a(i) = a(i + 1) \geq 0$ and $a(i) < a(i - 1)(a(i + 1) < a(i + 2))$, then $a(i - 2) = a(i - 1) = a(i + 2) = a(i + 3)$.*

**Lemma 2.9.** (i) *If for some $i \in V_n$, $a(i) = a(i+1)$ and $a(i) > a(i-1)$, then $a(i-1) = 0$.*

(ii) *If for some $i \in V_n$, $a(i - 1) = a(i)$ and $a(i) > a(i + 1)$, then $a(i + 1) = 0$.*

**Lemma 2.10.** (i) *If $a(i - 1) < a(i)$ (resp. $a(i - 1) > a(i)$) for some $i \in V_n$, then $a(i) \leq a(i + 1)$ (resp. $a(i - 2) \geq a(i - 1)$).*

(ii) *If $a(i - 1) < a(i) < a(i + 1)$ (resp. $a(i - 1) > a(i) > a(i + 1)$) for some $i \in V_n$, then*

(1) $a(i) = 0$,

(2) $a(i - 1) = -a(i + 1)$.

## 3. Proof of Theorem 2.2

Let $ax \geq a_0$ be a facet defining inequality of $P_{ID}(C_n)$ different from (1), (2), (4) and (5). We will show that $ax \geq a_0$ is necessarily of type (6). To this end, let $M$ denote $\max\{a(j), j \in V_n\}$.

**Lemma 3.1.** *There exists $i \in V_n$ such that $a(i-1) < a(i)$.*

**Proof.** Suppose that $a(j) = M$ for all $j \in V_n$. If $M > 0$ (resp. $M < 0$), then $ax \geq a_0$ is of type (5) (resp. (4)) which contradicts the hypothesis. □

Now, let us denote by $I_a$ the set of nodes $i \in C_n$ such that $a(i) = M$ and $a(i-1) < M$. Note that by Lemma 3.1, $I_a \neq \emptyset$. Let $s = |I_a|$, and $I_a = \{i_1, \ldots, i_s\}$ such that $1 \leq i_1 < \cdots < i_s \leq n$. Furthermore, as

$$a(i_l - 1) < a(i_l), \tag{10}$$

we have, by Lemma 2.10(i), that

$$a(i_l) = a(i_l + 1), \qquad \text{for } l = 1, \ldots, s. \tag{11}$$

**Lemma 3.2.** (i) *For each $l = 1, \ldots, s$, there exists an integer $k_l \geq 0$ such that $a(j) = M$, for $j = i_l, \ldots, i_l + 3k_l + 1$, and $M > a(i_l + 3k_l + 2)$.*
(ii) $M > 0$.

**Proof.** (i) is a direct consequence of (10) and (11) together with Lemma 2.8(iii).

(ii) By (10) and (11) with respect to node $i_1$, Lemma 2.8(iii) yields $a(i_1 - 1) \geq 0$. As $M = a(i_1)$, by (10) it follows that $M > 0$. □

Denote the set $C_n(i_l, i_l + 3k_l + 1)$ by $I_l$ for $l = 1, \ldots, s$.

**Lemma 3.3.** $|C_n(i_l + 3k_l + 2, i_{l+1} - 1)| \geq 2$, *for $l = 1, \ldots, s$.*

**Proof.** If $s = 1$, then the result is a direct consequence of Lemma 2.5. If not, by (10) and (11) with respect to node $i_{l+1}$, Lemma 2.8(iii) yields $a(i_{l+1} - 2) \leq a(i_{l+1} - 1)$, for $l = 1, \ldots, s$ (here the indices are modulo $s$). Hence, $i_{l+1} - 2 \notin C_n(i_l, i_l + 3k_l + 1)$ and the lemma follows. □

**Lemma 3.4.** $a(i_l - 1) = a(i_l + 3k_l + 2) = 0$ *for $l = 1, \ldots, s$.*

**Proof.** Let $l \in \{1, \ldots, s\}$. By (10) and (11) together with Lemma 2.9(i), we have that $a(i_l - 1) = 0$. As $a(i_l + 3k_l) = a(i_l + 3k_l + 1)$ and $a(i_l + 3k_l + 2) < a(i_l + 3k_l + 1)$, by Lemma 2.9(ii) we obtain $a(i_l + 3k_l + 2) = 0$. □

Let $J_l = C_n(i_l + 3k_l + 3, i_{l+1} - 2), l = 1, \ldots, s$, that is $J_l$ is the set of nodes of $C_n$ between $I_l$ and $I_{l+1}$, different from $i_l + 3k_l + 2$ and $i_{l+1} - 1$. Let $r_l = |J_l|$.

**Lemma 3.5.** $a(j) = -M$ *for $j \in J_l, l = 1, \ldots, s$.*

**Proof.** Let $l \in \{1, \ldots, s\}$. If $J_l = \emptyset$, then there is nothing to prove. Suppose now that $r_l \geq 1$. Let $j_l = i_l + 3k_l + 2$.

As $a(j_l - 2) = a(j_l - 1)$ and $a(j_l - 1) > a(j_l)$, by Lemma 2.8(iv) with respect to node $j_l - 1$, we have that $a(j_l + 1) \leq a(j_l)$. Moreover, by Lemma 3.4, $a(j_l) = 0$.

Hence, if $a(j_l+1) = a(j_l)$, then by Lemma 2.8(v) with respect to node $j_l$, we obtain that $a(j_l+2) = a(j_l-1) = M$. This implies that $j_l+2 = i_{l+1}$. So we obtain $|C_n(j_l, i_{l+1}-1)| = 2$, that is $r_l = 0$, a contradiction. Thus, $a(j_l+1) < a(j_l)$. Now, by Lemma 2.10(ii), $a(j_l+1) = -a(j_l-1) = -M$.

So, the statement holds if $r_l = 1$. Now, suppose that $r_l \geq 2$.

**Claim 1.** *If* $a(j) = -M, j = j_l+1, \ldots, j_l+t$, *for some* $1 \leq t \leq r_l - 1$, *then* $a(j_l+t+1) = -M$.

**Proof.** Assume that $a(j) = -M$, for $j = j_l+1, \ldots, j_l+t$ for some integer $1 \leq t < r_l$. By Lemma 2.4(iii), it follows that

$$\begin{aligned} a(j_l+t+1) &\geq \min(a(j_l+t), a(j_l+t) - a(j_l+t-1)) \\ &= \min(-M, -M - a(j_l+t-1)). \end{aligned}$$

Furthermore, as $a(j_l) = 0$ and $a(j_l+1) = \cdots = a(j_l+t) = -M$, we have that $a(j_l+t-1) \leq 0$. It then follows that $a(j_l+t+1) \geq -M$.

Suppose that $a(j_l+t+1) > -M = a(j_l+t)$. Thus, by Lemma 2.10(i), $a(j_l+t+2) \geq a(j_l+t+1)$. If $a(j_l+t+2) = a(j_l+t+1)$, then by Lemma 2.8(iii) with respect to node $j_l+t+1$, we obtain that $a(j_l+t) = -M \geq 0$, a contradiction. If $a(j_l+t+2) > a(j_l+t+1)$, then Lemma 2.10(ii) yields $a(j_l+t+1) = 0$ and $a(j_l+t+2) = -a(j_l+t) = M$. Therefore, $j_l+t+2 = i_{l+1}$. But, since $t < r_l$, this contradicts the fact that $j_l+t+2 \leq j_l+r_l+1 = i_{l+1}-1$. Consequently, $a(j_l+t+1) = -M$, and the claim is proved. ♦

As $a(j_l+1) = -M$, the lemma follows. □

As $M > 0$, we can suppose without loss of generality that $M = 1$. Thus, the facet defining inequality $ax \geq a_0$ can be written as

$$\sum_{l=1}^{s} \sum_{j \in I_l} x(j) - \sum_{l=1}^{s} \sum_{j \in J_l} x(j) \geq a_0 \tag{12}$$

with $a_0 \in \mathbb{Z}$. Let $\alpha_0$ denote the right hand side of inequality (6). Now to complete the proof, it suffices to show the following.

**Lemma 3.6.** $a_0 = \alpha_0$.

**Proof.** First, note that, as by Theorem 2.1 inequality (6) is valid for $P_{ID}(C_n)$, we have that $a_0 \geq \alpha_0$. In what follows, we are going to exhibit an independent dominating set $S \in C_n$ such that $ax^S = \alpha_0$.

Without loss of generality, we may suppose that $I_a = \{i_1 = 1, i_2, \ldots, i_s\}$. Let $C_p = \{v_1, v_2, \ldots, v_p\}$ be the cycle deduced from $C_n$ by contracting the edges $(i_l+3q, i_l+1+3q)$, for $q = 0, \ldots, k_l,\ l = 1, \ldots, s$. Remark that a node $v_j \in C_p$ corresponds either to an edge $(i, i+1)$ or to a node $i$ of $C_n$. As $i_1 = 1$, $v_1$ corresponds to edge $(1, 2)$, $v_{p-1}$ to node $n-1$ and $v_p$ to node $n$. Also note also that $p = n - \sum_{l=1}^{s}(k_l+1)$ and $\tilde{S} = \{v_1, v_3, \ldots, v_{2\lfloor \frac{p}{2} \rfloor - 1}\}$ is an independent dominating set of $C_p$. Let $A_1$ be the set of nodes $i \in C_n$ such that there exists $v_j \in \tilde{S}$ where $i = v_j$. And let $A_2$ be the set of nodes of $C_n$ of the form $i_l+1+3q, 0 \leq q \leq k_l$ and $1 < l \leq s$ such that there exists a node $v_j \in \tilde{S}$ which corresponds to edge $(i_l + 3q, i_l + 1 + 3q)$. Let $S = A_1 \cup A_2 \cup \{1 + 3t, t = 0, \ldots, k_1\}$. We claim that $S$ is an independent dominating set of $C_n$. In fact, first, note that as $\tilde{S}$ is an independent set, $S$ is so. Now, we shall show that $S$ is a dominating set of $C_n$, that is $x^S(N(i)) \geq 1$, for all $i \in C_n \setminus S$. As node 1 belongs to $S$, $x^S(N(n)) \geq 1$. Moreover, as $i_1 = 1$, we have that $n-1, n \notin C_n(i_s, i_s+3k_s+1)$. Thus, If $p$ is even, then $v_{2\lfloor \frac{p}{2} \rfloor -1} = n-1$, and hence $x^S(N(n-1)) = 1$. If $p$ is odd, then $n - 1 \notin S$. So, if $v_{p-2} = n - 2$, since $n - 2$ belongs to $A_1$ and hence to $S$, it follows that $x^S(N(n-1)) = x^S(n-2) = 1$. If this is not the case, then $v_{p-2}$ corresponds to $(n - 3, n - 2)$. As by Lemma 2.5, $n - 3, n - 2 \notin C_n(i_1, i_1 + 3k_1 + 1)$, we have that $n - 2$ belongs to $A_2$ and hence to $S$. This implies that $x^S(N(n-1)) = x^S(n-2) = 1$.

Now, let $i \in C_n \setminus (S \cup \{n-1, n\})$. Suppose first that $i = v_j$ for some $v_j \in C_p$. If either $v_{j-1} = i-1$ or $v_{j+1} = i+1$, as $\{i-1, i+1\} \cap S \neq \emptyset$, $x^S(N(i)) = x^S(i-1)+x^S(i+1) \geq 1$. If not, then $v_{j-1}$ would correspond to the edge $(i - 2, i - 1)$ and $v_{j+1}$ to $(i + 1, i + 2)$. Moreover, we have $v_{j-1}, v_{j+1} \in \tilde{S}$. So, if $i \in C_n(i_1, i_1 + 3k_1 + 1)$, then $i - 2, i + 1$ belong to $S$, and hence $x^S(N(i)) = x^S(i + 1) = 1$. If not, then $i \in C(i_l, i_l + 3k_l + 1)$, for some $1 < l \leq s$. Thus $i - 1, i + 2 \in A_2$ and hence $x^S(N(i)) = x^S(i - 1) = 1$.

Now, suppose that $v_j = (i, i + 1)$ and $v_j \notin \tilde{S}$ (otherwise either $i$ or $i - 1$ is in $S$). Then, $v_{j-1} = i-1,\ v_{j+1} = i+2$ and $v_{j-1}, v_{j+1} \in \tilde{S}$. So, $i-1$ and $i+2$ belong to $A_1$ and hence to $S$. Thus, $x^S(N(i)) = x^S(i - 1) = 1$ ($x^S(N(i + 1)) = x^S(i + 2) = 1$). This implies that $S$ is a dominating set of $C_n$.

Now, it remains to show that $ax^S = \alpha_0$. For this, note first that, if $p$ is even (resp. odd), then the incidence vector of $S$, $x^S$, satisfies as an equation the inequalities

$$x^S(N(i)) \geq 1, \qquad i = i_l + 3q, i_l + 1 + 3q, \quad q = 0, \ldots, k_l,\ l = 1, \ldots, s,$$

and the inequalities

$$x^S(i) + x^S(i + 1) \leq 1,\ i = i_l + 3k_l + 2, \ldots, i_{l+1} - 2,\ l = 1, \ldots, s. \quad (\text{resp. } i = i_l + 3k_l + 2, \ldots, i_{l+1} - 2,\ l = 1, \ldots, s - 1,\ \text{and } i = i_s + 3k_s + 2, \ldots, n - 2).$$

By adding these inequalities, we obtain

$$2 \sum_{l=1}^{s} \sum_{j \in I_l} x^S(j) - 2 \sum_{l=1}^{s} \sum_{j \in J_l} x^S(j) = 2 \sum_{l=1}^{s} (k_l + 1) - \sum_{l=1}^{s} (r_l + 1)$$

$$\left(\text{resp. } 2 \sum_{l=1}^{s} \sum_{j \in I_l} x^S(j) - 2 \sum_{l=1}^{s} \sum_{j \in J_l} x^S(j) = 2 \sum_{l=1}^{s} (k_l + 1) - \left(\sum_{l=1}^{s} (r_l + 1) - 1\right)\right).$$

As $n = \sum_{l=1}^{s} 3(k_l+1)+\sum_{l=1}^{s}(r_l+1)$, $\sum_{l=1}^{s}(r_l+1)$ is even (resp. odd). So, by dividing by 2 the above equality, we obtain that $ax^S = \alpha_0$.

In consequence, we have that $a_0 = \alpha_0$ which ends the proof of the claim. $\square$

By (12) together with Lemma 3.6, it follows that the inequality $ax \geq a_0$ is of type (6), and the proof of our theorem is complete.

## 4. Separation and algorithmic consequences

The separation problem for a class of inequalities consists in deciding whether a given vector $\bar{x} \in \mathbb{R}^n$ satisfies the inequalities, and if not, in finding an inequality that is violated by $\bar{x}$. An algorithm that solves this problem is called a *separation algorithm*. A fundamental result in combinatorial optimization is the well known equivalence between optimization and separation. That is, there exists a polynomial time algorithm for optimizing over a class of inequalities if and only if the separation problem for this class can be solved in polynomial time. Thus, if for a class of inequalities there exists a polynomial time separation algorithm, then it can be used efficiently in the framework of a cutting plane algorithm for solving the corresponding optimization problem.

Clearly, the separation problem for inequalities (1), (2), (4) and (5) can be solved in polynomial time. In what follows we shall show that inequalities (6) can also be separated in polynomial time. As it will turn out, the separation problem for these inequalities reduces to a shortest path problem in an appropriate directed graph.

**Theorem 4.1.** *Inequalities (6) can be separated in polynomial time on $C_n$.*

**Proof.** Let $\bar{x} \in \mathbb{R}^n$. We may suppose that $\bar{x}$ satisfies inequalities (1), (2), (4) and (5). Hence for the proof we can only consider inequalities (6) where $\sum_{l=1}^{s}(r_l + 1)$ is odd. The inequalities with $\sum_{l=1}^{s}(r_l + 1)$ even are redundant with respect to inequalities (1) and (2). An inequality of type (6) with $\sum_{l=1}^{s}(r_l + 1)$ odd can then be written as

$$\sum_{l=1}^{s} \sum_{j=i_l}^{i_l+3k_l+1} x(j) - \sum_{j=i_l+3(k_l+1)}^{i_{l+1}-2} x(j) \geq \sum_{l=1}^{s}(k_l + 1) - \frac{\sum_{l=1}^{s}(r_l + 1) - 1}{2}. \tag{13}$$

As

$$2 \sum_{j=i_l}^{i_l+3k_l+1} x(j) = \sum_{q=0}^{k_l} (x(N(i_l + 3q)) + x(N(i_l + 3q + 1))) - x(i_l - 1) - x(i_l + 3k_l + 2),$$

and

$$-2 \sum_{j=i_l+3(k_l+1)}^{i_{l+1}-2} x(j) = - \sum_{j=i_l+3k_l+2}^{i_{l+1}-2} (x(j) + x(j + 1)) + x(i_l + 3k_l + 2) + x(i_{l+1} - 1)$$

for $l = 1, \ldots, s$, inequality (13) can be written as

$$\sum_{l=1}^{s}\left(\sum_{q=0}^{k_l}(x(N(i_l+3q))+x(N(i_l+3q+1))-2)\right.$$
$$\left.-\sum_{j=i_l+3k_l+2}^{i_{l+1}-2}(x(j)+x(j+1)-1)\right)\geq 1. \tag{14}$$

Now, consider the directed graph $G=(U\cup V,E)$ such that

$$U=\{u_1,\dots,u_n\},$$
$$V=\{v_1,\dots,v_n\},$$
$$E=E_1\cup E_2,$$

where

$$E_1=\left\{(u_j,u_{j+3}),(v_j,v_{j+3});\ j=1,\dots,n\right\},$$
$$E_2=\left\{(u_j,v_{j+1});\ j=1,\dots,n\right\}\cup\left\{(v_j,u_{j+1});\ j=2,\dots,n-1\right\}.$$

Here the indices are taken modulo $n$.

Graph $G$ is constructed so that an arc of type either $(u_j,u_{j+3})$ or $(v_j,v_{j+3})$ corresponds to the valid inequality $x(N(j+1))+x(N(j+2))-2\geq 0$, and an arc of type $(u_j,v_{j+1})$ or $(v_j,u_{j+1})$ to the inequality $-x(j)-x(j+1)+1\geq 0$.

With an arc $e\in E_1$ of type $e=(u_j,u_{j+3})$ or $e=(v_j,v_{j+3})$ we associate the weight $w(e)=\bar{x}(N(j+1))+\bar{x}(N(j+2))-2$. And with an edge $e\in E_2$ of type either $e=(u_j,v_{j+1})$ or $e=(v_j,u_{j+1})$ we associate the weight $w(e)=1-\bar{x}(j)-\bar{x}(j+1)$. Note that as $\bar{x}$ satisfies inequalities (1) and (2), $w(e)\geq 0$ for all $e\in E$.

As it will turn out, the separation problem for inequalities (6) reduces to a shortest path problem in $G$. We are going to show that $\bar{x}$ satisfies all inequalities (6) if and only if there does not exist a path between two nodes $u_j$ and $v_j$ of length $<1$. Indeed consider an inequality of type (6) induced by $s$ pairwise disjoint subsets $I_1,\dots,I_s$ of $C_n$.

For $l=1,\dots,s$, let $P_l$ be the (unique) path of $G$ given by

$$(u_{i_l-1},u_{i_l+2},\dots,u_{i_l+3k_l+2},v_{i_l+3k_l+3},u_{i_l+3k_l+4},\dots,v_{i_l+3k_l+3+r_l-1},u_{i_l+3k_l+3+r_l})$$
$$(\text{resp. }(u_{i_l-1},u_{i_l+2},\dots,u_{i_l+3k_l+2},v_{i_l+3k_l+3},u_{i_l+3k_l+4},\dots,u_{i_l+3k_l+3+r_l-1},$$
$$v_{i_l+3k_l+3+r_l})),$$

if $r_l$ is odd (resp. even). And denote by $Q_l$ the path obtained from $P_l$ by replacing $u$ by $v$. Note that path $P_l$ (resp. $Q_l$) is the union of a path in $U$ $(V)$ of length $k_l+1$ and an alternative path between $U$ and $V$ of length $r_l+1$. Also note that $i_l+3k_l+3+r_l=i_{l+1}-1$. Now let $L_1,\dots,L_s$ be the paths defined in a recursive way as follows:

$L_1\leftarrow P_1$,
**for** $l=1,\dots,s-1$ **do**
  **if** $T(L_l)\in U$ **then**
    $L_{l+1}=P_{l+1}$
  **else**
    $L_{l+1}=Q_{l+1}$

where $T(L_l)$ is the end node (tail) of $L_l$. Let

$$L = \bigcup_{i=1}^{s} L_i.$$

As each path $P_l$ ($Q_l$) contains exactly $r_l + 1$ arcs between $U$ and $V$, $L$ contains $\sum_{l=1}^{s} r_l + s$ arcs between $U$ and $V$. Hence, $L$ is a path from $u_{i_1-1}$ to $v_{i_1-1}$. Moreover, its weight is equal to

$$\begin{aligned} w(L) = \sum_{l=1}^{s} &\left( \sum_{q=0}^{k_l} \bar{x}(N(i_l + 3q)) + \bar{x}(N(i_l + 3q + 1)) - 2 \right) \\ &+ \sum_{l=1}^{s} \left( \sum_{j=i_l+3k_l+2}^{i_{l+1}-2} (1 - \bar{x}(j) - \bar{x}(j+1)) \right). \end{aligned}$$

So if inequality (13) is violated, by (14), one should have $w(L) < 1$.

Conversely, given a path $L$ in $G$ from a node $u_j$ to $v_j$ for $j = 1, \ldots, n$, one can associate an inequality of type (6) in such a way that the left hand side of the corresponding inequality (14) is equal to the weight of $L$. In fact, let $L'_1, \ldots, L'_s$ be the subpaths of $L$ that are either contained in $U$ or in $V$. Let $t_{i_l-1}$ be the initial node of $L'_l$, for $l = 1, \ldots, s$. Here $t$ stands for either $u$ or $v$. Note that each path $L'_l$ is of length $k_l + 1$ for some $k_l \geq 0$. Now, let

$$I_l = \{i_l, i_l + 1, \ldots, i_l + 3k_l + 1\}, \qquad \text{for } l = 1, \ldots, s.$$

It is not hard to see that constraint (13) associated with $\{I_l, l = 1, \ldots, s\}$ has a left hand side equal to $w(L)$.

In consequence, to separate inequalities (6), one can compute the shortest path in $G$ between $u_j$ and $v_j$, for $j = 1, \ldots, n$ with respect to the weights $\{w(e), e \in E\}$. And then consider the shortest path among these paths. If the length of such a path is $\geq 1$, then no constraint is violated. If not, then that path yields a violated inequality of type (6).

As the shortest path problem with non-negative weights can be solved in polynomial time [10], the theorem follows. □

From [16], we then have the following:

**Corollary 4.2.** *The MWIDSP is polynomially solvable on a cycle.*

## 5. Twin operation

In this section, we introduce a lifting operation called twin operation and discuss some polyhedral consequences. In particular, we shall show that if $G'$ is a graph obtained from a graph $G$ by the twin operation, then an inequality defines a nontrivial facet of $P_{ID}(G)$ if and only if the lifted one defines a nontrivial facet of $P_{ID}(G')$.

Let $G = (V, E)$ be a graph (not necessarily a cycle) and $v$ a node of $V$. We say that a graph $G' = (V', E')$ is obtained from $G$ by the *twin operation* with respect to $v$ if there is a node $v' \in V'$ such that

(i) $V' = V \cup \{v'\}$,
(ii) $E' = E \cup \{(v', v)\} \cup \{(v', w);\ (v, w) \in E\}$.

The nodes $v$ and $v'$ are called twins.

Let $G = (V, E)$ be a graph and $G' = (V', E')$ a graph obtained from $G$ by the twin operation with respect to a node $v \in V$. Let $v'$ be the twin of $v$. Suppose also that $P_{ID}(G)$ is full dimensional. It is then not hard to see that $P_{ID}(G')$ is also full dimensional. We have the following lemmas; for the proof see [20].

**Lemma 5.1.** *If $ax \geq \alpha$ is an inequality that defines a nontrivial facet of $P_{ID}(G')$, then $a(v) = a(v')$.*

**Lemma 5.2.** *Let $ax \geq \alpha$ be a facet defining inequality of $P_{ID}(G)$, different from $x(v) \geq 0$. Set*

$$\begin{aligned} a'(u) &= a(u) && \text{if } u \in V' \setminus \{v'\}, \\ a'(u) &= a(v) && \text{if } u = v', \\ \alpha' &= \alpha. \end{aligned}$$

*Then $a'x \geq \alpha'$ defines a facet for $P_{ID}(G')$.*

Suppose now that $P_{ID}(G)$ is given by a system $S$ of inequalities of the form

$$S \begin{cases} a_i x \geq \alpha_i, & \text{for } i \in I \\ x(u) \geq 0, & \text{for all } u \in V, \end{cases}$$

where $I$ is an index set. Hence any 0–1 solution of $S$ is the incidence vector of an independent dominating set of $G'$. Let $S'$ be the system given by

$$S' \begin{cases} a_i x + a_i(v)\, x(v') \geq \alpha_i, & \text{for } i \in I \\ x(u) \geq 0, & \text{for all } u \in V', \end{cases}$$

and denote by $a'_i x \geq \alpha'_i$ inequality $a_i x + a_i(v) x(v') \geq \alpha_i$ for $i \in I$. Note that $a'_i x \geq \alpha'_i$ is the inequality obtained from $a_i x \geq \alpha_i$ by the lifting procedure of Lemma 5.2. The following lemmas are given without proof; for the proof see [20].

**Lemma 5.3.** *Let $bx \geq \beta$ be an inequality valid for $P_{ID}(G)$. Then $bx + b(v)x(v') \geq \beta$ is redundant in $S'$.*

**Lemma 5.4.** *Every 0–1 solution of $S'$ is the incidence vector of an independent dominating set of $G'$.*

Let $P$ be the polytope given by $S'$. We can now state the main result of this section.

**Theorem 5.5.** $P_{ID}(G') = P$.

**Proof.** By Lemma 5.2 we have that $P_{ID}(G') \subseteq P$. In what follows we shall show that $P \subseteq P_{ID}(G')$. By Lemma 5.4, it suffices to show that the extreme points of $P$ are integral.

Suppose, on the contrary, that $P$ has a fractional extreme point, say $y'$. Let $y \in \mathbb{R}^V$ be the solution given by

$$y(u) = \begin{cases} y'(u) & \text{if } u \in V \setminus \{v\}, \\ y'(v) + y'(v') & \text{if } u = v. \end{cases}$$

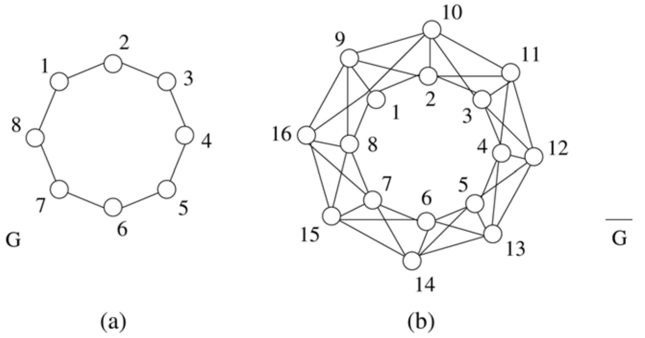


Fig. 1. $G$ and $\bar{G}$.

As $y'$ is a solution of $S'$, it follows that $y$ is a solution of $S$. Consequently, $y$ can be written as a convex combination of 0–1 extreme points, say $y_1, \ldots, y_t$, of $P_{ID}(G)$. We distinguish two cases.

**Case 1.** $y'(v) + y'(v') > 0$.

Then $y(v) > 0$ and therefore there is $t_0 \in \{1, \ldots, t\}$ such that $y_{t_0}(v) = 1$. Let $\bar{y}' \in \mathbb{R}^{V'}$ given by

$$\bar{y}'(u) = \begin{cases} y_{t_0}(u) & \text{if } u \in V' \setminus \{v, v'\}, \\ 1 & \text{if } u = v \text{ and } y'(v) > 0, \\ 1 & \text{if } u = v' \text{ and } y'(v) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that every inequality of $S'$ which is satisfied with equality by $y'$ is also satisfied with equality by $\bar{y}'$. In fact, this is clear for the non-negativity inequalities. Consider an inequality $a'_i x \geq \alpha'_i$. If $a'_i y' = \alpha'_i$, then $a_i y = \alpha_i$, and hence $a_i y_{t_0} = \alpha_i$. As $\bar{y}'(v) + \bar{y}'(v') = 1$, $a'_i(v) = a'_i(v') = a_i(v)$ and $\alpha'_i = \alpha_i$, it follows that $a'_i \bar{y}' = \alpha'_i$. Therefore $\bar{y}'$ satisfies the same equality system as $y'$. As $y' \neq \bar{y}'$, this is impossible.

**Case 2.** $y'(v) + y'(v') = 0$.

Then $y(v) = 0$, and hence $y_j(v) = 0$, for $j = 1, \ldots, t$. Let $\hat{y}' \in \mathbb{R}^{V'}$ be given by

$$\hat{y}'(u) = \begin{cases} y_1(u) & \text{if } u \in V' \setminus \{v, v'\}, \\ 0 & \text{if } u \in \{v, v'\}. \end{cases}$$

It is easy to see that $\hat{y}'$ satisfies the same equality system as $y'$. Since $y \neq \hat{y}'$, we have again a contradiction, which ends the proof of the theorem. $\square$

In order to illustrate the above constructions, consider the graph $G = (V, E)$ of Fig. 1(a). Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph of Fig. 1(b) obtained from $G$ by recursive applications of the twin operation on the nodes $1, 2, \ldots, 8$, respectively. From constraints (6) and Theorem 2.2, it follows that the constraints of $P_{ID}(G)$ different from (1) and (2) that may define facets are the following:

$$\sum_{j=1}^{8} x(j) \geq 3,$$

$$x(i) + x(i+1) - \sum_{j=i+3}^{i+6} x(j) \geq -1, \qquad \text{for } i = 1, \ldots, 8 \text{ (modulo 8).}$$

From Theorem 5.5, it follows that $P_{ID}(\bar{G})$ is given by inequalities (2) and (3) together with the inequalities

$$x(i) + x(i+1) + x(i+8) + x(i+9) \leq 1, \quad \text{for } i = 1, \ldots, 8 \text{ (modulo 16),}$$

$$x(i) + x(i+1) + x(i+8) + x(i+9) - \sum_{j=i+3}^{i+6} (x(j) + x(j+8)) \geq -1, \quad \text{for } i = 1, \ldots, 8 \text{ (modulo 16),}$$

$$\sum_{j=1}^{16} x(j) \geq 3.$$

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